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On Weibull-Inverse Weibull distribution and its Application in Lifetime Data Analysis

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ABSTRACT

In this article, a new four parameter generalization of Inverse Weibull model is introduced using the generator technique. A comprehensive account of the different structural properties including reliability analysis, moments, order statistics, Renyi entropy and quantile function is provided. The estimates of the parameters are computed using the technique of maximum likelihood estimation. The flexibility and the usefulness of the distribution for modeling the lifetime data is illustrated using the real life data sets.

Keywords: Inverse Weibull distribution, Moments, Order Statistics, Reliability analysis, Renyi entropy, Weibull-G distribution.

I. INTRODUCTION

In the past few years, many generalization techniques were introduced in the statistical literature by adding an additional parameter to the classical model so as to provide an adequate fit to the real data sets. This induction one or more additional shape parameters to the baseline distribution to generate new distributions proved beneficial in exploring tail properties and also for improving the goodness-of-fit of the proposed generator family. The well established generators in the statistical distributional theory are Marshall-Olkin G by Marshall and Olkin [1], Beta-G by Eugene et al. [2], Transmuted–G by Shaw and Buckley[3], Kumaraswamy-G (K-G) by Cordeiro and de Castro [4], McDonald-G (Mc-G) by Alexander et. al. [5], Gamma-G (type 1) by Zografos and Balakrishanan [6], gamma-G (type 2) by Ristic and Balakrishanan [7], exponentiated generalized G by Cordeiro et al. [8], Transformed- transformer (T-X) by Alzaatreh et al. [9] and Lomax G (LG) by Cordeiro et al. [10]. In other words, adding an extra parameter to the already existing distributions can be very useful in analyzing lifetime data.

II. WEIBULL-INVERSE WEIBULL DISTRIBUTION

The Weibull distribution is a lifetime probability model named after Walladi Weibull, a Swedish physicist. This versatile distribution is widely used for analyzing lifetime data in reliability engineering, medicine, automobile industry, computing technology and aerospace. Despite its variety of applications, Weibull distribution is unable to analyze the lifetime data sets which have non monotonic failure rates such as bathtub and unimodal hazard rates. As such, several generalizations of the Weibull model have been introduced in the statistical literature. The inverse Weibull distribution was introduced by Keller et al. [11] for analyzing reliability and failure of mechanical components. This distribution finds its variety of applications in reliability engineering, aeronautics, hydrology, physics, biomedical sciences, agriculture, pharmaceutical sciences, psychology, metrology, economics and actuarial sciences etc.
Bourguignon et al. [12] introduced the Weibull G family of distributions. They derived some new special distribution from this family by assuming Weibull model as a base distribution. They replaced the variable x with the term $\frac{G(x,\Lambda)}{1-G(x,\Lambda)}$ and obtained the distribution function of Weibull generalized distribution as:

$$F(x,\alpha,\beta,a,b)=\int_{0}^{\frac{G(x,\Lambda)}{1-G(x,\Lambda)}}\alpha\beta\, t^{\beta-1}e^{-\alpha\, t^{\beta}}dt\;=1-e^{-\alpha\left[\frac{G(x,\Lambda)}{1-G(x,\Lambda)}\right]^{\beta}}=1-e^{-\alpha\left[\frac{G(x,\Lambda)}{1-G(x,\Lambda)}\right]^{\beta}}. \quad (1)$$

Then, the corresponding probability density function turns out to be:

$$f(x,\alpha,\beta,\Lambda)=\alpha\beta g(x,\Lambda)\frac{[G(x,\Lambda)]^{\beta-1}}{[1-G(x,\Lambda)]^{\beta+1}}e^{-\alpha\left[\frac{G(x,\Lambda)}{1-G(x,\Lambda)}\right]^{\beta}}. \quad (2)$$

where $G(x,\Lambda)$ and $g(x,\Lambda)$ are the distribution function and density function of the base model respectively.

In this manuscript, four parameter Weibull Inverse Weibull model is proposed by assuming $G(x,\Lambda)$ and $g(x,\Lambda)$ as cdf and pdf of Inverse Weibull distribution. The corresponding probability density function and cumulative distribution function of Inverse Weibull are respectively given in the equation (1) and (2):

$$g(x) = abx^{-(b+1)}\exp\left(-ax^{-b}\right) \qquad (3)$$

$$G(x) = \exp\left(-ax^{-b}\right) \qquad (4)$$

The Cumulative distribution function of the proposed new four parameter Weibull-Inverse Weibull distribution using the equation (1) and (4) is given by:

$$\Rightarrow\ F(x,\alpha,\beta) = 1 - e^{-\alpha\left[\frac{\exp\left(-\frac{a}{x^b}\right)}{1-\exp\left(-\frac{a}{x^b}\right)}\right]^{\beta}} = 1 - e^{-\alpha\left[e^{\frac{a}{x^b}}-1\right]^{-\beta}}. \qquad (5)$$

The graphical plotting of the distribution function for different values of parameters of Weibull Inverse Weibull distribution is shown in Figure 1. It can be seen that the cumulative distribution function is an increasing function in every case.

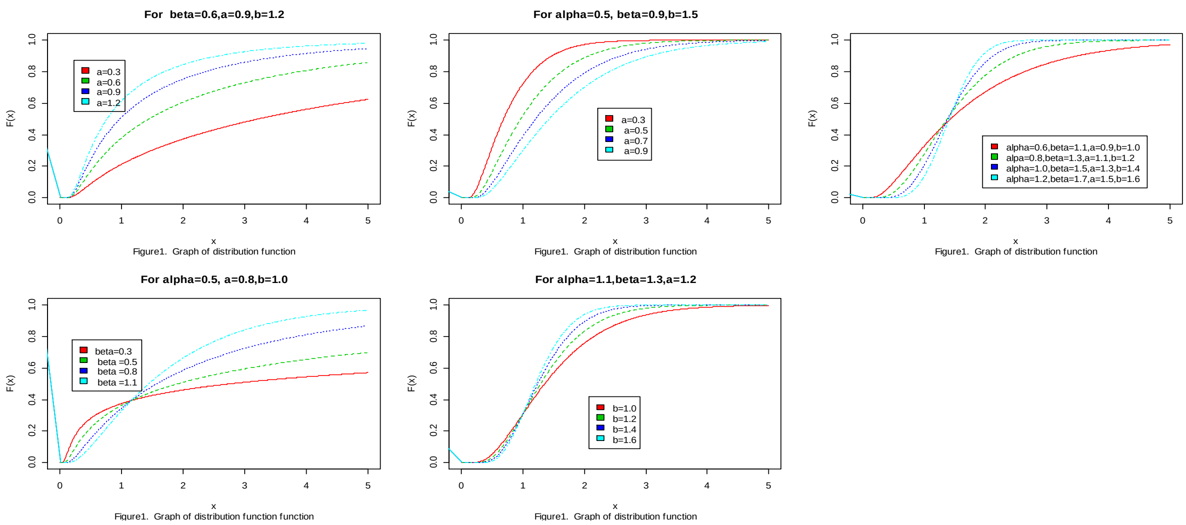

Figure1. Graph of distribution function

Then, the consequent pdf of the Weibull Inverse Weibull distribution using the equations (2), (3) and (4) is as follows:

$$f(x,\alpha,\beta,a,b) = \alpha\beta abx^{-(b+1)}e^{\frac{a}{x^b}}\left[e^{\frac{a}{x^b}}-1\right]^{-(\beta+1)} e^{-\alpha\left[e^{\frac{a}{x^b}}-1\right]^{-\beta}}. \qquad (6)$$

Figure 2 gives the description of density function for different values of the four parameters of the proposed Weibull Inverse Weibull model.

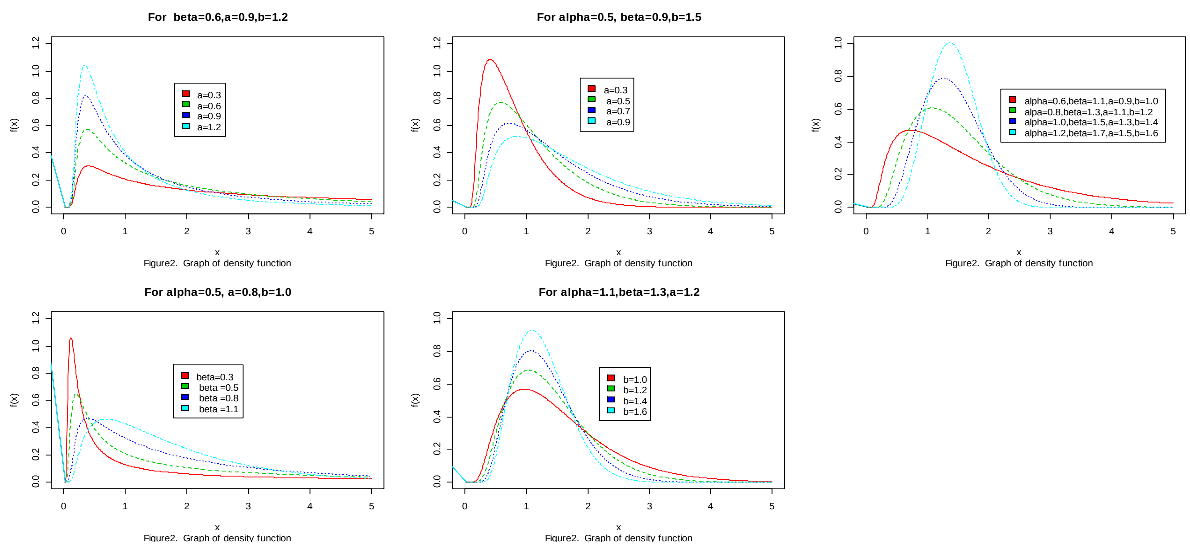

Figure2. Graph of density function

This research article is further organized as follows: The section 3 describes the reliability analysis of the proposed model. Section 4 deals with the calculation of quantile function and the three quartiles. The structural properties associated with the newly developed model including moments, harmonic mean, m.g.f and characteristic function are discussed in section 5. Further, the comprehensive description of order statistics and Renyi entropy of the postulated distribution is given in section 6 and 7 respectively. The maximum likelihood estimates of four unknown parameters along with the observed Fisher Information matrix are provided in section 8. Finally, the three real life data sets are used for the analyzing the proposed model in the section 9.

III. RELIABILITY ANALYSIS

In this section, the survival function, hazard rate, reverse hazard rate, mills ratio and mean residual time of the proposed model have been discussed.

3.1 Reliability function

It is also termed as survivor function or survival function of the model. Denoted by R(x), it can be defined as the probability that an item does not fail prior to sometime t. It is complement to the distribution function and can be mathematically obtained as:

$$R(x,\alpha,\beta,a,b)=1-F(x,\alpha,\beta,a,b)=e^{-\alpha\left[e^{\frac{a}{x^b}}-1\right]^{-\beta}}. \qquad (7)$$

The description of the reliability function is depicted in figure 3 which indicates that it is the decreasing function for every possible value of the parameters of the proposed model.

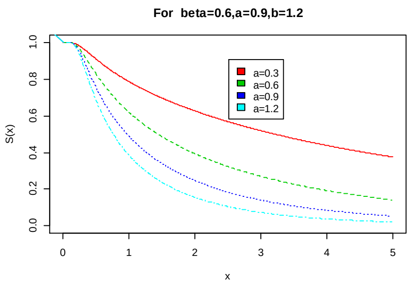

Figure3. Graph of Reliability function

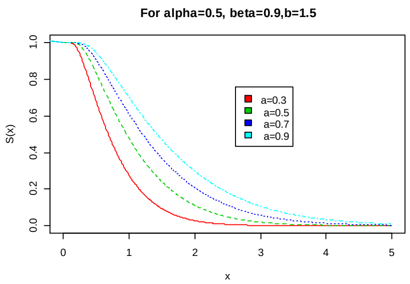

Figure3. Graph of Reliability function

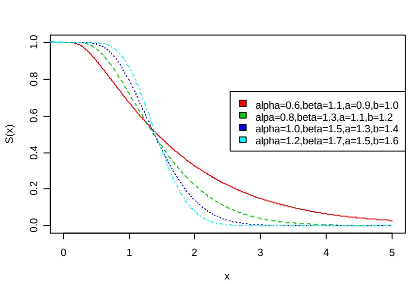

Figure3. Graph of Reliability function

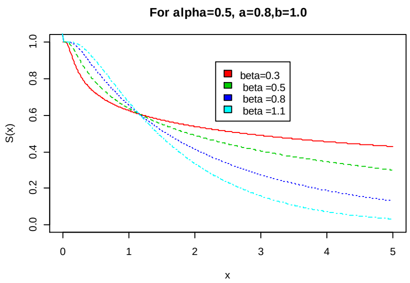

Figure3. Graph of Reliability function

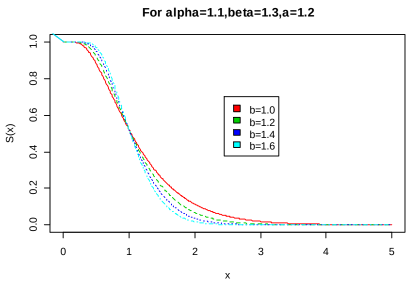

Figure3. Graph of Reliability function

3.2 Hazard rate

The hazard function of the system is also termed as the hazard rate, failure rate or force of mortality. Denoted by $h(x)$, it can be derived as the ratio of the probability density function and the reliability function. It can be mathematically computed as:

$$H(x)=\frac{f(x)}{R(x)}=\alpha\beta abx^{-(b+1)}e^{\frac{a}{x^b}}\left[e^{\frac{a}{x^b}}-1\right]^{-(\beta+1)} \qquad (8)$$

The hazard rate for the proposed model is given in the figure 5 for the several values of the parameters.

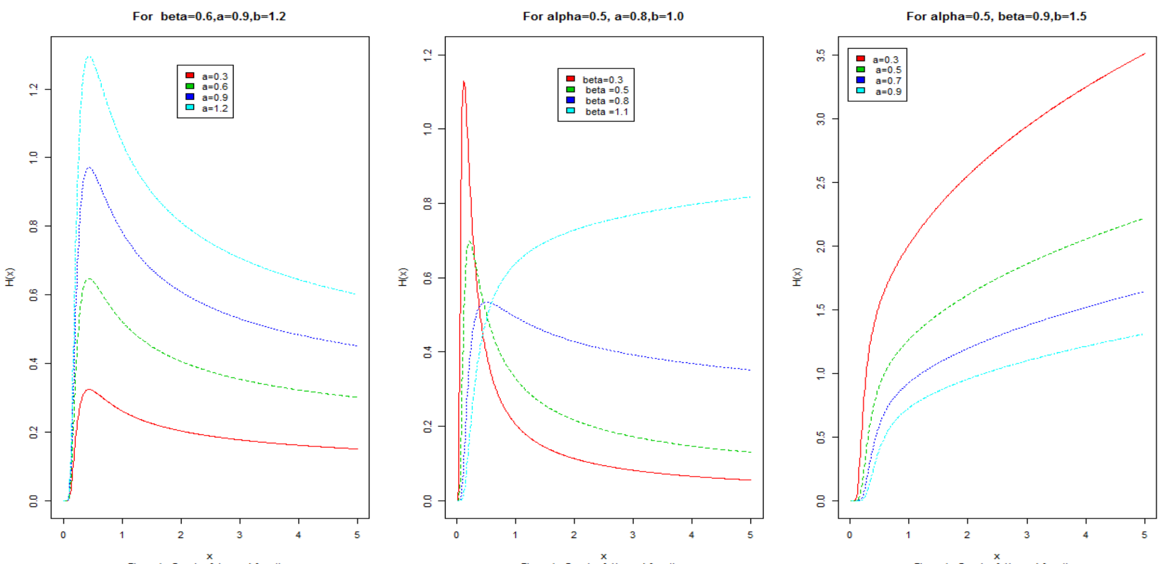

Figure4. Graph of hazard function   Figure4. Graph of Hazard function   Figure4. Graph of Hazard function

3.3 Reverse hazard rate

This is also an important feature which characterizes life phenomenon. It is computed as the ratio of the probability density function and the cumulative distribution function. Denoted by $\phi(x)$, the reverse hazard rate is given as follows:

$$\phi(x)=\frac{f(x)}{F(x)}=\frac{\alpha\beta abx^{-(b+1)}e^{\frac{a}{x^b}}\left[e^{\frac{a}{x^b}}-1\right]^{-(\beta+1)}e^{-\alpha\left[e^{\frac{a}{x^b}}-1\right]^{-\beta}}}{1-e^{-\alpha\left[e^{\frac{a}{x^b}}-1\right]^{-\beta}}}. \qquad (9)$$

IV. QUANTILE FUNCTION

This section deals with obtaining the quantile function and the first three quartiles of the Weibull Inverse Weibull distribution. The quantile function of any distribution is obtained by the method of inversion. In this method, the cdf of the distribution is equated to the number u drawn itself from $U(0,1)$. The quantile function of the model under study is given as:

$$Q(u)=F^{-1}(u), \qquad 0<u<1 \qquad (10)$$

$$Q(u)=\left[\frac{a}{\log\left[1+\left[(-\alpha)^{-1}\log(1-u)\right]^{\frac{-1}{\beta}}\right]}\right]^{\frac{1}{b}}. \qquad (11)$$

When $u=\frac{1}{4},\frac{1}{2}$ and $\frac{3}{4}$, we get the first, second (median) and the third quartile respectively.
Once the quantile function of the distribution is computed, we can generate the random numbers for the distribution under discussion using the quantile function.

V. STRUCTURAL PROPERTIES OF WEIBULL-INVERSE WEIBULL DISTRIBUTION

In this section, the statistical properties comprising of moments, harmonic mean, moment generating function and characteristic function of Weibull Inverse Weibull distribution are discussed.

5.1 Moments

The k^{th} moment of the continuous random variable X drawn from the proposed Weibull inverse Weibull distribution with density function f (x) given in equation (6) can be computed as follows:

$$E\left(x^k\right)=\int^{\infty} x^k f(x)dx \quad =\int_0^{\infty} x^k \alpha\beta ab x^{-(b+1)} e^{\frac{-a}{x^b}} \frac{\left[e^{\frac{-a}{x^b}}\right]^{\beta-1}}{\left[1-e^{\frac{-a}{x^b}}\right]^{\beta+1}} e^{-\alpha\left[\frac{e^{\frac{-a}{x^b}}}{1-e^{\frac{-a}{x^b}}}\right]^{\beta}} dx. \qquad (12)$$

Using the expansion of the exponential term

$$e^{-\alpha\left[\frac{G(x,\Lambda)}{1-G(x,\Lambda)}\right]^{\beta}} =\sum_{i=0}^{\infty}\frac{(-1)^i\alpha^i}{i!}\left[\frac{G(x,\Lambda)}{1-G(x,\Lambda)}\right]^{\beta i} =\sum_{i=0}^{\infty}\frac{(-1)^i\alpha^i}{i!}\left[\frac{e^{-\frac{a}{x^b}}}{1-e^{\frac{-a}{x^b}}}\right]^{\beta i}$$

