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Approximate solutions for certain Integrodifferential equations of Volterra_Fredholm type

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ABSTRACT

The main aim of this work is to investigate the approximate solutions for Volterra Fredholm integro_differential equations with given initial conditions. A certain integral inequality with explicit estimate is employed to obtain the results.

Keywords: approximate solutions, Integrodifferential equations, explicit estimate, closeness of solutions.

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1. INTRODUCTION

Let R^nbe areal n-dimensional Euclidean space with an appropriate norm denoted by $|.|$ and $R_+ = [0, \infty)$be the given subsets of R,the set of real numbers.

In this work we study the following initial value problem (IVP, for short)for the Volterra –Fredholm integrodifferential equations of the form

$$x'(t) = f(t, x(t), x'(t), Hx(t), Kx(t)) \quad (1.1)$$

let t∈ $R_+ = [0, \infty)$,with a given initial condition

$$x(0) = x_0, \quad (1.2)$$

Where

$$Hx(t) = \int_0^t h\left(t, \sigma, x(\sigma), x'(\sigma)\right) d\sigma \quad (1.3)$$

$$Kx(t) = \int_0^\beta k(t, \sigma, x(\sigma), x'(\sigma)) d\sigma, \quad and \quad (1.4)$$

$f \in C(R_+ \times R^n \times R^n \times R^n \times R^n, R^n)$, $h, k \in C(R_+^2 \times R^n \times R^n, R^n)$ known functions for $\sigma \le t$, x is the unknown function has to be found , and ´denotes the derivative.

The special case of IVP such as

$$x'(t) = f(t, x(t), Hx(t)) \qquad x(0) = x_0 \quad and$$

$$x'(t) = f\left(t, x(t), x'(t), Hx(t)\right) \qquad x(0) = x_0 \ ,$$

have been studied by many authors see [1,4,5,6,7] and references there in.

Our aim in this paper is the extension of the resultsof[3]where the approximate solutions of (1.1)_(1.2) and other properties are estimate to get the results.

2- MAIN RESULTS

Let $x_i'(t) \in C(R_+, R^n)(i = 1,2)$ be functions such that $x_i'(t)$ exist for $t \in R_+$ and satisfy the inequalities
$$\left|x_i'(t) - f(t, x_i(t), x_i'(t), Hx_i(t), Kx_i(t))\right| \leq \in_i \quad (2.1)$$
For given constants $\in_i \geq 0$, where it is assumed that the initial conditions
$$x_i(0) = x_i, \quad (2.2)$$
are fulfilled .then we call $x_i(t)$ the $\in_{i\_}$ approximate solutions with respect to IVP(1.1)_(1.2) .
We need the following integral inequality, we establish it by the same technique usedin[2].

Lemma .let $u, a, b \in c(R_+, R_+)$ and for $s \leq t; e(t,s), Q(t,s), k(t,s), \frac{\partial}{\partial t}e(t,s), \frac{\partial}{\partial t}Q(t,s) \in C(R_+^2, R_+)$ and *a(t)* is no decreasing for $t \in R_+$. If
$$u(t) \leq a(t) + \int_0^\beta Q(t,\sigma)u(\sigma)d\sigma + \int_0^t [\, b(s)u(s) + e(t,s)u(s) + \int_0^s k(s,\sigma)u(\sigma)d\sigma + \int_0^\beta Q(s,\sigma)u(t)ds]ds \quad (2.3)$$
for $t \in R_+$, then
$$u(t) \leq \frac{a(t)}{1-d} exp(\int_0^t [b(s) + A(s)]ds) \quad (2.4)$$
For $t \in R_+$,where
$$d = \int_0^\beta Q(0,\sigma)\exp(\int_0^\sigma [b(s) + A(s)]ds)d\sigma \quad (2.5)$$
$$A(t) = e(t,t) + \int_0^t [k(t,\sigma) + \frac{\partial}{\partial t}e(t,\sigma)] + \int_0^\beta [Q(t,\sigma) + \frac{\partial}{\partial t}Q(t,\sigma)]d\sigma \quad (2.6)$$

