International Journal of Enhanced Research in Science Technology & Engineering, ISSN: 2319-7463
Vol. 2 Issue 11, November-2013, pp: (228-241), Available online at: www.erpublications.com

Three Problems of Weyl's Theorem

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ABSTRACT

Quantum mechanics is a part of quantum theory. The latter was initiated in 1900, when Max Planck announced the concept of a quantum which brought a revolution. This year and this decisive event is referred to as the dividing point between the classical physics and modern or quantum physics. The new period of physics was caused by many new basic discoveries: X-ray, the electron, radioactivity etc. Quantum mechanics, based upon the consideration of the space $L^2(-\infty,\infty)$, where elements are called states and self-adjoint operators called observables, provided much inputs to study the operators, particularly the self adjoint operator and non-self adjoint operators over Hilbert spaces. H. Weyl [40], in 1909, observed that for a self-adjoint operator in a Hilbert space, perturbation by self-adjoint compact operator leaves an 'essential' part of the spectrum invariant. Precisely that part of the spectrum which contains the limit points of spectrum and the points of infinite-multiplicity.

This part of the spectrum, later on was termed after H. Weyl, as Weyl spectrum and this observation became a classical version of Weyl's theorem. Thus, classically speaking, for a bounded self adjoint operator, the complement in the spectrum of this part of self-adjoint operators coincide with the isolated points of the spectrum which are the eigen-values of finite multiplicity and in the present form we say that a bounded linear operator is said to satisfy Weyl's theorem, if the complement of the Weyl's spectrum in the spectrum equals the set of isolated eigenvalues of finite multiplicity.

Ever since its formulation, for its large number of applications to physics, the problem of identifying operators satisfying Weyl's theorem has been a subject of research for a host of mathematicians throughout the world. Notable contributions, among others are from, L.A. Coburn [6], S.K. Berberian [2,3,4], V. Istrttescu [23], Karl Gustafson [14], K.K. Oberai [29, 30], S.C. Arora [1], W.Y. Lee and S.H. Lee [25, 26, 27, 28], D.R. Farenick [11], Youngoh Yang [41, 42, 43, 44].
In this introduction, We set and present notations, terminology to be used and a brief summary.

Unless stated otherwise H will denote an infinite dimensional Hilbert space and C, the space of complex-numbers. $||x||$, denotes the norm of the vector x. By a subspace of H, we mean a closed linear manifold of H. If M is a subspace of H, $M^{\perp}$ denotes the orthogonal complement of M in H. By an operator T on H, we shall mean a bounded linear transformation of H into H. We write B(H) for the algebra of operators on H. For T in B(H), T* denotes the adjoint of T. R(T) (Ran T) stands fbr the range space and N(T) (KerT, $T^{-1}(0)$ for the null space of T. A subspace M is said to be invariant under T if T(M)⊑M. If both M and $M^{\perp}$ are invariant under T, we say that M reduces T. If $M^{\perp}$ is invariant under T, $T|_M$ denotes the restriction of T to M. If S and T are operators on the Hilbert spaces H and K respectively, then the operator S ⊕ T is an operator on H ⊕ K, defined by

$$(S \oplus T)(x,y) = (Sx, Ty).$$

We now proceed to give various definitions pertaining to the spectrum and its parts. The **spectrum σ (T)** of an operator T is defined as

$$\sigma(T) = \{\lambda \in C : T - \lambda I \text{ is not invertible in } B(H)\}.$$

The**resolvent set ρ(T)** of an operator T is defined as

$$\rho(T) = \{\lambda \in \mathbb{C} : R_\lambda = (T-\lambda I)^{-1}, \text{ exists, is bounded and is defined on a set which is dense in } H\}.$$

The approximate point spectrum π(T) of T is the set of all λ in $\mathbb{C}$ such that $S(T-\lambda I) \neq I$ for any operator S on H. Equivalently, $\lambda \in \pi(T)$ if and only if there exists a sequence $\langle x_n \rangle$ of unit vectors in H such that

$$\|(T-\lambda I)x_n\| \to 0 \text{ as } n \to \infty.$$

A scalar λ is called an eigenvalue of T if there exists a nonzero vector x such that (T- λI) x=0. The set of all eigenvalues of T, denoted by π_0(T), is called the **point spectrum** of T. The null space N(T- λI) of T- λI is called the **eigenspace** corresponding to the scalerλ and dimension of N(T- λI) is called the **multiplicity** of the eigenvalue λ. π_{0f}(T) denotes the set of those eigenvalues which are of finite multiplicity,π_{0i} (T) denotes the set of eigenvalues of infinite multiplicity. π_{00}(T) denotes the **isolatedpoint spectrum** of T, that is, the set of all isolated eigenvalues of a(T) which are of finite multiplicity, and **isoσ(T)** denotes the set of all isolated points of σ(T). Also, **accσ-(T)** denotes the set of all accumulation points of σ(T).

