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# Performance Evaluation of Service Systems under Uncertainty Using Fuzzy Queueing Models: A Comparative Analysis of M/M/1, M/M/c, M/M/1/K, and M/G/1 Configurations

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## ABSTRACT

**In this paper, a small analytical model of queueing systems with inaccurate arrival and service parameters is created using triangular fuzzy numbers. Four canonical models are examined; M/M/1, M/M/c, M/M/1/K, and M/G/1. The classical steady states relations are initially in closed forms, followed by their generalization to fuzzy performance measures by alpha-cut propagation. Utilization, queue length, system size, waiting times, probability of delay, probability of blocking and effective arrival rate are examined. The paper obtains powerful fuzzy stability theorems, alpha-stability levels, the sensitivity laws and the comparative theorems using monotonicity arguments and corner evaluation. It was found out that M/M/1 is most sensitive around the saturation, M/M/c becomes strong in the case of pooling, M/M/1/K reduces internal congestion by introducing blocking, and M/G/1 further increases uncertainty by varying the service-time. Analytical findings are supported by numerical illustrations at various levels of uncertainty.**

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## I. INTRODUCTION

The queueing models are still used as a basic tool to study service systems in telecommunications, call centres, shared services, manufacturing interfaces, healthcare operations, and in capacity planning [1]-[8], [22]-[26]. They are useful because they have compact analytical connections between demand intensity, service capacity, congestion and delay. These models transform a small parameter vector to a rich set of performance measures in their crisp form. In single server exponential queue, the system is characterized by the land service rate m and arrival rate. The number of servers in the queue focuses in the multi-server queue. The K be truncation level of the state-space is structurally significant in the finite-capacity queue. The first and second moments of the service time determine waiting and congestion in the single-server general-service queue [3]- [8], [24], [25].

This is also the limitation in their operation as these classical queueing formulas are analytical. These formulas are typically appraised at point estimates, despite actual arrival and service data being typically interval-based as opposed to exact. The error of aggregation, the misspecification of a forecast, the effect of campaigns, variation by the time of the day, and demand bursts influence arrival rates [1]-[3], [22], [23]. The heterogeneity of the operators, the interruptions in the tasks, learning, fatigue, and job complexity change depending on the service rates. When these numbers are squeezed into single numbers the resultant queueing outputs can be seen to be more accurate than the data available warrants. This is particularly devastating when close to the stability threshold where the queueing characteristics are non-linear with the left-over capacity slack and minor changes in parameter might cause significant alterations in waiting time and queue length [4], [24], [29], [30].

This type of parameter uncertainty can be described using fuzzy set theory in a straight forward and mathematically consistent manner [9]-[12]. A formulation of fuzzy queueing is that which preserves the stochastic queueing law, except that an input parameter is considered fuzzy, as opposed to being a point. It is no longer therefore a single value, but an alpha-indexed family of intervals. This technique has been formulated of basic possibility-based queues, finite capacity systems, bulk-service queues, unreliable-server queues, retrial models, and multi-channel fuzzy queues [13]- [19]. It has the merit of preserving the analytical benefit that standard formulae of queueing may still be applied, but they must be properly propagated by alpha-cut calculus and interval monotonicity.

This paper uses triangular fuzzy numbers due to the fact that they are easy, understandable and closed to the linear transformations required in most queueing computations. Suppose the fuzzy arrival and service rates are:

$$\tilde{\lambda}=(\lambda_1,\lambda_2,\lambda_3)\;\tilde{\mu}=(\mu_1,\mu_2,\mu_3)$$

with alpha-cuts

$$\lambda(\alpha)=[\lambda_L(\alpha),\lambda_U(\alpha)]\;\mu(\alpha)=[\mu_L(\alpha),\mu_U(\alpha)]\lambda_L(\alpha)=\lambda_1+\alpha(\lambda_2-\lambda_1)\;\lambda_U(\alpha)=\lambda_3-\alpha(\lambda_3-\lambda_2)$$

$$\mu_L(\alpha)=\mu_1+\alpha(\mu_2-\mu_1)\;\mu_U(\alpha)=\mu_3-\alpha(\mu_3-\mu_2)$$

For $0\leq\alpha\leq1$. Deterministic parameter rectangle is defined by each alpha-level. A crisp measure of queueing is then processed throughout the rectangle and the lowest and highest figures outline the fuzzy lower and upper limits. In case monotonicity is provided, it becomes corner evaluation [10], [13]-[19].

The mathematical objective of the current paper is not merely to rephrase classical relations on queueing with fuzzy inputs, but also to compare the ways of propagation of uncertainty through particular queueing structures. The comparison is structural. In M/M/1, m-l is the most dominant denominator. In M/M/c, pooling modifications slacking to cm-l with addition of Erlang-C delay likelihood. In M/M/1/K, there is limited capacity around internal congestion, yet blocking is observed. M/G/1 maintains the same single-server slack, but service variability adds the second moment $E[S^2]$, which increases delay. These differences define the behaviour of fuzzy spreads around the stability boundary.

The research value of the paper is fourfold. First, it provides a single alpha-cut model to four canonical queueing design. Second, it obtains precise fuzzy equations in monotone instances and compact alpha-cut min-max equations in instances that are more cumbersome. Third, it derives strong fuzzy stability tests, alpha-stability limits and derivatives sensitive laws. Fourth, it demonstrates relative findings that pooling generates less sensitivity to uncertainty, finite capacity generates internal congestion by transferring it to blocking, and variability of service-time scales up fuzzy waiting compared to the exponential standard [4], [13]-[19], [24]-[29], [30].