Proof: From(2.3), we have
$$\frac{u(t)}{a(t)} \leq 1 + \int_0^\beta Q(t,\sigma)\frac{u(\sigma)}{a(t)}d\sigma + \int_0^t [\, b(s)\frac{u(s)}{a(t)} + e(t,s)\frac{u(s)}{a(t)} + \int_0^s k(s,\sigma)\frac{u(\sigma)}{a(t)}d\sigma + \int_0^\beta Q(s,\sigma)\frac{u(\sigma)}{a(t)}d\sigma]ds\,,$$

and since $\frac{u(s)}{a(t)} \leq \frac{u(s)}{a(s)}$
$$\frac{u(t)}{a(t)} \leq 1 + \int_0^\beta Q(t,\sigma)\frac{u(\sigma)}{a(\sigma)}d\sigma + \int_0^t [\, b(s)\frac{u(s)}{a(s)} + e(t,s)\frac{u(s)}{a(s)} + \int_0^s k(s,\sigma)\frac{u(\sigma)}{a(\sigma)}d\sigma + \int_0^\beta Q(s,\sigma)\frac{u(\sigma)}{a(\sigma)}d\sigma]ds \quad (2.7)$$
Let $w(t)$ denote the right hand side of (2.7), then $w(0) \geq 0, \frac{u(t)}{a(t)} \leq w(t), w(t)$ is no decreasing in $t \in R_+$ and
$$w(t) \leq 1 + \int_0^\beta Q(t,\sigma)w(\sigma)d\sigma + \int_0^t [b(s)w(s) + e(t,s)w(s) + \int_0^s k(s,\sigma)w(\sigma)d\sigma + \int_0^\beta Q(s,\sigma)w(\sigma)d\sigma]ds,$$
and
$$w'(t) \leq \int_0^\beta \frac{\partial}{\partial t}Q(t,\sigma)w(\sigma)d\sigma + b(t)w(t) + e(t,t)w(t) + \int_0^t \frac{\partial}{\partial t}e(t,s)w(s)ds + \int_0^t k(t,\sigma)w(\sigma)d\sigma + \int_0^\beta Q(t,\sigma)w(\sigma)d\sigma$$
$$w'(t) \leq b(t)w(t) + e(t,t)w(t) + \int_0^t [k(t,\sigma) + \frac{\partial}{\partial t}e(t,\sigma)]w(\sigma)d\sigma + \int_0^\beta [Q(t,\sigma) + \frac{\partial}{\partial t}Q(t,\sigma)]w(\sigma)d\sigma$$
$$w'(t) \leq b(t)w(t) + A(t)w(t)$$
$$\leq [b(t) + A(t)]w(t) \quad (2.8)$$

Integrating(2.8)from 0 to t gives
$$w(t) \leq w(0)\, exp(\int_0^t [b(s) + A(s)]\, ds) \quad (2.9)$$
$$w(0) \leq 1 + \int_0^\beta Q(0,\sigma)w(\sigma)d\sigma$$
$$w(0)[1 - \int_0^\beta Q(0,\sigma)exp(\int_0^\sigma [b(s) + A(s)]ds)d\sigma\,)] \leq 1$$
$$w(0) \leq \frac{1}{1-d} \quad (2.10)$$

Using (2.10)in(2.9) and the fact that $\frac{u(t)}{a(t)} \leq w(t)$,we obtain the required inequality in (2.4).

In the following theorem deals with estimate the difference between the two approximate solutions of equation (1.1) with (2.2).

Theorem1. Assume that the functionsf, h, k in equation (1.1) satisfy the conditions

$$|f(t,x,y,z,w) - f(t,\bar{x},\bar{y},\bar{z},\bar{w})| \leq M[|x-\bar{x}| + |y-\bar{y}|] + |z-\bar{z}| + |w-\bar{w}| \quad (2.11)$$

$$|h(t,\sigma,x,y) - h(t,s,\bar{x},\bar{y})| \leq q_1(t,s)[|x-\bar{x}| + |y-\bar{y}|] \quad (2.12)$$

$$|k(t,s,x,y) - k(t,s,\bar{x},\bar{y})| \leq q_2(t,s)[|x-\bar{x}| + |y-\bar{y}|] \quad (2.13)$$

Where $M \geq 0$ is a constant such that $M < 1$ and for$s \leq t$; $q_1(t,s), q_2(t,s), \frac{\partial}{\partial t} q_1(t,s), \frac{\partial}{\partial t} q_2(t,s) \in C(R_+^2, R_+)$.

