Existence and Uniqueness Solution of Fractional Integro-Differential Equations

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ABSTRACT

In this paper we study the existence and uniqueness solution of fractional integro-differential equation, by using both methods Picard approximation and Banach fixed point theorem. Also we extend the results of Butris.

Keywords: Fractional integro – Differentional Equations, Existence and, uniqueness solutions

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I. INTRODUCTION

Fractional differential equations have been of great interest recently. It is caused both by the intensive development of theory of fractional calculus itself and by application of such constructions in various sciences such as physics, chemistry, mechanics, engineering, For details, [2, 4, 5, 6, 7]. As for the theory, the investigations in clued the existence, uniqueness of solution, asymptotic behavior, stability, etc. for example [2, 5, 7, 8]. Butris has study a solution of integro-differential equation of fractional order which has the form:

$$x^{\alpha}(t) = f\left(t, x, \int_{-\infty}^{t} G(t,s)\, g\left(s, x(s)\right) ds\right) \quad , \quad x^{\alpha-1}(0) = x_0 \quad , \quad 0 < \alpha < 1$$

Where: $x \in D_{\alpha} \subseteq [0, T]$ and $D_{1\alpha}$ is a closed and bounded domain.

Our work we extended the results of Butris[3] and D^{α} is the standard Riemann – Liouville fractional derivative.

Definition 1.

Let f be a function which is defined a. e. (almost every where) on $[a, b]$. For $\alpha > 0$, we define:

$${}_{a}^{b}\mathrm{I}^{\alpha} f = \frac{1}{\Gamma(\alpha)} \int_{a}^{b} (b-s)^{\alpha-1} f(s)\, ds$$

Provided that this integral (Lebesgue) exists.

Definition 2.[2].

If $\alpha > 0$, then Gamma's function is denote by (Γ) and defined by the form: $\Gamma(\alpha) = \int_{0}^{\infty} e^{-s} s^{\alpha-1} ds$

Lemma 1[1].

If $\{f_n\}_{n=1}^{\infty}$ is a sequences of functions is defined on the set $E \subseteq R$ such that $|f_n| \leq M_n$, where M_n is a positive number, then $\sum_{n=1}^{\infty} f_n$ is uniformaly convergent on E if $\sum_{n=1}^{\infty} M_n$ is convergent.

Lemma 2[1].

Let $E_\alpha(m;x) = \sum_{m=1}^{\infty} \frac{m^{n-1} x^{n\alpha-1}}{\Gamma(n\alpha)}$, where m = R, then:

1. the series converges for $x \neq 0$ and $\alpha > 0$.
2. the series converges everywhere when $\alpha \geq 0$.
3. if $\alpha = 0$, then $E_1(m,x) = \exp(mx)$.

Lemma 3[3].

If K_1 and K_2 be a positive constant, and f be a continuous function on $a \leq t \leq b$, such that:

$$f(t) \leq K_1 + K_2 \int_a^t f(s)ds$$

Then

$$f(t) \leq K_1 \exp\left(K_2(t-a)\right)$$

This paper deals with the existence and uniqueness solution of fractional integro-differential equation which has the form:-

$$x^\alpha(t) = f\left(t, x(t), h(t)x(t), \int_{-\infty}^{t} G(t,s)P\left(s, x(s), h(s)x(s)\right)ds\right) \quad \dots (1\text{-}1)$$

$$x_0^{\alpha-1}(0) = x_0 \quad , \quad 0 < \alpha < 1$$

The function $f\left(t, x(t), h(t)x(t), Q(t)\right)$ is defined, continuous on the domain:

$$(t,x) \in [0,T] \times D_\alpha \quad \dots\dots (1.2)$$

Where: $x \in D_\alpha \subseteq [0,T]$, D_α is a closed and bounded domain subset of R.