, the above equation reduces to

$$E\left(x^k\right)=\alpha\beta ab\sum_{i=0}^{\infty}\frac{(-1)^i\alpha^i}{i!}\int_0^{\infty} x^{k-b-1}\frac{\left[e^{\frac{-a}{x^b}}\right]^{\beta(i+1)}}{\left[1-e^{\frac{-a}{x^b}}\right]^{\beta(i+1)+1}}dx. \qquad (13)$$

Also, from the generalized binomial theorem

$$\left[1-e^{\frac{-a}{x^b}}\right]^{-\{\beta(i+1)+1\}} =\sum_{j=0}^{\infty}\frac{\Gamma(\beta(i+1)+1+j)}{j!\,\Gamma(\beta(i+1)+1)}\left[e^{\frac{-a}{x^b}}\right]^{j}$$

, the equation (13) turns out to be

$$E\left[x^k\right]=\alpha\beta ab\sum_{i=0}^{\infty}\frac{(-1)^i\alpha^i}{i!}\sum_{j=0}^{\infty}\frac{\Gamma(\beta(i+1)+1+j)}{\Gamma(\beta(i+1)+1)\,j!}\int_0^{\infty} x^{k-b-1}\left[e^{\frac{-a}{x^b}}\right]^{\beta(i+1)+j} dx. \qquad (14)$$

Setting $\frac{a}{x^b}=u,\ x=\left(\frac{u}{a}\right)^{-\frac{1}{b}}, dx=\frac{du}{-ab\left(\frac{u}{a}\right)^{\frac{b+1}{b}}}$, The expression of kth moment finally reduces to

$$E\left[x^k\right]=\alpha^{1+\frac{k}{b}}\beta ab\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\delta_{i,j}\frac{\Gamma\left(1-\frac{k}{b}\right)}{\left[\beta(i+1)+j\right]^{1-\frac{k}{b}}}. \qquad (15)$$

When k=1, the expected value of the new model is obtained as:

$$E(x)=\alpha^{1+\frac{1}{b}}\beta ab\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\delta_{i,j}\frac{\Gamma\left(1-\frac{1}{b}\right)}{\left[\beta(i+1)+j\right]^{1-\frac{1}{b}}}. \qquad (16)$$

$$\text{and for } k=2,\ E\left(x^2\right)=\alpha^{1+\frac{2}{b}}\beta ab\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\delta_{i,j}\frac{\Gamma\left(1-\frac{2}{b}\right)}{\left[\beta(i+1)+j\right]^{1-\frac{2}{b}}}. \qquad (17)$$

$$\text{where } \delta_{i,j}=\frac{(-1)^i\alpha^i}{i!}\frac{\Gamma(\beta(i+1)+1+j)}{\Gamma(\beta(i+1)+1)\,i!}.$$

The variance of the distribution is calculated using the expression (17) and (18) as:

$$V(X)=E\left(X^2\right)-\left[E(X)\right]^2 \qquad (18)$$

5.2 Harmonic mean

By the definition, the harmonic mean denoted by (H.M) can be mathematically worked out as follows:

$$H.M=\int_0^{\infty}\frac{1}{x}f(x,\alpha,\beta,a,b)dx\;=\alpha^{1-\frac{1}{b}}\beta ab\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\delta_{i,j}\frac{\Gamma\left(1+\frac{1}{b}\right)}{[\beta(i+1)+j]^{1+\frac{1}{b}}}. \qquad (19)$$

where $\delta_{i,j}=\frac{(-1)^i\alpha^i}{i!}\frac{\Gamma(\beta(i+1)+1+j)}{\Gamma(\beta(i+1)+1)i!}$.

5.3 Moment generating function

The moment generating function (m.g.f) of the random variable X drawn from the new Weibull Inverse Weibull distribution, denoted by $M_X(t)$ can be derived as:

$$M_X(t)=E(e^{tx})=\int_0^{\infty}e^{tx}f(x,\alpha,\beta,a,b)dx.$$

Using the Taylor series expansion we have

$$M_X(t)=\int_0^{\infty}\left(1+tx+\frac{(tx)^2}{2!}+\cdots\right)f(x,\alpha,\beta,a,b)dx=\sum_{k=0}^{\infty}\frac{t^k}{k!}\int_0^{\infty}x^k f(x)dx.=\sum_{k=0}^{\infty}\frac{t^k}{k!}E(X^k)$$

$$\Rightarrow \qquad M_X(t)=\frac{t^k}{k!}\alpha^{1+\frac{k}{b}}\beta ab\sum_{r=0}^{\infty}\sum_{i,j=0}^{\infty}\delta_{i,j}\frac{\Gamma\left(1-\frac{k}{b}\right)}{[\beta(i+1)+j]^{1-\frac{k}{b}}}. \qquad (20)$$

where $\delta_{i,j}=\frac{(-1)^i\alpha^i}{i!}\frac{\Gamma(\beta(i+1)+1+j)}{\Gamma(\beta(i+1)+1)i!}$.

5.4 Characteristic function

The characteristic function of the continuous random variable X is denoted by $\phi_X(t)$ and can be defined as:

$$E(e^{itx})=\int_0^{\infty}e^{itx}f(x,\alpha,\beta,a,b)dx.$$

Using the Taylor series expansion, we have

$$\phi_X(t)=\int_0^{\infty}\left(1+itx+\frac{(itx)^2}{2!}+\cdots\right)f(x,\alpha,\beta,a,b)dx\,.=\sum_{k=0}^{\infty}\frac{(it)^k}{k!}\int_0^{\infty}x^k f(x)dx.=\sum_{k=0}^{\infty}\frac{(it)^k}{k!}E(X^k)$$

$$\Rightarrow \qquad \phi_X(it)=\sum_{k=0}^{\infty}\frac{(it)^k}{k!}\alpha^{1+\frac{k}{b}}\beta ab\sum_{r=0}^{\infty}\sum_{i,j=0}^{\infty}\delta_{i,j}\frac{\Gamma\left(1-\frac{k}{b}\right)}{[\beta(i+1)+j]^{1-\frac{k}{b}}}. \qquad (21)$$

where $\delta_{i,j}=\frac{(-1)^i\alpha^i}{i!}\frac{\Gamma(\beta(i+1)+1+j)}{\Gamma(\beta(i+1)+1)i!}$.