An operator Ton H is said to be **compact** if it maps every bounded set onto relatively compact sets. Equivalently, the image of every bounded sequence contains a convergent subsequence. **K(H)** denotes the ideal of all compact operators on H. The quotient algebra$\frac{B(H)}{K(H)}$ which is a Banach algebra is known as the calking algebra. Let $\hat{T} = T + B(H)$ denote the canonical image of T in the calkin algebra. Then the spectrum $\sigma(\hat{T})$ of $\hat{T}$ as an element of calkin algebra is called the calkin or the essential-spectra of T and is denoted by $\sigma_e(T)$. An operator T is called **Fredholm operator** if

(i) R(T) is closed, and
(ii) N(T) and N(T) are finite dimensional.

The **index i(T)** of a Fredholm operator T is defined as

$$i(T) = dim\, N(T) - dim\, N(T*)$$

where dim M is the dimension of the subspace M. The Atkinson theorem [15, Problem 142] gives an elegant characterization of $\sigma_e(T)$ as

$$\sigma_e(T) = \{\lambda \in \mathbb{C}: \text{T-}\lambda\text{I is not a Fredholm operator}\}.$$

The **Weyl spectrum** ω **(T)** of T is defined as

$$\omega(T) = \bigcap_{K \in K(H)} \sigma(T+K).$$

Equivalently,

$$\omega(T) = \{\lambda \in \mathbb{C}: \text{T-}\lambda\text{I is not a Fredholm operator of index zero}\}.$$

Any Fredholm operator of index zero is called a **Weyl operator**.

Therefore,

$$\omega(T) = \{\lambda \in \mathbb{C}: \text{T-}\lambda\text{I is not Weyl}\}.$$

The **Convex hull** convS of a set $S \subseteq \mathbb{C}^n$ is the intersection of all convex sets containing S. $\lambda \in \mathbb{C}$ is said to be a **semibare** point of σ(T) if it lies on the circumference of some closed disk that contains no other point of σ(T). A subset of complex measure is said to be **thin** if its planer lebesgue measure is zero. Also, an operator T in B(H) is said to be of**finite rank** if R(T) is finite dimensional. An operator T in B(H) is said to have **finite ascent** if there exists some non negative integer m such that

$$N(T^m) = N(T^{m+1}).$$

The smallest non negative integer m satisfying this condition is called the **ascent** of T. An operator T in B(H) is said to have **finite descent** if there exists some non negativeinteger m such that

$$R(T^m)^{\perp} = R(T^{m+1})^{\perp}.$$

The smallest non negative integer m satisfying this condition is called the descent of T.

An operator T in B(H) is called **self-adjoint** if T=T*, **normal** if TT* = T*T, **essentially normal** if T*T - TT* is compact, **unitary** if TT*=T*T=I, **isometry** if T*T=I and coisometry if TT*=I.

We now proceed to define various classes of non-normal operators. An operator T is called **hyponormal** if its self commutator [T*T] = T*T-TT* is positive, **M-hyponormal** if there exists M>0 such that $||(T-Z)\,x\,|| \leq M\,|\,|(T-Z)x||$, for all x in H and for all Z in **analytic quasihyponormal** if there exists a function f analytic on a neighbourhood of σ(T) such that f(T)* (T*T-TT*) f(T)$\geq$0, **seminormal**if either T or T* is hyponormal, **algebraically-hyponormal** if there exist a nonconstant polynomial p such that p(T) is hyponormal. An operator T in B(H) is said to be of class W if essential-spectrum of T equals the Weyl spectrum of T, that means, $\sigma_e(T) = \omega(T)$. T is said to satisfy **Growth condition** (G_1) if

$$||(T-\lambda I)^{-1}|| \leq \frac{1}{d(\lambda, \sigma(T))}, \qquad \lambda \notin \sigma(T) \quad \mathbb{C},$$

and is said to satisfy **reduction-(G_1)** if every direct summand of T satisfies (G_1). This means that if $T = T_1 \oplus T_2$ then T_1 and T_2 both satisfy condition (G_1) as operators on their respective domains.