We denote to $\int_{-\infty}^{t} G(t,s)P\left(t, x(t), h(t)x(t)\right)dt$ by $Q(t)$,

Suppose that the functions $f\left(t, x(t), h(t)x(t), Q(t)\right)$, $h(t)$ satisfies the following inequalities:

$$\left\| f\left(t, x(t), h(t)x(t), Q(t)\right)\right\| \leq M \quad , \quad \dots\dots (1.3)$$

$$\left\| f\left(t, x_1, hx_1, Q_1(t)\right) - f\left(t, x_2, hx_2, Q_2(t)\right)\right\| \leq L\|x_1 - x_2\| + NL\|x_1 - x_2\| + K\left(L\|x_1 - x_2\| + NL\|x_1 - x_2\|\right) \dots (1.4)$$

…

For all $t \in [0,T]$, and $x, x_1, x_2 \in D_\alpha$, where L, M, N, K, are positive constants. Here h(t) is a continuous function in t provided that:

$$\|h(t)\| \leq N \quad , \quad N>0 \quad \dots\dots (1.5)$$

Also the matrix G(t,s) is nonnegative, continuous in t , s and satisfies the following inequalities:

$$\left\|G(t,s)\right\| \leq \delta e^{-\lambda(t-s)} \qquad \dots\dots (1.6)$$

Where: $-\infty < 0 \leq s \leq t \leq T < \infty$, $\alpha, \beta > 0$.

We define the non-empty sets as follows:

$$D_{\alpha f} = D_{\alpha} - \frac{T^{\alpha}}{\Gamma(\alpha+1)} M \quad , \quad D_{\alpha f} \neq \varphi \qquad \dots\dots (1.7)$$

Moreover, we suppose the value of the following equation:

$$\Phi = \frac{T^{\alpha}}{\Gamma(\alpha+1)} \Psi < 1 \qquad \dots\dots (1.8)$$

Where: $\Psi = \left(1 - \frac{\delta}{\lambda}\right)(1+N)L$, $\|.\| = \max |.|$.

II. EXISTENCE OF SOLUTION

Theorem 1.

Let the vector function $f(t, x(t), h(t)x(t), Q(t))$ be defined in the domain (1.2), continuous in t, x and satisfy the inequalities (1.3), (1.4), and (1.5), then the function:

$$x(t,x_0) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t \left[f\left(s, x(s,x_0), h(s)x(s,x_0), \int_{-\infty}^{s} G(s,\tau) P(\tau, x(\tau,x_0), h(\tau)x(\tau,x_0)) d\tau\right)\right] (t-s)^{\alpha-1} ds$$

is a solution of (1.1).

Proof:

Let

$$x_{m+1}(t,x_0) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t \left[f\left(s, x_m(s,x_0), h(s)x_m(s,x_0), \int_{-\infty}^{s} G(s,\tau) P(\tau, x_m(\tau,x_0), h(\tau)x_m(\tau,x_0)) d\tau\right)\right] (t-s)^{\alpha-1} ds \qquad \dots\dots (2.1)$$

With:

$$x_0^{\alpha-1}(t,x_0) = x_0 \quad , \quad m = 0,1,2,\dots$$

be a sequence of functions which is defined on the domain:

$$(t, x_0) \in [0,T] \times D_{\alpha f} \qquad \dots\dots (2.2)$$

We will divide the proof as follows:

(i) $x_m(t,x_0) \in D_{\alpha}$, for all $t \in [0,T]$, $x_0 \in D_{\alpha f}$.

(ii) $x_m(t,x_0) \in D_{\alpha}$, is uniformly convergent to the function $x(t,x_0)$ on the domain (2.2), for all $t \in [0,T]$, $x_0 \in D_{\alpha f}$.

(iii) $x(t,x_0) \in D_{\alpha}$, for all $t \in [0,T]$, $x_0 \in D_{\alpha f}$.

proof (i):