Let $x_i(t)(i = 1,2)$be respectively $\in_i$ _a.pproximate solutions of equation (1.1)with(2.2)on R_+. such that

$$|x_1 - x_2| \leq \delta \quad (2.14)$$

Where $\delta \geq 0$ is a constant . Then

$$|x_1(t) - x_2(t)| + |x_1'(t) - x_2'(t)| \leq \alpha(t)exp(\int_0^t [\frac{M}{1-M} + A_0(s)]ds) \quad (2.15)$$

for$t \in R_+$, where

$$\propto (t) = \frac{(\epsilon_1 + \epsilon_2)(t+1) + \delta}{1-M}, \quad (2.16)$$

$$A_0(t) = \frac{1}{1-M}[q_1(t,t) + \int_0^t [q_1(t,\sigma) + \frac{\partial}{\partial t} q_1(t,\sigma)]d\sigma + \int_0^\beta [q_2(t,\sigma) + \frac{\partial}{\partial t} q_2(t,\sigma)]d\sigma] \quad (2.17)$$

Proof: since $x_i(t)(i = 1,2)$ for $t \in R_+$ are respectively ϵ_i-approximate solutions of equation (1.1) with (2.2), we have (2.1). Integrating both sides with respect to t from 0 tot ,we have

$$\begin{aligned}
\epsilon_i t &\geq \int_0^t |x'_i(s) - f(s, x_i(s), x'_i(s), Hx_i(s), kx_i(s))| ds \\
&\geq \left| \int_0^t \{x'_i(s) - f(s, x_i(s), x'_i(s), Hx_i(s), kx_i(s))\} ds \right| \\
&= \left| \{x_i(t) - x_i(0) - \int_0^t f(s, x_i(s), x'_i(s), Hx_i(s), kx_i(s)) ds\} \right|, \quad (2.18)
\end{aligned}$$

For $i = 1,2$. from (2.18) and using the elementary inequalities

$$|v - z| \leq |v| + |z| \quad , \quad |v| - |z| \leq |v - z| \quad (2.19)$$

We observe that

$$\begin{aligned}
(\in_1+\in_1)t &\geq \left| \{x_1(t) - x_1(0) - \int_0^t f(s, x_1(s), x'_1(s), Hx_1(s), kx_1(s)) ds\} \right| \\
&\quad + \left| \{x_2(t) - x_2(0) - \int_0^t f(s, x_2(s), x'_2(s), Hx_2(s), kx_2(s)) ds\} \right| \\
&\geq \left| \{x_1(t) - x_1(0) - \int_0^t f(s, x_1(s), x'_1(s), Hx_1(s), kx_1(s)) ds\} - \{x_2(t) - x_2(0) \right. \\
&\quad \left. - \int_0^t f(s, x_2(s), x'_2(s), Hx_2(s), kx_2(s)) ds\} \right| \\
&\geq |x_1(t) - x_2(t)| - |x_1(o) - x_2(o)| \\
&\quad - \left| \int_0^t f(s, x_1(s), x'_1(s), Hx_1(s), kx_1(s)) ds \right. \\
&\quad \left. - \int_0^t f(s, x_2(s), x'_2(s), Hx_2(s), kx_2(s)) ds \right| \quad (2.20)
\end{aligned}$$

Moreover, from(2.1) and using the elementary inequalities in (2.19), we observe that

$$\begin{aligned}
(\in_1+\in_2) &\geq |x_1'(t) - f(t, x_1(t), x'_1(t), Hx_1(t), kx_1(t))| + |x_2'(t) - f(t, x_2(t), x'_2(t), Hx_2(t), kx_2(t))| \\
&\geq |\{x_1'(t) - f(t, x_1(t), x'_1(t), Hx_1(t), kx_1(t))\} - \{x_2'(t) - f(t, x_2(t), x'_2(t), Hx_2(t), kx_2(t))\}| \\
&\geq |x_1'(t) - x_2'(t)| \\
&\quad - |f(t, x_1(t), x'_1(t), Hx_1(t), kx_1(t)) \\
&\quad - f(t, x_2(t), x'_2(t), Hx_2(t), kx_2(t))|. \quad (2.21)
\end{aligned}$$

Let $u(t) = |x_1(t) - x_2(t)| + |x_1'(t) - x_2'(t)|$for $t \in R_+$.From(2.20), (2.21) and using the hypotheses ,we observe that

$$u(t) \leq (\in_1+\in_2)t + |x_1(o) - x_2(o)| + (\in_1+\in_2) + \int_0^t \left|f\big(s, x_1(s), x'_1(s), Hx_1(s), kx_1(s)\big) - f\big(s, x_2(s), x'_2(s), Hx_2(s), kx_2(s)\big)\right| + \left|f\big(t, x_1(t), x'_1(t), Hx_1(t), kx_1(t)\big) - f\big(t, x_2(t), x'_2(t), Hx_2(t), kx_2(t)\big)\right|$$

$$\leq (\in_1+\in_2)(t+1) + \delta + \int_0^t \{Mu(s) + \int_0^s q_1(s,\sigma)u(\sigma)d\sigma + \int_0^\beta q_2(s,\sigma)u(\sigma)d\sigma\}ds + Mu(t) + \int_0^t q_1(t,\sigma)u(\sigma)d\sigma + \int_0^\beta q_2(t,\sigma)u(\sigma)d\sigma \tag{2.22}$$