Let **T** = {z∈ℂ : | Z | = 1} denote the unit circle in the complex plane ℂ, μ the normalised lebesgue measure on **T** and **L^2(T)** = **L^2**, the Hilbert space of complex-valued measurable square integrable functions on **T**. L^2 has a canonical orthonormal basis given by the trigonometric functions $e_n(z) = z^n$ for each n in I, the set of integers. The **Hardy space** **H^2 (T) = H^2** is the closed linear span of {e_n : n = 0, 1, 2, ...}. An element $f \in L^2$ is referred to as **analytic** if $f \in H^2$ and **coanalytic** if $f \in L^2 \ominus H^2$. If $P : L^2 \rightarrow H^2$ denotes the projection operator, then for every $\phi \in$ **L^∞ (T)**, the space of essentially bounded measurable functions, the operator T_ϕ on H^2 defined by

$$T_\phi\, g = P\,(\phi g)$$

for each g in H^2 is called the **Toeplitz operator** with symbol ϕ. **C(T)** denotes the set of all continuous complex-valued functions on the unit circle **T** and **H^∞(T) = $L^\infty \cap H^2$**. Both C(T) and H^∞(T) are Banach algebras. The elements of the closed self-adjoint subalgebra **QC**, which is defined to be

$$QC = (H^\infty(T)+C(T)) \cap \overline{(H^\infty(T)+C(T))}$$

are called **quasicontinuous functions.**

The three problems regarding Weyl's raised by K.K. Oberoi [30] in the year 1977 are discussed in the following three sections.

The first problem raised by K.K. Oberoi was the following:-

Let T in $B(H^2)$ be Toeplitz. Then does Weyl's theorem hold for T^2 ?

The problem remained open for nineteen years and was finally answered by D.R. Farenick and W.Y. Lee [11] negatively in the year 1996. In the process, they proved the following :

If ϕ is continuous and f is any analytic function defined on some open set containing $\sigma(T_\phi)$, then in general, $\sigma(T_{fo\phi}) \subseteq f(\sigma(T_\phi))$.

Also, $\sigma(T_{fo\phi}) = f(\sigma(T_\phi))$ if and only if Weyl's theorem holds for $f(T_\phi)$. This means $\omega(f(T_\phi)) = \sigma(f(T_\phi))$. In addition, if ϕ is defined on the unit circle as

$$\phi = \begin{cases} -e^{2i\theta} + 1 & (0 \leq \theta \leq \pi) \\ e^{-2i\theta} - 1 & (\pi \leq \theta \leq 2\pi) \end{cases}$$

then, $\sigma(T_{\phi^2}) \neq \{\sigma(T_\phi)\}^2$.

This establishes that square of a Toeplitz operator need not satisfy the Weyl's theorem. All this comprises of section 1 of the dissertation.

Section 2 discusses the second problem raised by K.K. Oberai. The problem states the following :

Let T in B(H) be hyponormal. Then does Weyl's theorem hold for T^2 ?

This problem was answered in affirmative by W.Y. Lee and S.H. Lee [28]. They proved [Theorem 3.3, 28] that if T in B(H) is hyponormal, then Weyl's theorem holds for T^2.

In section 3 we study the third problem raised by K.K. Oberai. The problem states the following :

Let T be in B(H). If Weyl's theorem holds for T and F is a finite rank operator commuting with T, then does Weyl's theorem hold for T+F?

The answer which comprises of section 2.3 given by W.Y. Lee and S.H. Lee [28] in the year 1996, presents an example of an operator T in B(H) and a finite rank operator F in B(H), commuting with T, such that Weyl's theorem holds for T, but it does not hold for T+F and thus answers the problem in negative.

In the year 1977, Kirti K. Oberai [30] raised the following three problems:

Problem 1: Let T in $B(H^2)$ be Toeplitz. Then, does Weyl's theorem hold for T^2?

Problem 2 : Let T in $B(H^2)$ be Hyponormal. Then, does Weyl's theorem hold for T^2?

Problem 3 : Let T be in B(H). If Weyl's theorem holds for T and F is a finite rank operator commuting with T, then does Weyl's theorem hold for T+F?

In this work, our aim is to discuss the solutions of the above mentioned problems obtained during these years. Accordingly, this chapter has been divided into three sections, discussing each problem in the respective section.

Section 1 :Toeplitz Operators

The first problem raised by K.K. Oberai [30] as mentioned is the following :-Let T in $B(H^2)$ be Toeplitz. Then, does Weyl's theorem hold for T^2?