Set m=0 and use (2.1), we get:

$$\left\|x_1(t,x_0)-x_0\right\|=\left\|x_0+\frac{1}{\Gamma(\alpha)}\int_0^t\left[f\left(s,x_1(s,x_0),h(s)x_1(s,x_0),\int_{-\infty}^{s}G(s,\tau)P\left(\tau,x_1(\tau,x_0),h(\tau)x_1(\tau,x_0)\right)d\tau\right)\right](t-s)^{\alpha-1}ds-x_0\right\|$$

$$\leq\frac{1}{\Gamma(\alpha)}\int_0^t\left\|f\left(s,x_1(s,x_0),h(s)x_1(s,x_0),\int_{-\infty}^{s}G(s,\tau)P\left(\tau,x_1(\tau,x_0),h(\tau)x_1(\tau,x_0)\right)d\tau\right)\right\|(t-s)^{\alpha-1}ds$$

$$\leq\frac{1}{\Gamma(\alpha)}\int_0^t M\,(t-s)^{\alpha-1}ds$$

$$\leq\frac{t^{\alpha}}{\Gamma(\alpha+1)}M\qquad,\qquad t\in[0,T]$$

$$\left\|x_1(t,x_0)-x_0\right\|\leq\frac{T^{\alpha}}{\Gamma(\alpha+1)}M\qquad\ldots\ldots(2.3)$$

That is $x_1(t,x_0)\in D_{\alpha}$, for all $t\in[0,T]$, $x_0\in D_{\alpha f}$.

By mathematical induction we have:

$$\left\|x_m(t,x_0)-x_0\right\|=\left\|x_0+\frac{1}{\Gamma(\alpha)}\int_0^t\left[f\left(s,x_m(s,x_0),h(s)x_m(s,x_0),\int_{-\infty}^{s}G(s,\tau)P\left(\tau,x_m(\tau,x_0),h(\tau)x_m(\tau,x_0)\right)d\tau\right)\right](t-s)^{\alpha-1}ds-x_0\right\|$$

$$\left\|x_m(t,x_0)-x_0\right\|\leq\frac{T^{\alpha}}{\Gamma(\alpha+1)}M$$

where $x_m(t,x_0)\in D_{\alpha}$, for all $t\in[0,T]$, $x_0\in D_{\alpha f}$.

proof (ii):

Now, we shall prove that the sequence of functions (2.2) is uniformly convergent on (2.2). From (2.1), when m=1 we get:

$$\left\|x_2(t,x_0)-x_1(t,x_0)\right\|=\left\|x_0+\frac{1}{\Gamma(\alpha)}\int_0^t\left[f\left(s,x_1(s,x_0),h(s)x_1(s,x_0),\int_{-\infty}^{s}G(s,\tau)P\left(\tau,x_1(\tau,x_0),h(\tau)x_1(\tau,x_0)\right)d\tau\right)\right](t-s)^{\alpha-1}ds-\right.$$

$$\left.-x_0-\frac{1}{\Gamma(\alpha)}\int_0^t\left[f\left(s,x_0(s,x_0),h(s)x_0(s,x_0),\int_{-\infty}^{s}G(s,\tau)P\left(\tau,x_0(\tau,x_0),h(\tau)x_0(\tau,x_0)\right)d\tau\right)\right](t-s)^{\alpha-1}ds\right\|$$

$$\leq\frac{1}{\Gamma(\alpha)}\int_0^t\left[L\left\|x_1(\tau,x_0)-x_0(\tau,x_0)\right\|+NL\left\|x_1(\tau,x_0)-x_0(\tau,x_0)\right\|+\int_{-\infty}^{s}\delta e^{-\lambda(t-s)}\left(L\left\|x_1(\tau,x_0)-x_0(\tau,x_0)\right\|+NL\left\|x_1(\tau,x_0)-x_0(\tau,x_0)\right\|\right)d\tau\right](t-s)^{\alpha-1}ds$$

$$\leq\frac{1}{\Gamma(\alpha)}\int_0^t\left[(1+N)L\left\|x_1(\tau,x_0)-x_0(\tau,x_0)\right\|-\frac{\delta}{\lambda}(1+N)L\left\|x_1(\tau,x_0)-x_0(\tau,x_0)\right\|\right](t-s)^{\alpha-1}ds$$

$$=\frac{1}{\Gamma(\alpha)}\left(1-\frac{\delta}{\lambda}\right)(1+N)\int_0^t\left[L\left\|x_1(\tau,x_0)-x_0(\tau,x_0)\right\|\right](t-s)^{\alpha-1}ds$$