The paper is structured in the following way. The basic fuzzy calculational outcomes are in section 2. The fuzzy M/M/1 queue is developed in section 3. Section 4 introduces the fuzzy M/M/c queue. Section 5 provides the fuzzy M/M/1/K model. The fuzzy M/G/1 model is provided in section 6. Theoretical comparisons between configurations are in section 7. Section 8 entails calculations in numbers. Part 9 and 10 give a brief discussion and conclusion.

## II. Laws, Theorems, and Core Fuzzy Calculational Results

Let a triangular fuzzy number be written as

$$\tilde{x}=(x_1,x_2,x_3),\,x_1\leq x_2\leq x_3$$

Its alpha-cut is

$$x(\alpha)=[x_L(\alpha),x_U(\alpha)]$$

with

$$x_L(\alpha)=x_1+\alpha(x_2-x_1)\;x_U(\alpha)=x_3-\alpha(x_3-x_2)$$

For positive fuzzy numbers, the basic reciprocal and quotient laws are

$$\left(\frac{1}{\tilde{x}}\right)(\alpha)=\left[\frac{1}{x_U(\alpha)},\frac{1}{x_L(\alpha)}\right]\left(\frac{\tilde{x}}{\tilde{z}}\right)(\alpha)=\left[\frac{x_L(\alpha)}{z_U(\alpha)},\frac{x_U(\alpha)}{z_L(\alpha)}\right]$$

These relations are sufficient for the rational queueing formulas used later [10]–[12].
For the fuzzy rates,

$$\tilde{\lambda}=(\lambda_1,\lambda_2,\lambda_3)\;\tilde{\mu}=(\mu_1,\mu_2,\mu_3)$$

the alpha-cut domain is

$$R(\alpha)=[\lambda_L(\alpha),\lambda_U(\alpha)]\times[\mu_L(\alpha),\mu_U(\alpha)]$$

Let a queueing measure be written as

$$y=f(\lambda,\mu)$$

Then the fuzzy output is

$$y(\alpha)=[y_L(\alpha),y_U(\alpha)]$$

with

$$y_L(\alpha)=\min\{f(\lambda,\mu):(\lambda,\mu)\in R(\alpha)\cap D\}\ y_U(\alpha)=\max\{f(\lambda,\mu):(\lambda,\mu)\in R(\alpha)\cap D\}$$

where $D$ is the admissible stable region. If $f$ is increasing in $\lambda$ and decreasing in $\mu$, then

$$y_L(\alpha)=f(\lambda_L(\alpha),\mu_U(\alpha))\ y_U(\alpha)=f(\lambda_U(\alpha),\mu_L(\alpha))$$

If the monotonic directions reverse, the alpha-cut corners reverse accordingly [13]–[19].
The M/M/1 quantities used later are

$$\rho=\frac{\lambda}{\mu}\ L=\frac{\lambda}{\mu-\lambda}\ L_q=\frac{\lambda^2}{\mu(\mu-\lambda)}\ W=\frac{1}{\mu-\lambda}\ W_q=\frac{\lambda}{\mu(\mu-\lambda)}$$

Their partial derivatives are

$$\frac{\partial\rho}{\partial\lambda}=\frac{1}{\mu}>0\ \frac{\partial\rho}{\partial\mu}=-\frac{\lambda}{\mu^2}<0\ \frac{\partial L}{\partial\lambda}=\frac{\mu}{(\mu-\lambda)^2}>0\ \frac{\partial L}{\partial\mu}=-\frac{\lambda}{(\mu-\lambda)^2}<0\ \frac{\partial W}{\partial\lambda}=\frac{1}{(\mu-\lambda)^2}>0\ \frac{\partial W}{\partial\mu}=-\frac{1}{(\mu-\lambda)^2}<0$$

$$\frac{\partial L_q}{\partial\lambda}=\frac{\lambda(2\mu-\lambda)}{\mu(\mu-\lambda)^2}>0\ \frac{\partial L_q}{\partial\mu}=-\frac{\lambda^2(2\mu-\lambda)}{\mu^2(\mu-\lambda)^2}<0\ \frac{\partial W_q}{\partial\lambda}=\frac{1}{(\mu-\lambda)^2}>0\ \frac{\partial W_q}{\partial\mu}=-\frac{\lambda(2\mu-\lambda)}{\mu^2(\mu-\lambda)^2}<0$$

Thus, all principal M/M/1 congestion measures are increasing in $\lambda$ and decreasing in $\mu$. This derivative pattern is used repeatedly.
For M/G/1 with

$$m_1=E[S]\ m_2=E[S^2]$$

the Pollaczek-Khinchine relation is

$$W_q=\frac{\lambda m_2}{2(1-\lambda m_1)}$$

with derivatives

$$\frac{\partial W_q}{\partial\lambda}=\frac{m_2}{2(1-\lambda m_1)^2}>0\ \frac{\partial W_q}{\partial m_1}=\frac{\lambda^2 m_2}{2(1-\lambda m_1)^2}>0\ \frac{\partial W_q}{\partial m_2}=\frac{\lambda}{2(1-\lambda m_1)}>0$$

Hence M/G/1 delay increases with arrival intensity, mean service time, and second service moment.
The robust fuzzy stability conditions used in this paper are

$$\lambda_U(\alpha)<\mu_L(\alpha)$$

for M/M/1,

$$\lambda_U(\alpha)<c\mu_L(\alpha)$$

for M/M/c, and

$$\lambda_U(\alpha)E[S]_U(\alpha)<1$$

for M/G/1. Under triangular fuzzy rates, the support-endpoint tests become

$$\lambda_3<\mu_1$$

for M/M/1,

$$\lambda_3<c\mu_1$$

for M/M/c.
The alpha-stability threshold in M/M/1 is defined by

$$\lambda_U(\alpha^*)=\mu_L(\alpha^*)$$

which yields

$$\alpha^*=\frac{\lambda_3-\mu_1}{(\lambda_3-\lambda_2)+(\mu_2-\mu_1)}$$

For M/M/c the threshold is

$$\alpha_c^*=\frac{\lambda_3-c\mu_1}{(\lambda_3-\lambda_2)+c(\mu_2-\mu_1)}$$

These thresholds identify the alpha-level below which peripheral uncertainty intersects the instability boundary.
For uncertainty amplification, let

$$g(\alpha)=\mu(\alpha)-\lambda(\alpha)$$

with

$$g_L(\alpha)=\mu_L(\alpha)-\lambda_U(\alpha)\,g_U(\alpha)=\mu_U(\alpha)-\lambda_L(\alpha)$$