Recently, D.R. Farenick and W.Y. Lee [11] in 1996, answered this question negatively by giving an example of a Toeplitz operator whose square does not satisfy Weyl's theorem. Without mentioning, it may be understood that, in this section our space is H_2 and the functions are defined on unit circle. To get the needful accompalished, we begin with the following :

Lemma 1 [11]: Let ϕ be continous and T_ϕ be Toeplitz operator induced by ϕ. Also let f be an analytic fucntion defined on some open set containing
$\sigma(T_\phi)$. Then

$$\sigma(T_{fo\phi}) \subseteq f(\sigma(T_\phi))$$

and

$$\sigma(T_{fo\phi}) = f(\sigma(T_\phi))$$

if and only if Weyl's theorem holds for $f(T_\phi)$. Also, then

$$\omega(f(T_\phi)) = \sigma(f(T_\phi)).$$

Proof : We prove this Lemma in three steps.

Step 1: With T_ϕ, Toeplitz, ϕ in QC, the class of quasicontinuous functions, and f analytic on an open set containing $\sigma(T_\phi)$. We claim that

$$T_{f0\phi} - f(T_\phi) \textit{ is compact.}$$

We know that [10] if ψ is in $H^\infty + C(\mathbf{T})$, then T_ψ is Fredholm if and only if ψ is invertible in $H^\infty + C(\mathbf{T})$. With this in mind, we let $\lambda \notin \sigma(T_\phi)$. Then, both $\phi - \lambda$ and $\overline{\phi - \lambda}$ are invertible in $H^\infty + C(\mathbf{T})$. Hence

$$(\phi - \lambda)^{-1} \in QC \qquad ..... \qquad (1)$$

Also, the Toeplitz operators whose symbols belong to the class QC [11] satisfy the following algebraic relation :

$$T_{\psi} T_{\phi} - T_{\psi\phi} \in K(H^2) \qquad \ldots\ldots \quad (2)$$

for every $\phi \in H^{\infty}(\mathbf{T}) + C(\mathbf{T})$ and $\psi \in L^{\infty}(\mathbf{T})$, where $K(H^2)$ is the ideal of compact operators on H^2. From (1) and (2), we have that for ψ in L^{∞} and λ, μ in $\mathbb{C}$

$$T_{\phi-\mu} T_{\psi} T^{-1}_{\phi-\lambda} - T_{(\phi-\mu)\psi(\phi-\lambda)^{-1}} \in K(H^2)$$

whenever $\lambda \notin \sigma(T_{\phi})$.

The argument above extend to rational functions to yield the following:

If r is any rational function with all its poles away off $\sigma(T_{\phi})$, then

$$r(T_{\phi}) - T_{(r o \phi)} \in K(H^2).$$

Since f is analytic on an open set containing $\sigma(T_{\phi})$, by Runge's theorem, there exists a sequence of rational functions $\langle r_n \rangle$ such that the poles of each r_n lie outside of $\sigma(T_{\phi})$ and $r_n \to f$ uniformly on $\sigma(T_{\phi})$. Thus $r_n(T_{\phi}) \to f(T_{\phi})$ in the norm topology of $L(H^2)$. Furthermore, as

$$r_n \circ \phi \to f \circ \phi$$

uniformly, we have

$$T_{r_n o\phi} \to T_{fo\phi}$$

in the norm topology. Therefore

$$T_{fo\phi} - f(T_{\phi}) = \text{lt}\,(T_{r_n o\phi} - r_n(T_{\phi}))$$

which is compact. Hence the claim. This proves step 1.

Step 2 : Claim : $\omega(f(T_{\phi})) = \sigma(T_{fo\phi})$.

Since Weyl spectrum is stable under the compact perturbations, it follows from step 1, that

$$\omega(f(T_{\phi})) = \omega(T_{fo\phi}) = \sigma(T_{fo\phi}).$$

This proves the claim.

International Journal of Enhanced Research in Science Technology & Engineering, ISSN: 2319-7463
Vol. 2 Issue 11, November-2013, pp: (228-241), Available online at: www.erpublications.com

Step 3 : Claim : $\omega(f(T_\phi)) = \sigma(f(T_\phi)) \sim \pi_{00}(f(T_\phi))$

if and only if

$$\sigma(T_{fo\phi})) = f(\sigma(T_\phi)).$$

By Step 2, $\sigma(T_{fo\phi})) = \omega(f(T_\phi))$. Also

$$\omega(f(T_\phi)) \subseteq \sigma(f(T_\phi)) = f(\sigma(T_\phi)).$$

Since for $\phi \in L^\infty(\mathbf{T})$, $\sigma(T_\phi)$ is connected, so is $f(\sigma(T_\phi)) = \sigma(f(T_\phi))$, therefore, the set $\pi_{00}(f(T_\phi))$ is empty. So we conclude that

$$\omega(f(T_\phi)) = \sigma(f(T_\phi)) \sim \pi_{00}(f(T_\phi))$$

if and only if

$$\sigma(T_{fo\phi})) = f(\sigma(T_\phi)).$$

This proves the theorem. □

Remark 2 **[11]** : If ϕ is not continous, it is possible for Weyl's theorem to hold for some $f(T_\phi)$ without $\sigma(T_{fo\phi})$ being equal to $f(\sigma(T_\phi))$.