From (2.22), it is easy to observe that

$$u(t) \leq \propto (t) + \frac{1}{1-M}[\int_0^\beta q_2(t,\sigma)u(\sigma)d\sigma + \int_0^t \{Mu(s) + q_1(t,s)u(s) + \int_0^s q_1(s,\sigma)u(\sigma)d\sigma + \int_0^\beta q_2(s,\sigma)u(\sigma)d\sigma\}ds] \tag{2.23}$$

Where $\propto (t)$is given by (2.16). Clearly $\propto (t)$ is no decreasing for $t \epsilon R_+$.Now a suitable application of Lemma to (2.23) yields(2.15).

Remark 1. Note that the estimate obtained in (2.15) yields not only a bound for the difference between the two approximate of solutions of equation(1.1) with(2.2) but also abound on the difference between their derivatives . If$x_1(t)$ is a solution of equation (1.1) with$x_1(0) = x_1$, then we see that $x_2(t) \to x_1(t)$ as $\in_2 \to 0$ and$\delta \to 0$.

Consider the IVP (1.1)__(1.2) together with the following IVP

$$y(t) = g\left(t, y(t), y'(t), Hy(t), Ky(t)\right), \tag{2.24}$$

$$y(0) = y_0, \tag{2.25}$$

for $t \in R_+$, where H, K is given by (1.3), (1.4) and $g \in C(R_+ \times R^n \times R^n \times R^n \times R^n, R^n)$.

In the next theorem we provide conditions the closeness of the solution of IVP(1.1)_(1.2) and IVP(2.24)_(2.25).

Theorem 2. Assume that the function f, h, k in equation (1.1) satisfy the conditions (2.11), (2.12), (2.13) and there exist constants $\bar{\in} \geq 0, \bar{\delta} \geq 0$ such that

$$|f(t,x,y,z,u) - g(t,x,y,z,u)| \leq \bar{\in} \tag{2.26}$$

$$|x_0 - y_0| \leq \bar{\delta} \tag{2.27}$$

Where f, x_0 and g, y_0 are as in IVP (1.1)_(1.2) and IVP (2.24)_(2.25). Let $x(t) and\ y(t)$ be respectively solutions of IVP (1.1)_(1.2) and IVP (2.24)_(2.25)on R_+. Then

$$|x_1(t) - y_2(t)| + |x_1'(t) - y_2'(t)| \leq \gamma(t) exp(\int_0^t [\frac{M}{1-M} + A_0(s)]ds) \tag{2.28}$$

for $t \epsilon R_+$, where

$$\gamma(t) = \frac{\bar{\in}(t+1) + \bar{\delta}}{1-M}, \tag{2.29}$$

and $A_0(t)$is as in (2.17).

Proof: Let $r(t) = |x(t) - y(t)| + |x'(t) - y'(t)|$for $t \epsilon R_+$.Using that $x(t) and\ y(t)$ be respectively solutions of IVP (1.1)_(1.2) and IVP (2.24)_(2.25)and the assumption ,we observe that

$$r(t) \leq |x_0 - y_0| + \int_0^t \left|f\left(s, x(s), x'(s), Hx(s), Kx(s)\right) - f\left(s, y(s), y'(s), Hy(s), Ky(s)\right)\right| ds + \int_0^t \left|f\left(s, y(s), y'(s), Hy(s), Ky(s)\right) - g\left(s, y(s), y'(s), Hy(s), Ky(s)\right)\right| ds + \left|f\left(t, x(t), x'(t), Hx(t), Kx(t)\right) - f\left(t, y(t), y'(t), Hy(t), Ky(t)\right)\right| + \left|f\left(t, y(t), y'(t), Hy(t), Ky(t)\right) - g\left(t, y(t), y'(t), Hy(t), Ky(t)\right)\right|$$

$$\leq \bar{\delta} + \int_0^t \{Mr(s) + \int_0^s q_1(s,\sigma)r(\sigma)d\sigma + \int_0^\beta q_2(s,\sigma)r(\sigma)d\sigma\}ds + \bar{\in} t + Mr(t) + \int_0^t q_1(t,\sigma)r(\sigma)d\sigma + \int_0^\beta q_2(t,\sigma)r(\sigma)d\sigma + \bar{\in}$$

$$= \bar{\in}(t+1) + \bar{\delta} + Mr(t) + \int_0^\beta q_2(t,\sigma)r(\sigma)d\sigma + \int_0^t \{Mr(s) + q_1(t,s)r(s) + \int_0^s q_1(s,\sigma)r(\sigma)d\sigma + \int_0^\beta q_2(s,\sigma)r(\sigma)d\sigma\}ds \tag{2.30}$$