$$\leq\frac{1}{\Gamma(\alpha)}\left(1-\frac{\delta}{\lambda}\right)(1+N)\int_0^t\left[L\frac{T^{\alpha}}{\Gamma(\alpha+1)}M\right](t-s)^{\alpha-1}ds$$

$$\leq\left(1-\frac{\delta}{\lambda}\right)(1+N)L\left(\frac{T^{\alpha}}{\Gamma(\alpha+1)}\right)^2M$$

$$= \Psi \left(\frac{T^{\alpha}}{\Gamma(\alpha+1)} \right)^{2} M$$

And hence:

$$\left\| x_2(t, x_0) - x_1(t, x_0) \right\| \le \Psi \left(\frac{T^{\alpha}}{\Gamma(\alpha+1)} \right)^{2} M$$

Now when m=2 in (2.1) we get:

$$\left\| x_3(t,x_0) - x_2(t,x_0) \right\| = \left\| x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t \left[f\left(s, x_2(s,x_0), h(s)x_2(s,x_0), \int_{-\infty}^{s} G(s,\tau) P\left(\tau, x_2(\tau,x_0), h(\tau)x_2(\tau,x_0)\right) d\tau \right) \right] (t-s)^{\alpha-1} ds - \right.$$

$$\left. - x_0 - \frac{1}{\Gamma(\alpha)} \int_0^t \left[f\left(s, x_1(s,x_0), h(s)x_1(s,x_0), \int_{-\infty}^{s} G(s,\tau) P\left(\tau, x_1(\tau,x_0), h(\tau)x_1(\tau,x_0)\right) d\tau \right) \right] (t-s)^{\alpha-1} ds \right\|$$

$$\le \frac{1}{\Gamma(\alpha)} \int_0^t \left[L \left\| x_2(\tau,x_0) - x_1(\tau,x_0) \right\| + NL \left\| x_2(\tau,x_0) - x_1(\tau,x_0) \right\| + \right.$$

$$\left. + \int_{-\infty}^{s} \delta e^{-\lambda(t-s)} \left(L \left\| x_2(\tau,x_0) - x_1(\tau,x_0) \right\| + NL \left\| x_2(\tau,x_0) - x_1(\tau,x_0) \right\| \right) d\tau \right] (t-s)^{\alpha-1} ds$$

$$\le \frac{1}{\Gamma(\alpha)} \int_0^t \left[(1+N) L \left\| x_2(\tau,x_0) - x_1(\tau,x_0) \right\| - \frac{\delta}{\lambda} (1+N) L \left\| x_2(\tau,x_0) - x_1(\tau,x_0) \right\| \right] (t-s)^{\alpha-1} ds$$

$$\le \frac{1}{\Gamma(\alpha)} \left(1 - \frac{\delta}{\lambda} \right) (1+N) \int_0^t \left[L \Psi \left(\frac{T^{\alpha}}{\Gamma(\alpha+1)} \right)^{2} M \right] (t-s)^{\alpha-1} ds$$

$$\le \left(1 - \frac{\delta}{\lambda} \right) (1+N) L \Psi \left(\frac{T^{\alpha}}{\Gamma(\alpha+1)} \right)^{3} M$$

$$= \Psi^{2} \left(\frac{T^{\alpha}}{\Gamma(\alpha+1)} \right)^{3} M$$

Therefore:

$$\left\| x_3(t, x_0) - x_2(t, x_0) \right\| \le \Psi^{2} \left(\frac{T^{\alpha}}{\Gamma(\alpha+1)} \right)^{3} M$$

Then by mathematical induction we have:

$$\left\| x_{m+1}(t, x_0) - x_m(t, x_0) \right\| \le \left(\frac{T^{\alpha}}{\Gamma(\alpha+1)} \right)^{m+1} M \Psi^{m} \qquad \ldots \ldots (2.4)$$

For all m=0,1,2,… .