Then

$$\left(\frac{1}{\mu-\lambda}\right)(\alpha)=\left[\frac{1}{g_U(\alpha)},\frac{1}{g_L(\alpha)}\right]$$

and the interval width is

$$wid\left(\frac{1}{\mu-\lambda}\right)(\alpha)=\frac{g_U(\alpha)-g_L(\alpha)}{g_L(\alpha)\,g_U(\alpha)}$$

As $g_L(\alpha)$ approaches zero from the positive side, the width diverges. This is the fuzzy form of heavy-traffic amplification [4], [24], [29], [30].

A first-order sensitivity law is also useful. If $y=f(\theta_1,\ldots,\theta_k)$ is differentiable at the modal point, and the alpha-cut half-width of parameter $\theta_j$ is

$$h_j(\alpha)=\frac{\theta_{j,U}(\alpha)-\theta_{j,L}(\alpha)}{2}$$

then the output half-width satisfies the local approximation

$$h_y(\alpha)\approx\sum_{j=1}^{k}/\ \frac{\partial f}{\partial\theta_j}/\ h_j(\alpha)$$

This identifies the main drivers of fuzzy dispersion in the later queueing models.

## III. FUZZY M/M/1 QUEUE: FORMULAE, DERIVATIONS, AND PROOFS

### *A. Model assumptions*

The M/M/1 queue has Poisson arrivals of rate $\lambda$, exponential service of rate $\mu$, one server, FCFS discipline, and infinite waiting room. The stable region is

$$\lambda<\mu$$

and the utilization is

$$\rho=\frac{\lambda}{\mu}$$

For fuzzy inputs, each alpha-level is stationary if

$$\lambda_U(\alpha)<\mu_L(\alpha)$$

Robust fuzzy stability requires this inequality for every $\alpha$.

### *B. Classical steady-state structure*

Let $p_n=P(N=n)$. The balance equations are

$$\lambda p_n=\mu p_{n+1},n\geq 0$$

Hence

$$p_{n+1}=\rho p_n\,p_n=\rho^n p_0$$

Normalization gives

$$\sum_{n=0}^{\infty}p_n=1$$

so

$$p_0=1-\rho$$

and therefore

$$p_n=(1-\rho)\rho^n$$

The mean number in system is

$$L=\sum_{n=0}^{\infty}n\,p_n$$

which yields

$$L=\frac{\rho}{1-\rho}$$

hence

$$L=\frac{\lambda}{\mu-\lambda}$$

The mean queue length is

$$L_q=\frac{\rho^2}{1-\rho}$$

thus

$$L_q=\frac{\lambda^2}{\mu(\mu-\lambda)}$$

By Little's law,

$$L=\lambda W L_q=\lambda W_q$$

therefore

$$W=\frac{1}{\mu-\lambda} W_q=\frac{\lambda}{\mu(\mu-\lambda)}$$

[5]–[8].

### *C. Fuzzy utilization*

Since $\rho$ is increasing in $\lambda$ and decreasing in $\mu$, the alpha-cut bounds are

$$\rho(\alpha)=[\rho_L(\alpha),\rho_U(\alpha)]\rho_L(\alpha)=\frac{\lambda_L(\alpha)}{\mu_U(\alpha)}\ \rho_U(\alpha)=\frac{\lambda_U(\alpha)}{\mu_L(\alpha)}$$

The stable alpha-cut condition may also be written as

$$\rho_U(\alpha)<1$$

### *D. Fuzzy system size*

The fuzzy system-size interval is

$$L(\alpha)=[L_L(\alpha),L_U(\alpha)]$$

with

$$L_L(\alpha)=\frac{\lambda_L(\alpha)}{\mu_U(\alpha)-\lambda_L(\alpha)}\ L_U(\alpha)=\frac{\lambda_U(\alpha)}{\mu_L(\alpha)-\lambda_U(\alpha)}$$

The lower bound uses the smallest arrival rate and largest service rate. The upper bound uses the largest arrival rate and smallest service rate.

### *E. Fuzzy queue size*

The fuzzy queue-length interval is

$$L_q(\alpha)=[L_{q,L}(\alpha),L_{q,U}(\alpha)]$$

with

$$L_{q,L}(\alpha)=\frac{\lambda_L(\alpha)^2}{\mu_U(\alpha)[\mu_U(\alpha)-\lambda_L(\alpha)]}\ L_{q,U}(\alpha)=\frac{\lambda_U(\alpha)^2}{\mu_L(\alpha)[\mu_L(\alpha)-\lambda_U(\alpha)]}$$

The same monotonicity logic applies because $L_q$ rises with $\lambda$ and falls with $\mu$.

### *F. Fuzzy waiting time in system*

The fuzzy system-time interval is

$$W(\alpha)=[W_L(\alpha),W_U(\alpha)]$$

with

$$W_L(\alpha)=\frac{1}{\mu_U(\alpha)-\lambda_L(\alpha)} W_U(\alpha)=\frac{1}{\mu_L(\alpha)-\lambda_U(\alpha)}$$

The dominant denominator is the alpha-level residual capacity.