For example, let

$$\phi(e^{i\theta}) = e^{i\theta/3} \qquad (0 \le \theta < 2\pi)$$

be a piece-wise continuous function. The operator T_ϕ is invertible but T_{ϕ^2}

Since f is analytic on an open set containing $\sigma(T_\phi)$, by Runge's theorem, there exists a sequence of rational functions $\langle r_n \rangle$ such that the poles of each r_n lie outside of $\sigma(T_\phi)$ and $r_n \to f$ uniformly on $\sigma(T_\phi)$. Thus $r_n(T_\phi) \to f(T_\phi)$ in the norm topology of $L(H^2)$. Furthermore, as

$$r_n \circ \phi \;\to\; f \circ \phi$$

uniformly, we have

$$T_{r_n o\phi} \;\to\; T_{fo\phi}$$

in the norm topology. Therefore

$$T_{fo\phi} - f(T_\phi) = \mathrm{lt}\,(T_{r_n o\phi} - r_n(T_\phi))$$

which is compact. Hence the claim. This proves step 1.

Step 2 : Claim : $\omega(f(T_\phi)) = \sigma(T_{fo\phi})$.

Since Weyl spectrum is stable under the compact perturbations, it follows from step 1, that

$$\omega(f(T_\phi)) = \omega(T_{fo\phi}) = \sigma(T_{fo\phi}).$$

This proves the claim.

Step 3 : Claim : $\omega(f(T_\phi)) = \sigma(f(T_\phi)) \sim \pi_{00}(f(T_\phi))$

if and only if

$$\sigma(T_{fo\phi})) = f(\sigma(T_\phi)).$$

By Step 2, $\sigma(T_{fo\phi})) = \omega(f(T_\phi))$. Also

$$\omega(f(T_\phi)) \subseteq \sigma(f(T_\phi)) = f(\sigma(T_\phi)).$$

Since for $\phi \in L^\infty(\mathbf{T})$, $\sigma(T_\phi)$ is connected, so is $f(\sigma(T_\phi)) = \sigma(f(T_\phi))$, therefore, the set $\pi_{00}(f(T_\phi))$ is empty. So we conclude that

$$\omega(f(T_\phi)) = \sigma(f(T_\phi)) \sim \pi_{00}(f(T_\phi))$$

if and only if

$$\sigma(T_{fo\phi})) = f(\sigma(T_\phi)).$$

This proves the theorem. ❑

Remark 2 **[11]** : If ϕ is not continous, it is possible for Weyl's theorem to hold for some $f(T_\phi)$ without $\sigma(T_{fo\phi})$ being equal to $f(\sigma(T_\phi))$.

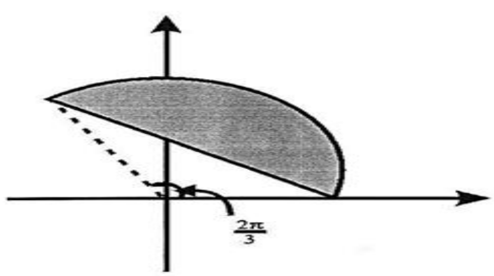
For example, let

$$\phi(e^{i\theta}) = e^{i\theta/3} \qquad (0 \le \theta < 2\pi)$$

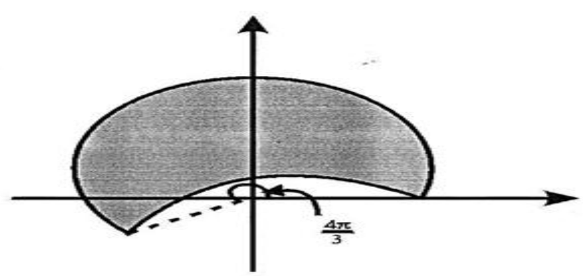
be a piece-wise continuous function. The operator T_ϕ is invertible but T_{ϕ^2} is not invertible. Hence

$$0 \in \sigma(T_{\phi^2}) \sim \{T(T_\phi)\}^2.$$

However, $\omega(T_\phi{}^2) = \sigma(T_\phi{}^2)$ and $\pi_\infty(T_\phi{}^2)$ is empty. Therefore, Weyl's-theorem holds for $T_\phi{}^2$.



