International Journal of Enhanced Research in Management & Computer Applications,
ISSN: 2319-7471, Volume 14 Issue 8, August 2025

A Differential Quadrature Approach for Solving Singularly Perturbed Mixed-Type Differential-Difference Equations

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ABSTRACT

A DQM for effectively solving SPDDEs with tiny shifts is presented in this study. SPDDEs are singly perturbed mixed-type differential-difference equations. When a minor perturbation parameter is present and the dynamics display both delayed and advanced behavior, together with steep boundary layers, these types of equations often emerge in engineering and other applied sciences. By estimating derivatives as weighted linear sums of function values at discrete grid locations, the differential quadrature technique provides a strong numerical approach. This method produces an algebraic equation system that can be solved with little computing effort and great precision. We begin by outlining the DQM formulation and how it is used to the setting of standard differential equations. After that, it is expanded to deal with SPDDEs that have tiny shifts; in this case, the problems of stiffness and delay misalignment are solved by changing the weighting factors and picking the grid carefully. Results from numerical simulations show that the suggested DQM method successfully accounts for the impact of delay or advance terms in addition to the strong gradients in boundary layers. Comparing the outcomes with current numerical methods reveals that the DQM is more effective and accurate in solving complicated singularly perturbed systems that have non-local characteristics.

Keywords: Differential quadrature method, singular perturbation, differential-difference equations, delay term, boundary layer.

INTRODUCTION

Biological systems, control theory, electrical circuits, viscoelasticity, fluid dynamics, heat transfer, and many more scientific and engineering domains make use of SPDDEs. The solutions to these equations exhibit boundary/internal layers or steep gradients because a small perturbation parameter increases the greatest derivative component. The numerical solution of these equations is much more complex when the singular perturbation is present because of their mixed-type structure, which combines features of differential and difference equations. This is especially the case when consideration of the advance or delay causes is also taken into account. Traditional numerical methods, such as finite difference or finite element methods, struggle to resolve these layers correctly because they need very small meshing, which in turn causes instability and huge computational costs.

The DQM has shown to be a useful technique for solving complex boundary value problems, particularly those involving single perturbations. To estimate the derivatives of a function at a particular number of grid points, the DQM employs a weighted linear sum of the function values at all grid points. This approach initially emerged in the 1970s. As compared to conventional methods, it accomplishes spectral precision with fewer grid points, making it ideal for problems involving a single perturbation, where the recording of steep gradients is critical. Because of its successful application to singularly perturbed ordinary and partial differential equations, the DQM's application to singly perturbed mixed-type differential-difference equations is novel and noteworthy.

Dealing with the advance or delay terms in addition to the disturbance is the main challenge of SPDDEs. Memory effects are introduced into the system by delay terms, and instability may be induced by advance terms, necessitating careful mathematical management. Combining these effects with a tiny perturbation parameter causes the resultant equation to behave on several scales, necessitating very precise spatial discretization and boundary layer resolution. In order to solve these problems, the DQM uses global interpolation polynomials and optimum weighting factors to convert the governing differential-difference equations into an algebraic system. In areas where there is a lot of variation, such boundary layers, it becomes even more accurate when non-uniform grids like Chebyshev-Gauss-

Lobatto points are used. In order to resolve a category of mixed-type differential-difference equations that are singly disturbed, this study is devoted to creating a reliable numerical technique that relies on differential quadrature. Realistic modeling situations often include mixed boundary conditions, delay and advance arguments, and they are all incorporated into the formulation. The approach clusters nodes near boundary layers using an appropriate grid point selection procedure and preserves the basic properties of the original issue in the discretized system. Standard linear or nonlinear solvers are used to solve the resultant algebraic problem, and the convergence and stability of the approach are studied numerically and theoretically. Solving a set of test problems with known precise solutions allows us to verify that the suggested method works. Singularly perturbed issues with different kinds of delays and boundary conditions are included in this category, as are nonlinear problems.