### *G. Fuzzy queue waiting time*

The fuzzy queue-waiting interval is

$$W_q(\alpha)=[W_{q,L}(\alpha),W_{q,U}(\alpha)]$$

with

$$W_{q,L}(\alpha)=\frac{\lambda_L(\alpha)}{\mu_U(\alpha)[\mu_U(\alpha)-\lambda_L(\alpha)]} W_{q,U}(\alpha)=\frac{\lambda_U(\alpha)}{\mu_L(\alpha)[\mu_L(\alpha)-\lambda_U(\alpha)]}$$

### *H. Derivatives with respect to* $\lambda$

The first derivatives are

$$\frac{\partial \rho}{\partial \lambda}=\frac{1}{\mu}\quad \frac{\partial L}{\partial \lambda}=\frac{\mu}{(\mu-\lambda)^2}\quad \frac{\partial L_q}{\partial \lambda}=\frac{\lambda(2\mu-\lambda)}{\mu(\mu-\lambda)^2}\quad \frac{\partial W}{\partial \lambda}=\frac{1}{(\mu-\lambda)^2}\quad \frac{\partial W_q}{\partial \lambda}=\frac{1}{(\mu-\lambda)^2}$$

All are positive on the stable region.

### *I. Derivatives with respect to* $\mu$

The first derivatives are

$$\frac{\partial \rho}{\partial \mu}=-\frac{\lambda}{\mu^2}\quad \frac{\partial L}{\partial \mu}=-\frac{\lambda}{(\mu-\lambda)^2}\quad \frac{\partial L_q}{\partial \mu}=-\frac{\lambda^2(2\mu-\lambda)}{\mu^2(\mu-\lambda)^2}\quad \frac{\partial W}{\partial \mu}=-\frac{1}{(\mu-\lambda)^2}\quad \frac{\partial W_q}{\partial \mu}=-\frac{\lambda(2\mu-\lambda)}{\mu^2(\mu-\lambda)^2}$$

All are negative on the stable region.

### *J. Monotonicity theorem*

Monotonicity theorem. On the stable region lambda < mu, the measures rho, L, Lq, W, and Wq are increasing in lambda and decreasing in mu.

Proof.
For utilization,
rho = lambda / mu
d rho / d lambda = 1 / mu > 0
d rho / d mu = - lambda / mu^2 < 0
For mean number in system,
L = lambda / (mu - lambda)
d L / d lambda = mu / (mu - lambda)$^2$ > 0
d L / d mu = - lambda / (mu - lambda)$^2$ < 0
For mean number in queue,
Lq = lambda$^2$ / [mu(mu - lambda)]
d Lq / d lambda = lambda(2mu - lambda) / [mu(mu - lambda)$^2$] > 0
d Lq / d mu = - lambda$^2$(2mu - lambda) / [mu$^2$(mu - lambda)$^2$] < 0
For mean time in system,
W = 1 / (mu - lambda)
d W / d lambda = 1 / (mu - lambda)$^2$ > 0
d W / d mu = - 1 / (mu - lambda)$^2$ < 0
For mean waiting time in queue,
Wq = lambda / [mu(mu - lambda)]
d Wq / d lambda = 1 / (mu - lambda)$^2$ > 0
d Wq / d mu = - lambda(2mu - lambda) / [mu$^2$(mu - lambda)$^2$] < 0
Hence all five measures increase with lambda and decrease with mu on the stable region. Therefore, for every alpha,
lower bound is obtained at (lambdaL(alpha), muU(alpha))
upper bound is obtained at (lambdaU(alpha), muL(alpha))

### *K. Instability-boundary theorem*

Instability-boundary theorem. If lambdaU(alpha) approaches muL(alpha) from below, then rhoU(alpha) approaches 1, and LU(alpha), LqU(alpha), WU(alpha), and WqU(alpha) all go to infinity.

Proof.
The upper alpha-cut bounds are
rhoU(alpha) = lambdaU(alpha) / muL(alpha)
LU(alpha) = lambdaU(alpha) / [muL(alpha) - lambdaU(alpha)]
LqU(alpha) = lambdaU(alpha)^2 / {muL(alpha)[muL(alpha) - lambdaU(alpha)]}
WU(alpha) = 1 / [muL(alpha) - lambdaU(alpha)]
WqU(alpha) = lambdaU(alpha) / {muL(alpha)[muL(alpha) - lambdaU(alpha)]}
Now let
muL(alpha) - lambdaU(alpha) -> 0+
Then
rhoU(alpha) -> 1
LU(alpha) -> infinity
LqU(alpha) -> infinity
WU(alpha) -> infinity
WqU(alpha) -> infinity
So the upper fuzzy bounds diverge as the stability boundary is approached.

### L. *Calculational summary*

The fuzzy M/M/1 model is fully characterized by

$$\rho(\alpha)=\left[\frac{\lambda_L(\alpha)}{\mu_U(\alpha)},\frac{\lambda_U(\alpha)}{\mu_L(\alpha)}\right] L(\alpha)=\left[\frac{\lambda_L(\alpha)}{\mu_U(\alpha)-\lambda_L(\alpha)},\frac{\lambda_U(\alpha)}{\mu_L(\alpha)-\lambda_U(\alpha)}\right]$$

$$L_q(\alpha)=\left[\frac{\lambda_L(\alpha)^2}{\mu_U(\alpha)[\mu_U(\alpha)-\lambda_L(\alpha)]},\frac{\lambda_U(\alpha)^2}{\mu_L(\alpha)[\mu_L(\alpha)-\lambda_U(\alpha)]}\right] W(\alpha)=\left[\frac{1}{\mu_U(\alpha)-\lambda_L(\alpha)},\frac{1}{\mu_L(\alpha)-\lambda_U(\alpha)}\right]$$

$$W_q(\alpha)=\left[\frac{\lambda_L(\alpha)}{\mu_U(\alpha)[\mu_U(\alpha)-\lambda_L(\alpha)]},\frac{\lambda_U(\alpha)}{\mu_L(\alpha)[\mu_L(\alpha)-\lambda_U(\alpha)]}\right]$$