$\sigma(T_\phi) = \omega(T_\phi)$



$\sigma(T_\phi{}^2) = \omega(T_\phi{}^2)$

The following example shows that Weyl's theorem may not hold for $T_\phi{}^2$ if T_ϕ is a Toeplitz operator.

Example 3 [11] : There exists a continuous function $\phi \in C(\mathbf{T})$ such that

$$\sigma(T_{\phi^2}) \neq \{\sigma(T_\phi)\}^2.$$

Let ϕ be defined by

$$\phi(e^{i\theta}) = \begin{cases} -e^{2i\theta} + 1 & (0 \le \theta \le \pi) \\ e^{-2i\theta} - 1 & (\pi \le \theta \le 2\pi) \end{cases}$$

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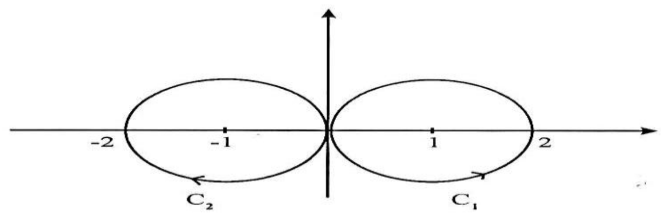
Vol. 2 Issue 11, November-2013, pp: (228-241), Available online at: www.erpublications.com

The orientation of the graph of ϕ is shown in the following figure :



Evidently, ϕ is continuous, ϕ has winding number +1, with respect to any hole of C_1 and winding number -1, with respect to any hole of C_2. Thus, we have

$$\sigma_e (T_\phi) = \phi(\mathbf{T})$$

and
$$\sigma(T_\phi) = \text{Conv}\, \phi(\mathbf{T}).$$

On the other hand a straight forward calculation shows that $\phi^2(\mathbf{T})$ is the cardiod $r = 2(1+\cos\theta)$. In particular, $\phi^2(\mathbf{T})$ traverses the cardioid once in a counter clockwise direction and then once in clockwise direction. Thus, winding number of $(\phi^2 - \lambda) = 0$ for each λ in the hole of $\phi^2(\mathbf{T})$. Hence $T_{\phi^2-\lambda}$ is a Weyl operator and is therefore invertible for each λ in the hole of $\phi^2(\mathbf{T})$. This implies that $\sigma(T_{\phi^2})$ is the cardioid $r = 2(1+\cos\theta)$.

As, $\{\sigma(T_\phi)\}^2 = \{\text{Conv}\, \phi(\mathbf{T})\}^2 = \{(r,\theta) : r \leq 2(1+\cos\theta)\}$

it follows that $\sigma(T_{\phi^2}) \neq \{\sigma(T_\phi)\}^2$. ❑

Section 2 : Hyponormal operators

In this section we obtain the soulution of the following second probelm raised by K.K. Oberai [30] :

Let T in B (H) be Hyponormal. Then does Weyl's theorem hold for T^2?

To begin our study, we first recall the following :

Definition 4 [10] : An operator T in B(H) is said to be **Hyponormal** if its self-commutator $[T^*T] = T^*T - TT^*$ is positive.

Definition 5 [30] : An operator T in B(H) is said to be **Isoloid** if the isolated points of $\sigma(T)$ are eigenvalues of T.

We begin answering the above question with the following lemma :

Lemma 6 [28] : Let T and S be commuting hyponormal operators. Then, TS is Weyl if and only if T and S both are Weyl.

Proof : Let S and T be Weyl. Then S and T are Fredholm and $i(S) = i(T) = 0$.

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Now, we know that [39, theorem 2.3] the product of two Fredholm operators is Fredholm and $i(ST) = i(S) + i(T)$. Therefore, it follows that ST is Fredholm and $i(ST) = 0$. Hence, ST is Weyl.

Conversely, let ST be Weyl. As ST=TS, therefore

$\text{Ker } S \cup \text{Ker } T \subseteq \text{Ker } ST$ and $\text{Ker } S^* \cup \text{Ker } T^* \subseteq \text{Ker } (ST)^*$. As, ST is Weyl, dim Ker S and dim Ker T both are finite and also dim Ker S* and dim Ker T* are finite. Again range of S and range of T are closed [16, theorem 3.2.2]. Hence S, T are Fredholm. Since S and T are hyponormal, $i(S) \leq 0$, $i(T) \leq 0$. But $i(ST) = 0$. Hence $i(S) + i(T) = 0$. This gives that $i(S) = 0 = i(T)$. Thus both S and T are Fredholm operators each of index 0. Therefore, S and T are both Weyl. ❑

The following theorem due to S.H.Lee and W.Y.Lee [28] provides a complete answer to the above-mentioned problem :

Theorem 7 [28] : Let T in B(H) be hyponormal. Then Weyl's theorem holds for T^2.