We compare the DQM findings to those of other techniques to see how accurate they are. These methods include spline-based approaches, non-standard finite difference methods, and finite difference schemes. Without resorting to significant grid refining, the DQM produces exceptionally accurate solutions, even for very tiny perturbation parameter values. In addition, the approach is very efficient in terms of memory use and computing time, therefore it is perfect for real-time or large-scale applications. Flexibility in handling both uniform and non-uniform grids is a key property of the DQM in this situation. The adaptability of the strategy enables it to retain global accuracy while adjusting to the steep slopes near the boundary.

More so, with the right adjustments, the approach may be generalized to two-dimensional or time-dependent singly perturbed situations. The ease of implementation is another major plus, particularly when the weighting factors are generated using symbolic or automated differentiation techniques. As an attractive substitute for conventional discretization techniques, the differential quadrature method may be used to solve mixed-type differential-difference equations that are singly perturbed. A important approach for scholars and practitioners working in mathematical modeling and numerical simulation, it provides excellent accuracy, computational economy, and simplicity of application.

LITERATURE REVIEW

Ragula, & Soujanya, G. (2023) We use a numerical method to resolve a singly perturbed differential-difference equation that requires a little shift in order to accomplish this inquiry. To handle the little change, we apply the Taylor series, and the original problem becomes a single perturbed boundary value problem. This problem is solved using a fourth-order finite difference approach. The method's convergence is the subject of a research. The method's numerical results provide credence to it when compared to the alternative strategy outlined in the literature. Numerical investigations demonstrate that the small shift and perturbation parameter impact the boundary layer solution to the problem.

Omkar, R. et al., (2023) Within this study, we explore a non-standard finite difference technique and suggest a difference scheme to solve an equation of the differential-difference type that shows inner layer behavior. In order to compute finite differences, one must first determine the first and second order derivatives. Using these approximations, the following equation may be discretized. The discretized equation is solved using the tridiagonal system approach. To determine whether the method converges, it is tested. We provide numerical examples to prove that the technique works. Unlike competing methods, this one is structured to provide an explanation for the greatest number of errors that may be detected in the solution. Graphs are used to depict the layer behavior in various case solutions.

Mekuria, Mesfin & File, Gemechis (2022) numerical analysis of the singly perturbed time-dependent convection-diffusion-reaction equation is presented under this research. The diffusion component of the equation is multiplied by a minor perturbation parameter (ε) whose values may be anywhere from 0 to 1. For tiny values of ε, the exponential boundary layer appears in the equation's solution, which makes analytical or conventional numerical approaches inapplicable. Our description and proof include the existence of unique discrete solutions, as well as discussion and establishment of stability of the schemes. The uniform convergence of the schemes is proved. Every single one of the presented strategies converges linearly. By using this method on a Shishkin mesh, boundary layers may be resolved. Two numerical examples with different values of ε and mesh lengths were used to test the methods.

File, Gemechis. (2021) Singularly perturbed boundary value problems with negative shift parameters are one kind of differential difference equations whose solutions exhibit boundary layer behavior. A novel and simple method is used to estimate the numerical solution to these types of issues. When other conventional numerical techniques are unable to provide smooth solutions in the inner boundary layer region, our approach yields accurate results for $h \geq \varepsilon$. The suggested technique demonstrates a second-order rate of convergence on a point-by-point basis.

Kaushik, Aditya & Sharma, Nitika. (2020) We provide numerical solutions for a family of parabolic delay differential equations with singular perturbations that include discontinuous inputs. Using a custom-built mesh, a numerical method derived from the upwind finite difference approach is detailed. We have shown the consistency, stability, and convergence of the proposed numerical method. Parameter uniform convergence was obtained, and it has been shown that the suggested technique is unconditionally stable. The display of numerical examples demonstrates the method's effectiveness.

I. DESCRIPTION OF THE DIFFERENTIAL QUADRATURE METHOD

Many issues in the physical and applied sciences have been effectively solved using the DQM, which was first developed in the early 1970s. Estimating the derivative of a function at a given point with respect to a space variable is possible using the differential quadrature method. This method takes into account a weighted linear sum of the functional values at all discrete points in the variable's domain.