## IV. FUZZY M/M/C QUEUE: FORMULAE, DERIVATIONS, AND PROOFS

### A. *Model assumptions*

The M/M/c queue has Poisson arrivals, $c$exponential servers of rate $\mu$, FCFS discipline, and infinite waiting room. Define

$$a=\frac{\lambda}{\mu}\ \rho=\frac{\lambda}{c\mu}$$

The stable region is

$$\lambda<c\mu$$

For fuzzy inputs, the robust alpha-cut stability condition is

$$\lambda_U(\alpha)<c\mu_L(\alpha)$$

### B. *Classical multi-server formulation*

The steady-state birth-death equations are

$$\lambda p_n=(n+1)\mu p_{n+1},0\le n\le c-1\lambda p_n=c\mu p_{n+1},n\ge c$$

Hence

$$p_n=\frac{a^n}{n!}p_0,0\le n\le c-1\,p_n=\frac{a^c}{c!}\rho^{n-c}p_0,n\ge c$$

### C. *Derivation of* $P_0$

Normalization gives

$$P_0=\left[\sum_{n=0}^{c-1}\frac{a^n}{n!}+\frac{a^c}{c!(1-\rho)}\right]^{-1}$$

This is the empty-system probability.

### D. *Derivation of Erlang-C waiting probability*

The probability of waiting is the probability that all $c$servers are busy:

$$P(\text{wait})=\sum_{n=c}^{\infty}p_n$$

Hence

$$P(\text{wait})=\frac{a^c}{c!(1-\rho)}P_0$$

The queue length is

$$L_q=\frac{P_0a^c\rho}{c!(1-\rho)^2}$$

The waiting-time and system-size relations are

$$W_q=\frac{L_q}{\lambda}\ W=W_q+\frac{1}{\mu}\ L=L_q+a$$

### E. *Fuzzy utilization*

The fuzzy utilization interval is

$$\rho(\alpha)=\left[\frac{\lambda_L(\alpha)}{c\mu_U(\alpha)},\frac{\lambda_U(\alpha)}{c\mu_L(\alpha)}\right]$$

The offered-load interval is

$$a(\alpha)=\left[\frac{\lambda_L(\alpha)}{\mu_U(\alpha)},\frac{\lambda_U(\alpha)}{\mu_L(\alpha)}\right]$$

*F. Derivation of fuzzy* $P_0$

Since $P_0$decreases as congestion increases, its alpha-cut bounds are reversed relative to the congestion measures:

$$P_0(\alpha)=[P_{0,L}(\alpha),P_{0,U}(\alpha)]P_{0,L}(\alpha)=P_0(\lambda_U(\alpha),\mu_L(\alpha);c)\ P_{0,U}(\alpha)=P_0(\lambda_L(\alpha),\mu_U(\alpha);c)$$

Expanded directly,

$$P_{0,L}(\alpha)=\left[\sum_{n=0}^{c-1}\frac{a_U(\alpha)^n}{n!}+\frac{a_U(\alpha)^c}{c![1-\rho_U(\alpha)]}\right]^{-1}\ P_{0,U}(\alpha)=\left[\sum_{n=0}^{c-1}\frac{a_L(\alpha)^n}{n!}+\frac{a_L(\alpha)^c}{c![1-\rho_L(\alpha)]}\right]^{-1}$$

*G. Derivation of fuzzy waiting probability*

The fuzzy Erlang-C interval is

$$P(\text{wait})(\alpha)=[P(\text{wait})_L(\alpha),P(\text{wait})_U(\alpha)]$$

with

$$P(\text{wait})_L(\alpha)=P(\text{wait})(\lambda_L(\alpha),\mu_U(\alpha);c)P(\text{wait})_U(\alpha)=P(\text{wait})(\lambda_U(\alpha),\mu_L(\alpha);c)$$

The direct closed forms are

$$P(\text{wait})_L(\alpha)=\frac{a_L(\alpha)^c}{c![1-\rho_L(\alpha)]}P_{0,U}(\alpha)P(\text{wait})_U(\alpha)=\frac{a_U(\alpha)^c}{c![1-\rho_U(\alpha)]}P_{0,L}(\alpha)$$

*H. Derivation of fuzzy queue length*

The fuzzy queue-length interval is

$$L_q(\alpha)=[L_{q,L}(\alpha),L_{q,U}(\alpha)]$$

with

$$L_{q,L}(\alpha)=\frac{P_{0,U}(\alpha)a_L(\alpha)^c\rho_L(\alpha)}{c![1-\rho_L(\alpha)]^2}\ L_{q,U}(\alpha)=\frac{P_{0,L}(\alpha)a_U(\alpha)^c\rho_U(\alpha)}{c![1-\rho_U(\alpha)]^2}$$

*I. Derivation of fuzzy waiting time*

The queue-waiting interval is

$$W_q(\alpha)=[W_{q,L}(\alpha),W_{q,U}(\alpha)]$$

with

$$W_{q,L}(\alpha)=\frac{L_{q,L}(\alpha)}{\lambda_L(\alpha)}W_{q,U}(\alpha)=\frac{L_{q,U}(\alpha)}{\lambda_U(\alpha)}$$

Using the slack form,

$$W_q=\frac{P(\text{wait})}{c\mu-\lambda}$$

the same bounds are

$$W_{q,L}(\alpha)=\frac{P(\text{wait})_L(\alpha)}{c\mu_U(\alpha)-\lambda_L(\alpha)}W_{q,U}(\alpha)=\frac{P(\text{wait})_U(\alpha)}{c\mu_L(\alpha)-\lambda_U(\alpha)}$$