Proof : We are to show that $\pi_{00}(T^2) = \sigma(T^2) \sim \omega(T^2)$.

That is, we show that

$$\lambda \in \sigma(T^2) \sim \omega(T^2) \text{ if and only if } \lambda \in \pi_{00}(T^2).$$

First, we observe that if μ is square root of λ, then

$\lambda \notin \text{acc}\,(\sigma(T^2))$ if and only if $\pm\mu \notin \text{acc}\,\sigma(T)$ ..... (1)

where acc $(\sigma(T))$ denotes the set of accumulation points of $\sigma(T)$.

Now, let $\lambda \in \sigma(T^2) \sim \omega(T^2)$. This gives that $T^2 - \lambda$ is Weyl but not invertible. Since we know that [2] for any operator T, $\sigma(T) \sim \omega(T)$ is either empty or consists of eigenvalues of finite multiplicity, we have, $\lambda \in \pi_0(T^2)$. Hence, it remains to show that λ is an isolated point of $\sigma(T^2)$. Now,

$$T^2 - \lambda = (T - \mu)(T + \mu)$$

Therefore, $T - \mu$ and $T + \mu$ are both Weyl. Since Weyl's theorem holds for T, it follows that $\pm\mu \notin \text{acc}\,\sigma(T)$. Thus we have $\lambda \notin \text{acc}\,\sigma(T^2)$. As $T^2 - \lambda$ is not invertible, therefore λ is an isolated point of $\sigma(T^2)$. This means that $\lambda \in \pi_{00}(T^2)$.

Conversely, let $\lambda \in \pi_{00}(T^2)$. Then $T^2 - \lambda$ is not invertible. We need to show that $T^2 - \lambda$ is Weyl. By our assumption, we have

$\lambda \in \text{iso}\,\sigma(T^2)$ and $0 < \dim (T^2-\lambda)^{-1}(0) < \infty$ .... (2)

By (1), we have $\pm\mu \notin \text{acc}\,\sigma(T)$. Since $(T \pm \mu)^{-1}(0) \subseteq (T^2 - \lambda)^{-1}(0)$, it follows from the second part of (2) that $\dim (T \pm \lambda)^{-1}(0) < \infty$. If $\pm\mu \in \text{iso}\,\sigma(T)$, then since T is isoloid [36], it follows that $T \pm \mu$ is not one-one. This gives that

$$0 < \dim (T \pm \mu)^{-1}(0) < \infty.$$

Since Weyl's theorem holds for T, it follows that $T - \mu$ and $T + \mu$ are both Weyl. Therefore, $T^2 - \lambda$ is Weyl. If $\mu \in \rho(T)$ or $-\mu \in \rho(T)$, then also a similar argument gives that $T^2 - \lambda$ is Weyl. □

Section 3 : Commuting finite rank operators

The third problem raised by K.K. Oberai [30] was :

Let T be in B(H). If Weyl's theorem holds for T and F is a finite rank operator commuting with T, then does Weyl's theorem hold for T+F?

In this section, we first give two perturbation theorems on Weyl's theorem and then discuss example which answers the above question by oberai negatively [28], [21].

We recall the following [28], [7], [12], [16]

If T is Weyl and K compact, then T+K is Weyl .... (1)

If T is Weyl, then T+K is invertible for some compact operator K .... (2)

Evidently, if T is Weyl and one-one then it is invertible and thus we have [2],

$$\sigma(T) \sim \omega(T) \subseteq \pi_0(T) \qquad \text{......... (3)}$$

Lemma 8 [28] : Let T be in B(H). If F is a finite rank operator, then

$$\dim T^{-1}(0) < \infty \text{ if and only if } \dim (T+F)^{-1}(0) < \infty \qquad \text{........ (4)}$$

Further if TF = FT, then

$$\lambda \in \text{acc}\, \sigma(T) \text{ if and only if } \lambda \in \text{acc}\, \sigma(T+F) \qquad \text{........ (5)}$$

Theorem 9 [28] : Let T in B(H) be isoloid. Let F be a finite rank operator commuting with T. If Weyl's theorem holds for T, then it holds for T+F also.

Proof : We are to show that

$$\sigma(T+F) \sim \omega(T+F) = \pi_{00}(T+F).$$

This means that

$$\lambda \in \sigma(T+F) \sim \omega(T+F) \text{ if and only if } \lambda \in \pi_{00}(T+F).$$

Without any loss of generality we may assume that $\lambda = 0$.