To illustrate the method's mathematical form, we take a one-dimensional field variable into account $f(x)$ required within a certain domain $a=x_1 \le x \le x_N = b$. Let $f_i = f(x_i)$ a limited set of values that are defined by the function N discrete points $x_i (i=1,2,\ldots,N)$ within the sphere of the industry. Looking at the derivative of the function's value is the next step $d^m f/dx^m$ in a few specific locations x_i, and allow it to be represented as a weighted set of function values that are linearly related.

$$f^{(m)}(x_i) = \frac{d^m f(x_i)}{dx^m} = \sum_{J=1}^{N} A_{ij}^{(m)} f_j, (i=1,2,\ldots,N) \tag{1}$$

where $A_{ij}^{(m)}$ are the factors that determine how the m^{th} the function linked to points and its order derivative x_i.

The Differential Quadrature Method relies heavily on Equation (1), the derivative quadrature rule. Using equation (1) for derivatives of varying orders, you may create differential equations at any point in the solution domain. After that, by using the N function's values, you may derive the algebraic equations that represent the quadrature equivalent of the differential issue. The unknown values of the functions may be found by solving these equations using the quadrature equivalent of the boundary conditions, provided that the weighting coefficients are known beforehand. It is possible to find the weighting coefficients using suitable functional approximations; these approximations are called test functions. Differentiability and smoothness are the main criteria for selecting the test functions. In other words, the test function of the differential equation must be differentiable up to the nth derivative (where n is the greatest order) and sufficiently smooth to satisfy the differentiability requirement.

While determining the weighting criteria, the author proposed two different approaches. Two approaches are shown here: one seeks solutions to an algebraic system, while the other uses a simple algebraic formulation, changing the roots of the shifted Legendre polynomial to the coordinates of the grid points. A fundamental shortcoming of the algebraic equation system is that its matrix is unconditioned when the order is huge. Thus, it is an extremely difficult process to calculate the weighting factors for a huge number of grid points. Researchers have made a lot of attempts to improve upon Bellman's methodology for calculating the weighting factors. Among the most useful is the approach that has been described. After then, a general approach based on linear vector space analysis and high-order polynomial approximation was published in the literature. This extended technique determines the weighting coefficients of the first-order derivative by use of an algebraic formulation that is both simple and free of restrictions on the choice of grid points. It uses a recurrence connection for derivatives of higher and second order.

The fundamental premise of the DQM is that a solution to a one-dimensional differential equation may be approximated by a high-degree polynomial with N terms:

$$f(x) = \sum_{k=1}^{N} c_k . x^{k-1} \tag{2}$$

given that ck is a fixed value.
To get the weighting coefficients, the generalized method employs two sets of base polynomials. For the Lagrange interpolated polynomials, the initial set of base polynomials is selected and expressed as

$$r_k(x) = \frac{M(x)}{(x-x_k) \cdot M^{(1)}(x_k)}; k=1,2,\ldots,N \tag{3}$$

Where

$$M(x) = (x-x_1).(x-x_2)\ldots\ldots(x-x_N)$$

and

$$M^{(1)}(x_k) = \prod_{j=1, j\neq k}^{N} (x_k - x_j)$$

as the first descendant of $M(x)$ at x_k.

Here $x_1, x_2, \ldots x_N$ the coordinates of the points on the grid, which may be picked at random but are separate.
The polynomials

$$r_k(x) = x^{k-1}, k=1,2,\ldots\ldots,N \tag{4}$$

Serve as the second group of polynomials used as foundations.

To make things easier, by adjusting

$$M(x) = N(x, x_k).(x - x_k),\ \ k = 1,2,....,N$$

with $N(x_i, x_j) = M^{(1)}(x_i).\delta_{ij}$, where δ_{ij} stands for the Kronecker operator, which simplifies equation (3) as:

$$r_k(x) = \frac{N(x, x_k)}{M^{(1)}(x_k)};\ \ k = 1,2,......,N \qquad (5)$$

Equation (5) was substituted into equation (1) for m = 1 and equation (4) was used to produce the following weighting factors for the discretization of the first order derivative:

$$A_{ij}^{(1)} = \frac{M^{(1)}(x_i)}{(x_i - x_j)M^{(1)}(x_j)},\ (i, j = 1,2,..,N; i \neq j)$$

$$A_{ii}^{(1)} = -\sum_{j=1; i\neq j}^{N} A_{ij}^{(1)},\quad (i = 1,2,......N) \qquad (6)$$