*J. Derivation of fuzzy system size*

The time in system is

$$W(\alpha)=[W_L(\alpha),W_U(\alpha)]$$

with

$$W_L(\alpha)=W_{q,L}(\alpha)+\frac{1}{\mu_U(\alpha)}W_U(\alpha)=W_{q,U}(\alpha)+\frac{1}{\mu_L(\alpha)}$$

The system size is

$$L(\alpha)=[L_L(\alpha),L_U(\alpha)]$$

with

$$L_L(\alpha)=L_{q,L}(\alpha)+a_L(\alpha)L_U(\alpha)=L_{q,U}(\alpha)+a_U(\alpha)$$

*K. Derivatives with respect to* $\lambda$

The basic derivatives are

$$\frac{\partial a}{\partial \lambda}=\frac{1}{\mu}\frac{\partial \rho}{\partial \lambda}=\frac{1}{c\mu}$$

Let

$$G(a,c)=\sum_{n=0}^{c-1}\frac{a^n}{n!}+\frac{a^c}{c!(1-a/c)}$$

Then

$$P_0=\frac{1}{G(a,c)}$$

and

$$\frac{\partial P_0}{\partial \lambda}=-\frac{G_a(a,c)}{\mu G(a,c)^2}<0$$

Since $P(\text{wait})$, $L_q$, $W_q$, and $L$ increase with $a$, they also increase with $\lambda$on the stable region.

### L. *Derivatives with respect to $\mu$*

The derivatives are

$$\frac{\partial a}{\partial \mu}=-\frac{\lambda}{\mu^2}\frac{\partial \rho}{\partial \mu}=-\frac{\lambda}{c\mu^2}$$

and

$$\frac{\partial P_0}{\partial \mu}>0\frac{\partial P(\text{wait})}{\partial \mu}<0\frac{\partial L_q}{\partial \mu}<0\frac{\partial W_q}{\partial \mu}<0\frac{\partial L}{\partial \mu}<0$$

Thus faster service reduces all delay and queueing measures.

### M. *Derivative effect of server count $c$*

Under continuous relaxation,

$$\frac{\partial \rho}{\partial c}=-\frac{\lambda}{c^2\mu}<0$$

For integer $c$, forward differences show

$$P(\text{wait})(c+1)<P(\text{wait})(c)\,L_q(c+1)<L_q(c)\,W_q(c+1)<W_q(c)$$

Pooling reduces both mean delay and sensitivity.

### N. *Stability theorem for fuzzy M/M/c*

The fuzzy M/M/c queue is robustly stable when $\lambda U(\alpha)<c\mu L(\alpha)$,for all $\alpha\in[0,1]$

At the instability boundary,

$$\lambda_U(\alpha)\uparrow c\mu_L(\alpha)$$

one gets

$$\rho_U(\alpha)\uparrow 1$$

and the upper bounds of $P(\text{wait})$, $L_q$, and $W_q$ diverge because their formulas contain factors involving $1-\rho$, or the same slack written as $c\mu-\lambda$ .

## V. Fuzzy M/M/1/K Queue: Formulae, Derivations, and Proofs

### A. *Model assumptions*

The M/M/1/K queue has Poisson arrivals, exponential service, one server, FCFS discipline, and total capacity $K$ . The state space is finite:

$$N\in\{0,1,\ldots,K\}$$

Define

$$\rho=\frac{\lambda}{\mu}$$

### B. *Classical state probabilities*

The steady-state probabilities are

$$p_n=p_0\rho^n,0\leq n\leq K$$

Normalization gives

$$p_0=\frac{1-\rho}{1-\rho^{K+1}},\rho\neq 1$$

Thus

$$p_n = \frac{(1-\rho)\rho^n}{1-\rho^{K+1}}$$

The blocking probability is

$$p_K = \frac{(1-\rho)\rho^K}{1-\rho^{K+1}}$$

*C. Effective arrival rate*

Blocked arrivals are lost, so the effective throughput is

$$\lambda_e = \lambda(1-p_K)$$

This quantity replaces $\lambda$ in the delay formulas.

*D. Mean system size and queue size*

The mean number in system is

$$L = \frac{\rho\left[1-(K+1)\rho^K + K\rho^{K+1}\right]}{(1-\rho)\left(1-\rho^{K+1}\right)}$$

The mean number in queue is

$$L_q = L-(1-p_0)$$

The waiting measures are

$$W = \frac{L}{\lambda_e}\, W_q = \frac{L_q}{\lambda_e}$$

*E. Fuzzy blocking and throughput*

The fuzzy utilization interval is

$$\rho(\alpha) = \left[\frac{\lambda_L(\alpha)}{\mu_U(\alpha)}, \frac{\lambda_U(\alpha)}{\mu_L(\alpha)}\right]$$

The blocking interval is

$$p_K(\alpha) = [p_{K,L}(\alpha), p_{K,U}(\alpha)]$$

with

$$p_{K,L}(\alpha) = p_K(\lambda_L(\alpha), \mu_U(\alpha), K)\ p_{K,U}(\alpha) = p_K(\lambda_U(\alpha), \mu_L(\alpha), K)$$

Thus the throughput interval is

$$\lambda_e(\alpha) = [\lambda_{e,L}(\alpha), \lambda_{e,U}(\alpha)]$$

with

$$\lambda_{e,L}(\alpha) = \lambda_L(\alpha)[1-p_{K,U}(\alpha)]\lambda_{e,U}(\alpha) = \lambda_U(\alpha)[1-p_{K,L}(\alpha)]$$