Thus, let $0 \in \sigma(T+F) \sim \omega(T+F)$. Hence T+F is Weyl, but not invertible. As

$$\sigma(T+F) \sim \omega(T+F) \subseteq \pi_0(T+F),\ 0 \in \pi_0(T+F),$$

therefore, it remains to show that 0 is isolated in $\sigma(T+F)$. By our assumption and (1), T is Weyl. Since Weyl's theorem holds for T, if follows that

$$0 \in \rho(T) \text{ or } 0 \in \text{iso}\, \sigma(T).$$

Thus by (5), we have $0 \notin \text{acc}\, \sigma(T+F)$. Since T+F is not invertible, we have

$$0 \in \text{iso } \sigma(T+F) \text{ and } \sigma(T+F) \sim \omega(T+F) \subseteq \pi_{00}(T+F).$$

Conversely, we suppose that $0 \in \pi_{00}(T+F)$. Hence T+F is not invertible. It is to be shown that T+F is Weyl. By our assumption we have

$$0 \in \text{iso } \sigma(T+F) \text{ and } 0 < \dim (T+F)^{-1}(0) < \infty.$$

By (4) and (5), we obtain

$$0 \notin \text{acc } \sigma(T) \text{ and } \dim T^{-1}(0) < \infty \quad \ldots\ldots\ldots\ldots \ (6)$$

If T is invertible, then T+F is Weyl. If T is not invertible, then by first part of (6), we have $0 \in \text{iso } \sigma(T)$. Since T is isoloid, it follows that T is not one-one. This together with the second part of (6) gives

$$0 < \dim T^{-1}(0) < \infty .$$

Thus, since Weyl's theorem holds for T, it follows that T is Weyl and hence, so is T+F. ❑

Theorem 10 [21] : Let T in B(H) be quasinilpotent. Let F be any finite rank operator commuting with T. If Weyl's theorem holds for T, then it holds for T+F also.

Proof : By Gelfand theory of commutative Banach alegbras, we get

$$\sigma(T+F) = \sigma(T) + \sigma(F) = \{0, \lambda_1, \lambda_2, \ldots, \lambda_\kappa\},$$

where $\lambda_i \neq 0$, $\lambda_i \in \sigma(F)$. Then

$$\{\lambda_1, \ldots, \lambda_\kappa\} \subseteq \pi_{00}(T+F).$$

Now, since $\omega(T+F) = \omega(T) = \{0\}$, it remains to show that $0 \notin \pi_{00}(T+F)$. As Weyl's theorem holds for T and T is quasinilpotent, $\sigma(T) = \{0\}$. Then $N(T) = \{0\}$ or N(T) is infinite-dimensional. Let $N(T) = \{0\}$. Therefore, T is one-one. Then, there is no finite rank operator commuting with T except $F = 0$ and so $0 \notin \pi_{00}(T+F)$. Next, let N(T) be infinite dimensional, and let $A = F|_{N(T)}$. Then

$$N(A) = N(T) \cap N(F).$$

So, we get that $N(T) \cap N(F)$ is infinite dimensional. Hence, $0 \notin \pi_{00}(T+F)$. Thus, Weyl's theorem holds for T+F. ❑

Now, if T is not assumed to be isoloid or quasinilpotent, then the above two theorems may fail to hold. Thus the following example answers the question of oberai negatively.

Example 11 [28] : There exists T in B(H) and a finite rank operator F, commuting with T, such that Weyl's theorem holds for T, but it does not hold for T+F.

Define

$$T = \begin{bmatrix} I & 0 \\ 0 & S \end{bmatrix} : l_2 \oplus l_2 \to l_2 \oplus l_2$$

and

$$F = \begin{bmatrix} K & 0 \\ 0 & 0 \end{bmatrix} : l_2 \oplus l_2 \to l_2 \oplus l_2$$

where $S : l_2 \to l_2$ is an injective quasinilpotent operator and $K : l_2 \to l_2$ is defined by $K(x_1, x_2, x_3, \ldots\ldots\ldots) = (-x_1, 0, 0, \ldots\ldots)$.

Then F is of finite rank and commutes with T. Also

$$\sigma(T) = \omega(T) = \{0,1\} \text{ and } \pi_{00}(T) = \phi.$$

This implies that Weyl's theorem holds for T. We, however, have

$$\sigma(T+F) = \omega(T+F) = \{0,1\} \text{ and } \pi_{00}(T+F) = \{0\}.$$

Consequently, Weyl's theorem does not hold for T+F. ❑

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