To get the weighting coefficients for the discretization of higher-order derivatives, the Shu's recurrence formula is provided as

$$A_{ij}^{(m)} = m[A_{ii}^{(m-1)} A_{ij}^{(1)} - \frac{A_{ij}^{(m-1)}}{(x_i - x_j)}],$$

$$(i, j = 1,2,......N;\ i \neq j; 2 \leq m \leq N-1)$$

$$A_{ii}^{(m)} = -\sum_{j=1; i\neq j}^{N} A_{ij}^{(m)}\ ,(i = 1,2,......N;\ m \geq 2) \qquad (7)$$

When discretizing any order derivative, the weighting coefficients may be easily calculated using equations (6) and (7). There is a benefit to using these explicit equations since they allow for the determination of weighting coefficients with a high degree of accuracy for a large number of randomly spaced sample points.

Choice of Sampling Points

The evenly spaced points are a practical and intuitive option for the sample spots. With unequally spaced sample sites, however, the Differential Quadrature solutions often provide more accurate findings. The zeroes of the orthogonal polynomials provide a reasonable foundation for the sample sites. The so-called Gauss-Lobatto-Chebyshev sampling points are a widely used kind of DQM sample points. If we have a no uniform grid with points that are unequally spaced and a domain where a≤x≤b, we may get the coordinate of each point i by:

$$x_i = a + \frac{1}{2}\left(1 - \cos\left(\frac{i-1}{N-1}\pi\right)\right)(b - a) \qquad (8)$$

II. APPLICATION TO DIFFERENTIAL EQUATION

An essential first step in DQM is to use equation (1) to approximatively find the derivatives of a differential equation. When we equalize both sides of the governing equations and substitute (1) into them, we get simultaneous equations that can be solved using Gauss elimination or other techniques. In other words, the following technique makes up DQM:

- Several function values at several randomly chosen sample points stand in for the function whose value is to be found. When solving a differential equation with N unknown values for the function, it is highly advised to use the Gauss-Lobatto-Chebyshev sampling points (8) to ensure numerical stability.
- Create a set of linear equations and then
- To get the answers you need, you need to solve a system of linear equations.

When solving differential equations numerically, it is crucial to apply boundary conditions correctly. Using DQM, one may estimate essential and natural boundary conditions. One has one equation for each point, for each unknown, since the governing equations are fulfilled at each sample point of the domain using the approach of solving differential equations. Every boundary condition stands in for its associated field equation in the DQM-derived system of algebraic equations. This process is simple if and only if we have distributed the sample points so that one point is at each boundary and there is exactly one boundary condition at each boundary.

APPLICATION OF DQM TO MIXED-TYPE SINGULARLY PERTURBED DIFFERENTIAL-DIFFERENCE EQUATIONS

In order to demonstrate the usefulness of DQM, we look at the boundary-value issues with a mixed-type singly perturbed differential-difference equation (i.e., with terms that have both positive and negative shifts) with minor shifts,

$$\varepsilon^2 y''(x)+\alpha(x)y(x-\delta)+w(x)y(x)+\beta(x)y(x+\eta)=f(x); \quad (9)$$

on [0,1], under the boundary conditions

$$y(x)=\phi(x); \quad -\delta \le x \le 0 \quad (10)$$

and

$$y(x)=\psi(x), \quad 1 \le x \le 1+\eta \quad (11)$$

where ε is a parameter for singular perturbations, (0< ε<<1) and δ η parameters that are always changing, with (0< δ <<1) and 0< η <<1. The functions $\alpha(x),\ \beta(x), \varphi(x), \psi(x)$ and $f(x)$ are taken to be functions in the interval [0,1] that are sufficiently differentiable continuously. On each side of the interval [0,1], the layer behavior is seen in the solution to the boundary-value issue (9) using (10) and (11).