VI. ORDER STATISTICS

In statistical distributional theory and modeling lifetime data, order statistics is found to be very useful. It is widely applicable in finding out the reliability of a system and life testing. If $X_{(1)},X_{(2)},.....,X_{(n)}$ denote the order statistics obtained from the random sample $X_1,X_2,.....,X_n$ drawn from new WIW distribution (α,β,a,b) with cumulative density function and probability density function given in the equations (5) and (6) respectively, then the probability density function of the order statistics is given as below:

$$f_r(x,\alpha,\beta,a,b)=\frac{1}{B(r,n-r+1)}f(x,\alpha,\beta,a,b)[F(x,\alpha,\beta,a,b)]^{r-1}[1-F(x,\alpha,\beta,a,b)]^{n-r}.$$

(22)

Since $[1-F(x,\alpha,\beta,a,b)]^{n-r}=\sum_{j=0}^{\infty}(-1)^j\left({}^{n-r}c_j\right)[F(x)]^j.$

Using in equation (22), we get:

$$f_r(x,\alpha,\beta,a,b)=\frac{1}{B(r,n-r+1)}\sum_{j=0}^{\infty}f(x,\alpha,\beta,a,b)[F(x,\alpha,\beta,a,b)]^{j+r-1}.$$

Also, $[F(x,\alpha,\beta,a,b)]^{j+r-1}=\left[1-e^{-\alpha\left[e^{\frac{a}{x^b}}-1\right]^{-\beta}}\right]^{j+r-1}=\sum_{k=0}^{\infty}(-)^k\left({}^{j+r-1}C_k\right)e^{-\alpha k\left[e^{\frac{a}{x^b}}-1\right]^{-\beta}}.$

$$f_r(x,\alpha,\beta,a,b)=\frac{1}{B(r,n-r+1)}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^{j+k}}{(k+1)}\left({}^{n-r}C_j\right)\left({}^{j+r-1}C_k\right)f(x,\alpha(k+1),\beta,a,b). \quad (23)$$

Put $r=1$ in equation (23), the pdf of the first order statistics is obtained as:

$$f_1(x,\alpha,\beta,a,b)=n\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^{j+k}}{(k+1)}\left({}^{n-1}C_j\right)\left({}^{j}C_k\right)f(x,\alpha(k+1),\beta,a,b). \quad (24)$$

Similarly, for $r=n$ in equation (23), the pdf of the n^{th} order statistics is obtained as:

$$f_n(x,\alpha,\beta,a,b)=n\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^{k}}{(k+1)}\left({}^{n-1}C_k\right)f(x,\alpha(k+1),\beta,a,b). \quad (25)$$

Also, when $n=2m$ in equation (23), the pdf of the median (m+1) is obtained as follows:

$$f_{m+1}(x,\alpha,\beta,a,b)=\frac{(2m+1)}{m!\,m!}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^{i+k}}{(k+1)}\left({}^{m}C_j\right)\left({}^{m+j}C_k\right)f(x,\alpha(k+1),\beta,a,b). \quad (26)$$

6.1 Joint Density function of i^{th} and j^{th} order statistics

The joint density function of (x_i,x_j) for $1\le i\le j\le n$ is given by:

$$f_{i:j:n}(x_i,x_j)=C[F(x_i)]^{i-1}[F(x_j)-F(x_i)]^{j-i-1}[1-F(x_j)]^{n-j}f(x_i)\,f(x_j), \quad (27)$$

where $C=\frac{n!}{(i-1)!(j-i-1)!(n-j)!}$.

$$f_{i,j}(x,\alpha,\beta,a,b)=C\left[1-e^{-\alpha\,h(x_i)^{-\beta}}\right]^{i-1}\left[e^{-\alpha\,h(x_i)^{-\beta}}-e^{-\alpha\,h(x_j)^{-\beta}}\right]^{j-i-1}e^{-\alpha(n-j)h(x_j)^{-\beta}}\alpha^2\beta^2a^2b^2x_i^{-(b+1)}x_j^{-(b+1)}$$
$$e^{a\left(\frac{1}{x_i}+\frac{1}{x_j}\right)}h(x_i)^{-(\beta+1)}h(x_j)^{-(\beta+1)}e^{-\alpha\,h(x_i)^{-\beta}}e^{-\alpha\,h(x_j)^{-\beta}}$$

$$f_{1,n}(x)=n(n-1)[F(x_n)-F(x_1)]^{n-2}f(x_1)f(x_n).$$

When $i=1$ and $j=n$, the joint density function of minimum and maximum order statistics can be computed as follows:

$$f_{1,n}(x,\alpha,\beta,a,b)=n(n-1)\left[e^{-\alpha\,h(x_1)^{-\beta}}-e^{-\alpha\,h(x_n)^{-\beta}}\right]^{n-2}\alpha^2\beta^2a^2b^2x_1^{-(b+1)}x_n^{-(b+1)}$$
$$e^{a\left(\frac{1}{x_1}+\frac{1}{x_n}\right)}h(x_1)^{-(\beta+1)}h(x_n)^{-(\beta+1)}e^{-\alpha\,h(x_1)^{-\beta}}e^{-\alpha\,h(x_n)^{-\beta}} \quad (28)$$

where $h(x)=\left(e^{\frac{a}{x^b}}-1\right)$.

VII. RENYI ENTROPY

This section deals with the computation of the entropy of the newly proposed distribution. The entropy is a measure of the variation of the uncertainty of a continuous random variable X. The increase in the value of the entropy is an indicator of the greater uncertainty in the data. Denoted by $I_R(\rho)$, the Renyi entropy (1960) for X with probability density WIW (α,β,a,b) is defined and computed as:

$$I_r(\rho)=\frac{1}{1-\rho}\log\int_0^\infty f(x)^\rho\,dx, \text{ where } \rho>0 \text{ and } \rho\neq 1. \quad (29)$$

where the expression for f(x) is given in the equation (6):

$$\text{Let } \mu(x)=\int_0^\infty f(x)^\rho\,dx.=\int_0^\infty (\alpha\beta ab)^\rho x^{-\rho(b+1)}\frac{e^{\frac{-a\beta\rho}{x^b}}}{\left(1-e^{\frac{-a}{x^b}}\right)^{\rho(\beta+1)}}e^{-\alpha\rho\left[\frac{e^{-\frac{a}{x^b}}}{1-e^{\frac{-a}{x^b}}}\right]^\beta}dx.$$