*F. Fuzzy system size, queue size, and waiting times*

The remaining measures are

$$L(\alpha) = \left[L(\lambda_L(\alpha), \mu_U(\alpha), K), L(\lambda_U(\alpha), \mu_L(\alpha), K)\right]$$

$$L_q(\alpha) = \left[L_q(\lambda_L(\alpha), \mu_U(\alpha), K), L_q(\lambda_U(\alpha), \mu_L(\alpha), K)\right]$$

$$W(\alpha) = \left[W(\lambda_L(\alpha), \mu_U(\alpha), K), W(\lambda_U(\alpha), \mu_L(\alpha), K)\right]$$

$$W_q(\alpha) = \left[W_q(\lambda_L(\alpha), \mu_U(\alpha), K), W_q(\lambda_U(\alpha), \mu_L(\alpha), K)\right]$$

*G. Finite-capacity bounding theorem*

The state space contains only $K+1$ states, the internal queueing measures are bounded by the capacity level. Specifically,

$$0 \le L \le K\ 0 \le L_q \le K-1$$

Thus, all fuzzy internal bounds remain finite for every admissible $\lambda$ and $\mu$.

*H. Delay-blocking trade-off theorem*

Decreasing $K$ reduces internal delay and queue size but increases blocking. Increasing $K$ reduces blocking but raises internal congestion.

Finite capacity therefore transforms part of congestion into loss rather than allowing unlimited internal accumulation [15], [27], [28].

## VI. FUZZY M/G/1 QUEUE: FORMULAE, DERIVATIONS, AND PROOFS

### *A. Model assumptions*

The M/G/1 queue has Poisson arrivals, one server, FCFS discipline, and general service-time distribution. Let

$$m_1 = E[S] \quad m_2 = E[S^2]$$

The stable region is

$$\rho = \lambda m_1 < 1$$

### *B. Classical Pollaczek-Khinchine formulas*

The queue-waiting relation is

$$W_q = \frac{\lambda m_2}{2(1-\lambda m_1)}$$

The remaining measures are

$$W = W_q + m_1 \quad L_q = \lambda W_q \quad L = \lambda W$$

### *C. Mean-rate and variability form*

Let the mean service rate be

$$\mu = \frac{1}{m_1}$$

and let

$$c_s^2 = \frac{Var(S)}{m_1^2}$$

Then

$$m_2 = \frac{1+c_s^2}{\mu^2}$$

Substituting into the waiting formula gives

$$W_q = \frac{\lambda(1+c_s^2)}{2\mu(\mu-\lambda)} \quad W = \frac{\lambda(1+c_s^2)}{2\mu(\mu-\lambda)} + \frac{1}{\mu} \quad L_q = \frac{\lambda^2(1+c_s^2)}{2\mu(\mu-\lambda)} \quad L = \lambda W$$

### *D. Fuzzy waiting and queueing measures*

For fixed $c_s^2$, the fuzzy queue-waiting interval is

$$W_q(\alpha) = [W_{q,L}(\alpha), W_{q,U}(\alpha)]$$

with

$$W_{q,L}(\alpha) = \frac{\lambda_L(\alpha)(1+c_s^2)}{2\mu_U(\alpha)[\mu_U(\alpha)-\lambda_L(\alpha)]} \quad W_{q,U}(\alpha) = \frac{\lambda_U(\alpha)(1+c_s^2)}{2\mu_L(\alpha)[\mu_L(\alpha)-\lambda_U(\alpha)]}$$

The system-time interval is

$$W(\alpha) = [W_L(\alpha), W_U(\alpha)]$$

with

$$W_L(\alpha) = W_{q,L}(\alpha) + \frac{1}{\mu_U(\alpha)} \quad W_U(\alpha) = W_{q,U}(\alpha) + \frac{1}{\mu_L(\alpha)}$$

The size intervals are

$$L_q(\alpha) = [L_{q,L}(\alpha), L_{q,U}(\alpha)] \quad L_{q,L}(\alpha) = \lambda_L(\alpha) W_{q,L}(\alpha) \quad L_{q,U}(\alpha) = \lambda_U(\alpha) W_{q,U}(\alpha) \quad L(\alpha) = [L_L(\alpha), L_U(\alpha)]$$

$$L_L(\alpha) = \lambda_L(\alpha) W_L(\alpha) \quad L_U(\alpha) = \lambda_U(\alpha) W_U(\alpha)$$

### *E. Derivatives and variability effect*

The derivatives of $W_q$ are

$$\frac{\partial W_q}{\partial \lambda} = \frac{1+c_s^2}{2(\mu-\lambda)^2} > 0 \quad \frac{\partial W_q}{\partial \mu} = -\frac{\lambda(1+c_s^2)(2\mu-\lambda)}{2\mu^2(\mu-\lambda)^2} < 0 \quad \frac{\partial W_q}{\partial c_s^2} = \frac{\lambda}{2\mu(\mu-\lambda)} > 0$$

Thus, delay rises with demand intensity, falls with service speed, and rises linearly with service-time variability.

### *F. Variability amplification theorem*

Relative to M/M/1, M/G/1 multiplies the waiting component by $(1+c_s^2)/2$.

Indeed,

$$W_q^{M/G/1}=\frac{1+c_s^2}{2}W_q^{M/M/1}$$

When $c_s^2>1$, the waiting interval is magnified beyond the exponential benchmark. When $c_s^2<1$, the waiting interval contracts.

## VII. COMPARATIVE THEOREMS ACROSS QUEUEING CONFIGURATIONS

### *A. Comparative amplification theorem*

Near saturation, the uncertainty amplification of waiting-time measures is strongest in M/M/1, weaker in M/M/$c$ because of pooling, bounded internally in M/M/1/$K$ because of finite capacity, and magnified in M/G/1 when $c_s^2>1$.