The use of the Taylor series expansion allows us to extend the terms including shift, as the solution to the boundary-value problem (9) obtained from equations (10 and 11) is continuous and continuously differentiable on the interval [0,1]. Lastly, we get

$$y(x-\delta) \approx y(x)-\delta\, y'(x) \quad (12)$$

$$y(x+\eta) \approx y(x)+\eta\, y'(x) \quad (13)$$

If we substitute equations (12) and (13) into equation (9) together with equations (10), we get

$$\varepsilon^2 y''(x)+[\beta(x)\eta-\alpha(x)\delta]y'(x)+[\alpha(x)+\beta(x)+w(x)]y(x)=f(x) \quad (14)$$

on [0,1], under the boundary conditions

$$y(0) \approx \phi(0) \quad (15)$$

and

$$y(1) \approx \psi(1) \quad (16)$$

The solution of the approximate differential equation (14) with (15) and (16) may be expressed using a different notation, such as u(x), as it is an approximation of equation (9) with (10) and (11). Consequently, the following singly perturbed boundary-value issue arises from the intersection of problems (14) with (15) and (16):

$$\varepsilon^2 u''(x)+[\beta(x)\eta-\alpha(x)\delta]u'(x)+[\alpha(x)+\beta(x)+w(x)]u(x)=f(x) \quad (17)$$

on [0,1], under the boundary conditions

$$u(0) \approx \phi(0) \quad (18)$$

and

$$u(1) \approx \psi(1) \quad (19)$$

We get the solution of the boundary-value issue (9) with the help of (10) and (11) on the interval [0,1] by solving this problem using DQM using (18) and (19).

We have used DQM to solve equation (17) with boundary conditions (18) and (19) by following this technique:

(i) Evaluate the interval [0,1] critically. such $0=x_1<x_2<x_3<\dots\dots<x_N=1$ where N is the total number of points on the sample or grid.

Denote $u_i=u(x_i),\quad f_i=f(x_i)$ etc.

(ii) Equation (17)'s derivatives may be approximated using the DQM, which leads to the discretized problem statement that follows:

$$\varepsilon^2\sum_{j=1}^{N}A_{i,j}^{(2)}u_j+[\beta_i\eta-\alpha_i\delta]\sum_{j=1}^{N}A_{i,j}^{(1)}u_j+[\alpha_i+\beta_i+w_i]u_i-f_i=0,\ i=1,2,\dots\dots,N \quad (20)$$

When faced with the limits

$$u_1=\phi(0); \qquad u_N=\psi(1) \quad (21)$$

(iii) For every interior location, plug the values into equation (20) $x_i, (i=2,3,\dots,N-1)$ such that N variables are involved in a set of (N-2) equations.

(iv) Put the limit values to use for u_1 and u_N step (iii) to construct a system of (N-2) equations with (N-2) unknowns,

using equation (21) in the acquired system of equations $u_i, (i = 2,3, \ldots, N-1)$.
(v) When you get the system of equations in step (iv), solve it for the unknowns $u_i, (i = 2,3, \ldots, N-1)$.

(vi)The whole answer may be obtained by applying the boundary values. For the unknowns in step (iv), we have solved the system of linear equations using the Gaussian elimination technique with partial pivoting and double precision Fortran. $u_2, u_3, \ldots u_{N-1}$.

CONCLUSION

One powerful numerical tool for solving the problems of singly perturbed mixed-type differential-difference equations is the Differential Quadrature Method (DQM). In particular, when dealing with steep boundary layers and multi-scale behavior caused by modest perturbation parameters, DQM achieves impressive accuracy with a minimal number of grid points by using global interpolation and weighted summing of function values. By effectively processing non-uniform grids and being flexible enough to handle delay and advance terms, the technique surpasses conventional numerical methods in terms of accuracy and computing effort. Even for very modest perturbation values, numerical studies on benchmark issues confirm that the technique outperforms other well-established methodologies. It is clear that DQM has the ability to be used more widely in many areas of science and engineering after its effective application to such complicated situations. It is also compatible with contemporary computational tools and has a straightforward formulation, therefore it is a good option for practical applications. In order to further establish it as a flexible tool in numerical analysis, future research may investigate its expansion to multi-dimensional and time-dependent singly perturbed situations.

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