Using the expansion of exponential term and the generalized binomial term, we get:

$$\mu(x)=(\alpha\beta ab)^\rho\,\delta_{i,j}\int_0^\infty x^{-\rho(b+1)}e^{\frac{-a\beta(\rho+i)+j}{x^b}}dx, \text{ where } \delta_{i,j}=\sum_{i=1}^\infty\sum_{j=0}^\infty\frac{(-1)^i(a\rho)^i}{i!}\frac{\Gamma(\beta(i+\rho)+\rho+j)}{j!\,\Gamma(\beta(i+\rho)+\rho)}$$

$$\text{Put } \frac{a}{x^b}=u,\; x=\left(\frac{u}{a}\right)^{-\frac{1}{b}}, dx=\frac{du}{-ab\left(\frac{u}{a}\right)^{\frac{b+1}{b}}}, \text{ we have}$$

$$I_R(\rho)=\frac{\rho}{1-\rho}\log\alpha+\frac{\rho}{1-\rho}\log\beta+\frac{1}{b}\log a-\log b$$
$$+\frac{1}{1-\rho}\log\left[\delta_{i,j}\frac{\Gamma\left(\frac{(b+1)(\rho-1)}{b}+1\right)}{[\beta(\rho+i)+j]^{\frac{(b+1)(\rho-1)}{b}+1}}\right]. \quad (30)$$

The β or q-entropy introduced by Havrda and Charvat (1967) is denoted by $I_H(q)$ and can be computed as:

$$I_H(q)=\frac{1}{q-1}\left\{1-\int_{-\infty}^\infty f(x)^q\,dx\right\}, \text{ where } q>0 \text{ and } q\neq 1$$

$$I_H(q)=\frac{1}{q-1}\left\{1-\alpha^q\beta^q a^{\frac{-(q-1)}{b}}b^{q-1}\nu_{i,j}\frac{\Gamma\left(\frac{(b+1)(q-1)}{b}+1\right)}{[\beta(q+i)+j]^{\frac{(b+1)(q-1)}{b}+1}}\right\}, \quad (31)$$

$$\text{where } \nu_{i,j}=\sum_{i=1}^\infty\sum_{j=0}^\infty\frac{\Gamma(\beta(i+q)+q+j)}{j!\,\Gamma(\beta(i+q)+q)}\frac{(-1)^i(\alpha q)^i}{i!}$$

VIII. PARAMETER ESTIMATION

In this section, the four unknown parameters are estimated and the observed fisher information matrix of the proposed model is derived.

8.1 Maximum Likelihood Estimation:

The procedure of maximum likelihood estimation is used for estimating the unknown parameters of probability density function. Let $x_1, x_2, x_3, ...x_n$ be the sample consisting of n observations with pdf given in equation ( ), then the likelihood function of the proposed distribution is given as: follows:

$$L(x\mid\alpha,\beta,a,b)=(\alpha\beta ab)^n\prod_{i=1}^{n}x_i^{-(b+1)}e^{\sum_{i=1}^{n}\frac{a}{x_i^b}}\prod_{i=1}^{n}\left(e^{\frac{a}{x_i^b}}-1\right)^{-(\beta+1)}e^{-\alpha\sum_{i=0}^{n}\left(e^{\frac{a}{x_i^b}}-1\right)^{-\beta}}. \quad (32)$$

The corresponding Log likelihood function of the equation (33) is:

$$\log\ L(x\mid\alpha,\beta,a,b)=n\log\alpha+n\log\beta+n\log a+n\log b-(b+1)\sum_{i=1}^{n}\log x_i+\sum_{i=0}^{n}\frac{a}{x_i^{\ b}}$$
$$-(\beta+1)\sum_{i=0}^{n}\log\left(e^{\frac{a}{x_i^b}}-1\right)-\alpha\sum_{i=1}^{n}\left(e^{\frac{a}{x_i^b}}-1\right)^{-\beta} \quad (33)$$

On differentiating the log likelihood function with respect to the unknown parameters of the Weibull Inverse Weibull model and equating to zero result in the following normal equations:

$$\frac{d}{d\alpha}\log L(x\mid\alpha,\beta,a,b)=\frac{n}{\alpha}-\sum_{i=1}^{n}\left(e^{\frac{a}{x_i^b}}-1\right)^{-\beta}=0 \quad (34)$$

$$\frac{d}{d\beta}\log L(x\mid\alpha,\beta,a,b)=\frac{n}{\beta}-\sum_{i=1}^{n}\log\left(e^{\frac{a}{x_i^b}}-1\right)+\alpha\left(e^{\frac{a}{x_i^b}}-1\right)^{-\beta}\log\left(e^{\frac{a}{x_i^b}}-1\right)=0 \quad (35)$$

$$\frac{d}{da}\log L(x\mid\alpha,\beta,a,b)=\frac{n}{a}+\sum_{i=1}^{n}x_i^{-b}-(\beta+1)\sum_{i=1}^{n}\frac{x_i^{-b}e^{\frac{a}{x_i^b}}}{\left(e^{\frac{a}{x_i^b}}-1\right)}=0. \quad (36)$$

$$\frac{d}{db}\log L(x\mid\alpha,\beta,a,b)=\frac{n}{b}-a\sum_{i=1}^{n}x_i^{-b}\log x+(\beta+1)\sum_{i=1}^{n}\frac{ax_i^{-b}e^{\frac{a}{x_i^b}}\log x}{\left(e^{\frac{a}{x_i^b}}-1\right)}-\alpha\beta\sum_{i=1}^{n}\frac{ax^{-b}e^{\frac{a}{x^b}}\log x}{\left(e^{\frac{a}{x^b}}-1\right)}=0. \quad (37)$$

It can be clearly seen that the equations are not in explicit form as such the estimates of the unknown parameters are obtained by solving the normal equations simultaneously using the Newton Raphson algorithm.