From (2.30), we have

$$r(t) \le \gamma(t) + \frac{1}{1-M}[\int_0^\beta q_2(t,\sigma)r(\sigma)d\sigma + \int_0^t \{Mr(s) + q_1(t,s)r(s) + \int_0^s q_1(s,\sigma)r(\sigma)d\sigma + \int_0^\beta q_2(s,\sigma)r(\sigma)d\sigma\}\, ds] \quad (2.31)$$

For $t\epsilon R_+$, Where$\gamma(t)$is given by (2.29). Clearly $\gamma(t)$ is no decreasing for $t\epsilon R_+$ .Now a suitable application of Lemma to (2.31) yields (2.28).

Remark 2. We note that the result given in theorem 2 relates the solutions of IVP(1.1)_(1.2) and IVP(2.24)_(2.25)in the sense that if f is close to g and $x_0$is close to y_0 , then the solutions of IVP (1.1)_(1.2) and IVP (2.24)_(2.25)are also close together.

The following theorem gives conditions for an estimate of the difference between the solutions of IVP (1.1)_(1.2) and IVP (2.24)_(2.25) .

Theorem 3: Assume that

$$|f(t,x,y,z,w) - g(t,\bar{x},\bar{y},\bar{z},\bar{w})| \le L[|x-\bar{x}| + |y-\bar{y}|] + |z-\bar{z}| + |w-\bar{w}| \quad (2.32)$$

Where $L \ge 0$ is a constant such that $L < 1$ and the conditions (2.12), (2.13),and (2.27) hold . Let $x(t)$and $y(t)$be respectively ,solutions of IVP (1.1)_(1.2) and IVP(2.24)_(2.25) on R_+. Then

$$|x(t) - y(t)| + |x'(t) - y'(t)| \le (\frac{\bar{\delta}}{1-L})exp(\int_0^t [\frac{L}{1-L} + A_1(s)]ds) \quad (2.33)$$

Where

$$A_1(t) = \frac{1}{1-L}[q_1(t,t) + \int_0^t [q_1(t,t) + \frac{\partial}{\partial t}q_1(t,\sigma)]d\sigma + \int_0^\beta [q_2(t,\sigma) + \frac{\partial}{\partial t}q_2(t,\sigma)]d\sigma] \quad (2.34)$$

Proof : Let $z(t) = |x(t) - y(t)| + |x'(t) - y'(t)|$ for $t \in R_+$. Using the facts that $x(t)$and $y(t)$ are respectively, solutions of IVP (1.1)_(1.2)and IVP (2.24)_(2.25) and the assumptions, we observe that

$$\begin{aligned} z(t) &\le |x_0 - y_0| + \int_0^t |f(s,x(s),x'(s),Hx(s),K(s)) - g(s,y(s),y'(s),Hy(s),Ky(s))|ds \\ &\quad + |f(s,x(s),x'(s),Hx(s),K(s)) - g(s,y(s),y'(s),Hy(s),Ky(s))| \\ &\le \bar{\delta} + \int_0^t \{Lz(s) + \int_0^s q_1(s,\sigma)z(\sigma)d\sigma + \int_0^\beta q_2(s,\sigma)z(\sigma)d\sigma\}ds + L\,z(t) \\ &\quad + \int_0^t q_1(t,\sigma)z(\sigma)d\sigma + \int_0^\beta q_2(t,\sigma)z(\sigma)d\sigma\,] \end{aligned} \quad (2.35)$$

From (2.35),we have

$$z(t) \le \frac{\bar{\delta}}{1-L} + \frac{1}{1-L}[\int_0^\beta q_2(t,\sigma)r(\sigma)d\sigma + \int_0^t \{Lz(s) + q_1(t,s)z(s) + \int_0^s q_1(s,\sigma)z(\sigma)d\sigma + \int_0^\beta q_2(s,\sigma)z(\sigma)d\sigma\}\, ds] \quad (2.36)$$

Now an application of Lemma to (2.36) yields (2.33).

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