The underlying structural factors are

$$\mu-\lambda$$

for M/M/1,

$$c\mu-\lambda$$

for M/M/c,

finite state-space truncation for M/M/1/K, and

$$\frac{1+c_s^2}{2}$$

times the single-server denominator in M/G/1.

### *B. Pooling dominance theorem*

For fixed $\lambda$and $\mu$, pooling capacity across $c$servers lowers both delay and fuzzy delay spread relative to the single-server system.

This follows because the pooled slack $c\mu-\lambda$ is larger and the Erlang-C delay probability decreases with $c$[4], [22], [24].

### *C. Finite-capacity transformation theorem*

Finite capacity limits internal congestion but generates blocking and reduces effective throughput.

Thus, internal robustness is obtained by truncation rather than by removing demand pressure.

### *D. Variability comparison theorem*

For fixed $\lambda$and $\mu$, M/G/1 exceeds M/M/1 in waiting time and queue length whenever $c_s^2>1$.

This follows from the multiplier $(1+c_s^2)/2$.

## VIII. NUMERICAL CALCULATION SECTION

Take the baseline parameters

$$\lambda=8,\mu=10,c=2,K=10,c_s^2=4$$

For M/M/1,

$$\rho=\frac{8}{10}=0.8\;L=\frac{8}{10-8}=4\;L_q=\frac{8^2}{10(10-8)}=\frac{64}{20}=3.2\;W=\frac{1}{10-8}=0.5\;W_q=\frac{8}{10(10-8)}=0.4$$

For M/M/c with $c=2$,

$$a=\frac{8}{10}=0.8\;\rho=\frac{8}{20}=0.4\;P_0=\left[1+0.8+\frac{0.8^2}{2(1-0.4)}\right]^{-1}=0.42857\;P(\text{wait})=\frac{0.8^2}{2(1-0.4)}P_0=0.22857$$

$$L_q=\frac{P_0a^c\rho}{c!(1-\rho)^2}=0.15238\;W_q=\frac{0.15238}{8}=0.01905\;W=0.01905+0.1=0.11905\;L=0.15238+0.8=0.95238$$

For M/G/1 with $c_s^2=4$,

$$W_q=\frac{8(1+4)}{2\cdot10\cdot(10-8)}=1\;W=1+0.1=1.1\;L_q=8\;L=8.8$$

Now consider the low-uncertainty case

$$\tilde{\lambda}=(7.6,8.0,8.4)\;\tilde{\mu}=(9.6,10.0,10.4)$$

At $\alpha=0$, the M/M/1 waiting-time interval is

$$W(0)=\left[\frac{1}{10.4-7.6},\frac{1}{9.6-8.4}\right]=[0.3571,0.8333]$$

The system-size interval is

$$L(0)=\left[\frac{7.6}{10.4-7.6},\frac{8.4}{9.6-8.4}\right]=[2.7143,7.0000]$$

For moderate uncertainty,

$$\tilde{\lambda}=(7.2,8.0,8.8)\tilde{\mu}=(9.2,10.0,10.8)$$

one obtains

$$W(0)=\left[\frac{1}{10.8-7.2},\frac{1}{9.2-8.8}\right]=[0.2778,2.5]$$

The increase in the upper bound is caused by the reduced slack $9.2-8.8=0.4$.
For high uncertainty,

$$\tilde{\lambda}=(6.4,8.0,9.6)\tilde{\mu}=(8.0,10.0,12.0)$$

the support-level M/M/1 stability condition fails because

$$9.6>8.0$$

Hence robust stability does not hold. The alpha-threshold is

$$\alpha^*=\frac{9.6-8.0}{(9.6-8.0)+(10.0-8.0)}=\frac{1.6}{3.6}=0.4444$$

Thus only alpha-levels above $0.4444$ are stationary in this high-uncertainty M/M/1 case.

For the same high-uncertainty inputs, the M/M/c robust stability test with $c=2$ is

$$9.6<16.0$$

which holds. This confirms the pooling effect analytically.

For M/G/1 with $c_s^2=4$, the waiting interval is multiplied by $2.5$. Thus, in the low-uncertainty case,

$$W_q^{M/G/1}(0)=2.5\times W_q^{M/M/1}(0)$$

and therefore, the M/G/1 queue-waiting interval is much wider than the corresponding M/M/1 interval. This directly verifies the variability amplification theorem.

## IX. DISCUSSION

The principal pattern of analysis in the paper is preponderance of the residual capacity slack in the propagation of uncertainty. In cases where the denominator contains $\mu-\lambda$, $c\mu-\lambda$, or $1-\rho$, the upper fuzzy bound grows sharply as the system becomes saturated. Pooling has the advantage of swamping this effect by increasing the effective slack. Finite capacity eliminates internal divergence by constraining the state space, but transfers some amount of congestion to blocking. The second service moment is that of general service, and thus it escalates uncertainty whenever service-time variability is larger than exponential benchmark. It is these structural mechanisms which make uncertainty not spread homogeneously across queueing models.

## X. CONCLUSION

The present paper formulated formula-based fuzzy queueing models of M/M/1, M/M/c, M/M/1/K and M/G/1 systems in case of uncertainty in parameters. An interval-valued performance measure was to be developed using triangular fuzzy numbers and alpha-cut propagation to be used in the utilization, delay, queue length, system size, blocking, and effective throughput. The analysis developed precise fuzzy formula under monotone, strong stability, alpha -thresholds and derivative-driven sensitivity regulations and comparative theorems among queueing structures. These findings include: single-server infinite-capacity queues are most susceptible to amplification of uncertainty at saturation, multi-server pooling reduces mean congestion and fuzzy spread and service time uncertainty increases fuzzy delay beyond the exponential-service condition. The framework gives a concise mathematical foundation of performance assessment when service-system parameters are not known but are rather uncertain.

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