8.2 Fisher Information Matrix

For the four parameters of WIW $(x;\alpha,\beta,a,b)$ all the second order derivatives of the log- likelihood function exist. Thus, the inverse dispersion matrix is given by:

$$\begin{pmatrix}\hat{\alpha}\\ \hat{\beta}\\ \hat{a}\\ \hat{b}\end{pmatrix}\sim N\left(\begin{pmatrix}\alpha\\ \beta\\ a\\ b\end{pmatrix},\begin{pmatrix}\hat{V}_{\alpha\alpha} & \hat{V}_{\alpha\beta} & \hat{V}_{\alpha a} & \hat{V}_{\alpha b}\\ \hat{V}_{\beta\alpha} & \hat{V}_{\beta\beta} & \hat{V}_{\beta a} & \hat{V}_{\beta b}\\ \hat{V}_{a\alpha} & \hat{V}_{a\beta} & \hat{V}_{aa} & \hat{V}_{ab}\\ \hat{V}_{b\alpha} & \hat{V}_{b\beta} & \hat{V}_{ba} & \hat{V}_{bb}\end{pmatrix}\right) \quad (38)$$

$$V^{-1} = -E\begin{pmatrix} \hat{V}_{\alpha\alpha} & \hat{V}_{\alpha\beta} & \hat{V}_{\alpha a} & \hat{V}_{\alpha b} \\ \hat{V}_{\beta\alpha} & \hat{V}_{\beta\beta} & \hat{V}_{\beta a} & \hat{V}_{\beta b} \\ \hat{V}_{a\alpha} & \hat{V}_{a\beta} & \hat{V}_{aa} & \hat{V}_{ab} \\ \hat{V}_{b\alpha} & \hat{V}_{b\beta} & \hat{V}_{ba} & \hat{V}_{bb} \end{pmatrix} \tag{39}$$

where $V_{\alpha\alpha} = \frac{\partial^2 L}{\partial\alpha\partial\alpha}$, $\hat{V}_{\beta\beta} = \frac{\partial^2 L}{\partial\beta\partial\beta}$, $\hat{V}_{aa} = \frac{\partial^2 L}{\partial a\partial a}$, $\hat{V}_{bb} = \frac{\partial^2 L}{\partial b\partial b}$, $V_{\alpha\beta} = \frac{\partial^2 L}{\partial\alpha\partial\beta}$, $\hat{V}_{\beta\alpha} = \frac{\partial^2 L}{\partial\beta\partial\alpha}$ and so on.

By deriving the inverse dispersion matrix, the asymptotic variances and covariances of the ML estimators for α, β, a and b are obtained.

IX. DATA ANALYSIS

In this section, the three real life data sets are considered to compare the flexibility of the proposed Weibull Inverse Weibull distribution with different models. In order to compare the different models the criteria like AIC (Akaike information criterion), BIC (Bayesian information criterion) and HQIC (Hannan-Quinn Information Criteria) have been considered. The distribution which provides us lesser values of AIC, BIC and HQIC is considered as best. The values of AIC, BIC and HQIC can be computed as follows:
AIC=2k-2logL, BIC=klogn-2logL and HQIC=2klog (log n)-2logL, where k is the number of parameters in the statistical model, n is the sample size and - 2logL is the maximized value of the log-likelihood function under the considered model. The analysis of all the data sets is performed through R software. The summary of the data sets I, II and III are given in table 1, 3 and 5. The MLEs of the parameters are obtained with standard errors shown in parentheses. Further, the different information measures corresponding to log-likelihood values, AIC, BIC and HQIC are displayed in Table 2, 4 and 6.

Data Set I: The first data set represents 84 observations of failure times (in hours) for a particular wind shield model reported by (Murthy et al. [13]): 0.040, 1.866, 2.385, 3.443, 0.301, 1.876, 2.481, 3.467, 0.309, 1.899, 2.610, 3.478, 0.557, 1.911, 2.625, 3.578, 0.943, 1.912, 2.632, 3.595, 1.070, 1.914, 2.646, 3.699, 1.124, 1.981, 2.661, 3.779, 1.248, 2.010, 2.688, 3.924, 1.281, 2.038, 2.823, 4.035, 1.281, 2.085, 2.890, 4.121, 1.303, 2.089, 2.902, 4.167, 1.432, 2.097, 2.934, 4.240, 1.480, 2.135, 2.962, 4.255, 1.505, 2.154, 2.964, 4.278, 1.506, 2.190, 3.000, 4.305, 1.568, 2.194, 3.103, 4.376, 1.615, 2.223, 3.114, 4.449, 1.619, 2.224, 3.117, 4.485, 1.652, 2.229, 3.166, 4.570, 1.652, 2.300, 3.344, 4.602, 1.757, 2.324, 3.376, 4.663

Table 1: Data summary of Set I

Min.	1st Qu.	Median	Mean	3rd Qu.	Max.	Standard deviation	Skewness	Kurtosis
0.040	1.839	2.354	2.557	3.393	4.663	1.1187	0.0994	2.3476

TABLE 2: MLEs of the Model Parameters Using Real Life Data Set I, the Resulting SEs in parentheses and also Criteria for Comparison

Distribution	MLE				Log-Likelihood	AIC	BIC	HQIC
	$\hat{\alpha}$	$\hat{\beta}$	$\hat{a}$	$\hat{b}$				
WIW	0.27782 (0.66356)	3.68856 (0.73083)	0.83379 (0.36647)	0.43344 (0.10217)	**270.4543**	**278.4543**	**288.1776**	**282.363**
IW			1.36455 (0.15162)	0.83870 (0.05211)	389.0733	393.0733	397.9349	395.0276
KIW	2.45109 (0.51935)	9.840455 (3.09715)	0.506791 (0.04567)	1.58032 (0.33485)	317.1457	325.1457	334.869	326.0772
WIE	0.01773 (0.03591)	1.64762 (0.24593)	0.22424 (0.19445)		279.985	285.985	293.2775	291.8937

Data Set II: The uncensored data set corresponding to intervals in days between 109 successive coal-mining disasters in Great Britain, for the period 1875-1951, published by Maguire et al. [14]. The sorted data are given as follows: 1, 4, 4 , 7 , 11 , 13 , 15 , 15 , 17 , 18 , 19 , 19, 20, 20, 22, 23, 28 ,29 , 31 , 32 , 36, 37 , 47 , 48 , 49 , 50 ,54 , 54 , 55 , 59 , 59, 61, 61, 66, 72, 72 ,75 ,78 ,78 ,81 ,93 ,96 ,99 ,108 ,113, 114, 120, 120, 120,123, 124, 129, 131, 137, 145, 151, 156, 171, 176, 182, 188, 189, 195, 203, 208, 215, 217, 217, 217, 224, 228,233, 255, 271, 275, 275, 275, 286, 291, 312, 312, 312, 315, 326, 326, 329, 330, 336, 338, 345, 348, 354, 361,364 ,369, 378, 390, 457, 467, 498, 517, 566, 644, 745, 871, 1312, 1357, 1613 ,1630

Table 3: Data Summary of Set II

Min.	1st Qu.	Median	Mean	3rd Qu.	Max.	Standard deviation	skewness	Kurtosis
1.0	54.0	145.0	233.3	312.0	1630.0	296.4344	2.9571	12.9943

TABLE 4: MLEs of the Model Parameters Using Real Life Data Set II, the Resulting SEs in parentheses and also Criteria for Comparison

Distribution	MLE				Log-likelihood	AIC	BIC	HQIC
	$\hat{\alpha}$	$\hat{\beta}$	$\hat{a}$	$\hat{b}$				
WIW	0.00559 (0.002013)	0.78465 (0.44811)	0.52299 (0.39122)	1.10876 (0.63748)	**1403.216**	**1411.216**	**1421.981**	**1415.582**
IW	–	–	5.08664 (0.64263)	0.45826 (0.03093)	1484.986	1488.986	1494.369	1491.169
KIW	3.46344 (1.48353)	2.61520 (0.71911)	0.45292 (0.05072)	3.46070 (1.48236)	1428.788	1436.788	1447.553	1438.062
WIE	0.01333 (0.01112)	0.72502 (0.05721)	0.50450 (0.49910)	–	1409.713	1415.713	1423.787	1422.079

Data Set III: The following data set is from Kotz and Johnson [15] and represents the survival times (in years) after diagnosis of 43 patients with a certain kind of leukemia: 0.019, 0.129, 0.159, 0.203, 0.485, 0.636, 0.748, 0.781, 0.869,1.175, 1.206, 1.219, 1.219, 1.282, 1.356, 1.362, 1.458, 1.564, 1.586, 1.592, 1.781, 1.923, 1.959, 2.134, 2.413, 2.466, 2.548, 2.652, 2.951, 3.038, 3.600, 3.655, 3.745, 4.203, 4.690, 4.888, 5.143, 5.167, 5.603, 5.633, 6.192, 6.655, 6.874

Table 5: Data Summary of Data set III

Min.	1st Qu.	Median	Mean	3rd Qu.	Max.	Standard deviation	Skewness	Kurtosis
0.019	1.212	1.923	2.534	3.700	6.874	1.927	0.7448	2.4363

TABLE 6: MLEs of the Model Parameters Using Real Life Data Set III, the Resulting SEs in parentheses and also Criteria for Comparison

Distribution	**MLE**				Log Likelihood	AIC	BIC	HQIC
	$\hat{\alpha}$	$\hat{\beta}$	$\hat{a}$	$\hat{b}$				
WIW	0.00236 (0.00041)	1.42731 (0.61655)	0.03212 (0.04673)	0.82395 (0.38257)	**163.6384**	**171.6384**	**178.6832**	**174.2363**
IW	–	–	0.90316 (0.15019)	0.62623 (0.06041)	204.7781	208.7781	212.3005	210.0771
KIW	1.58729 (1.59624)	5.26328 (2.19729)	0.39055 (0.05615)	1.57042 (1.57928)	184.7468	192.7468	199.7916	192.6952
WIE	0.01467 (0.01806)	0.97236 (0.12633)	0.03221 (0.03266)	–	166.5454	172.5454	181.5902	177.1433

Figure 5 (a), 5(b) and 5(c) show the fitted density functions of Weibull Inverse Weibull, Inverse Weibull, Kumaraswamy Inverse Weibull and Weibull Inverse exponential distributions

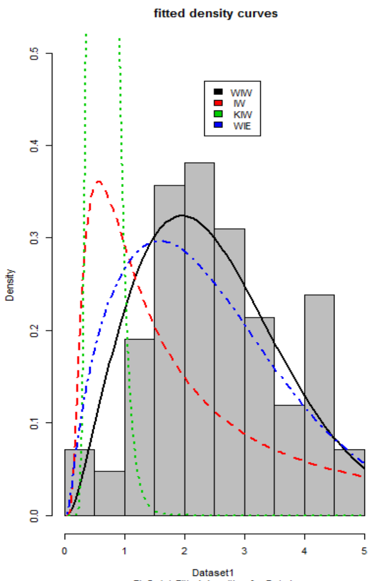

Fig5. (a) Fitted densities for Data1

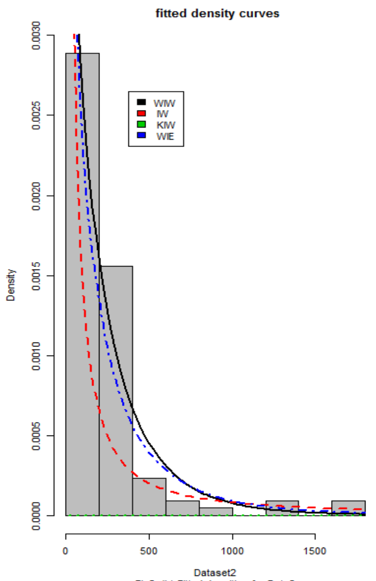

Fig5. (b) Fitted densities for Data2

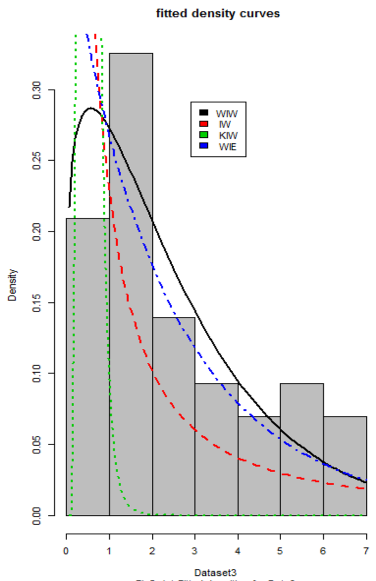

Fig5. (c) Fitted densities for Data3

X. CONCLUSION

This manuscript deals with the introduction of new Weibull Inverse Weibull distribution which is the obtained by Weibull G technique. The main aim of the paper is to study its different statistical properties like moments, harmonic mean, survival function, hazard rate, Renyi entropy and maximum likelihood estimation. Moreover, the postulated distribution is compared with the different models for flexibility and testing of better fit. This newly proposed model has been applied to the three real life data sets for competence. The results obtained are displayed in table 2, 4 and 6 respectively which show that the proposed distribution has lesser values of AIC, BIC and HQIC than the various models and same is also clearly depicted by the fitted probability densities of the different models. This proves that the newly developed model provides better fit for modeling lifetime data sets.

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