Now from (2.4), and for $p \ge 1$, we get:

$$\left\| x_{m+p}(t, x_0) - x_m(t, x_0) \right\| \le M \sum_{i=0}^{p-1} \left(\frac{T^{\alpha}}{\Gamma(\alpha+1)} \right)^{i+1} \Psi^{i} \qquad \ldots \ldots (2.5)$$

Where:

$$\left\| x_{m+p}(t, x_0) - x_m(t, x_0) \right\| = \left\| x_{m+p}(t, x_0) - x_{m+p-1}(t, x_0) \right\| +$$

$$+ \left\| x_{m+p-1}(t, x_0) - x_{m+p-2}(t, x_0) \right\| + \dots\dots\dots \; \dots. +$$
$$+ \left\| x_{m+1}(t, x_0) - x_m(t, x_0) \right\|$$
$$\leq \left(\frac{T^{\alpha}}{\Gamma(\alpha+1)} \Psi \right)^{m+p-1} \left\| x_1(t, x_0) - x_0 \right\| + \left(\frac{T^{\alpha}}{\Gamma(\alpha+1)} \Psi \right)^{m+p-2} \left\| x_1(t, x_0) - x_0 \right\| +$$
$$+ \dots\dots \; + \left(\frac{T^{\alpha}}{\Gamma(\alpha+1)} \Psi \right)^{m} \left\| x_1(t, x_0) - x_0 \right\|$$

So that

$$\left\| x_{m+p}(t, x_0) - x_m(t, x_0) \right\| \leq \left(\frac{T^{\alpha}}{\Gamma(\alpha+1)} \Psi \right)^{m} \left[1 + \left(\frac{T^{\alpha}}{\Gamma(\alpha+1)} \Psi \right) + \left(\frac{T^{\alpha}}{\Gamma(\alpha+1)} \Psi \right)^{2} + \dots. + \right.$$
$$\left. + \left(\frac{T^{\alpha}}{\Gamma(\alpha+1)} \Psi \right)^{p-2} + \left(\frac{T^{\alpha}}{\Gamma(\alpha+1)} \Psi \right)^{p-1} \right] \left\| x_1(t, x_0) - x_0 \right\|$$

… … (2.6)

We note that the right hand from (2.6) is bounded with the convergent geometric series and its summation to equals $\frac{1}{1-\Psi}$ and hence

$$\left\| x_{m+p}(t, x_0) - x_m(t, x_0) \right\| \leq \left(\frac{T^{\alpha}}{\Gamma(\alpha+1)} \Psi \right)^{m} \left[1 - \left(\frac{T^{\alpha}}{\Gamma(\alpha+1)} \Psi \right) \right]^{-1} \left\| x_1(t, x_0) - x_0 \right\|$$
$$\leq \left(\frac{T^{\alpha}}{\Gamma(\alpha+1)} \Psi \right)^{m} \left[1 - \left(\frac{T^{\alpha}}{\Gamma(\alpha+1)} \Psi \right) \right]^{-1} \frac{T^{\alpha}}{\Gamma(\alpha+1)} M_1 \Psi$$

… … (2.7)

But the inequality (1.8) is less than unity, then

$$\lim_{m \to \infty} \left(\frac{T^{\alpha}}{\Gamma(\alpha+1)} \Psi \right)^{m} = 0 \qquad \dots \dots (3.15)$$

Thus the right hand side of (2.7) equals to zero when $m \to \infty$. Suppose that $\varepsilon > 0$, we get a positive integer n such that $n < m$, and satisfied the next estimation for all m:

$$\left\| x_{m+p}(t, x_0) - x_m(t, x_0) \right\| < \varepsilon, \; p \geq 0.$$

Then according to the definition of uniformly convergent [1], we find that the sequence of function $\{x_m(t, x_0)\}_{m=0}^{\infty}$ is uniformly convergent to the function $x(t, x_0)$ and this function is continuous on the same interval.

Putting:

$$\lim_{m \to \infty} x_m(t, x_0) = x(t, x_0) \qquad \dots \dots (2.8)$$

Proof (iii):

to prove $x(t, x_0) \in D_{\alpha}$, for all $t \in [0, T]$, $x_0 \in D_{\alpha f}$ we assume that:

$$\left\| \frac{1}{\Gamma(\alpha)} \int_0^t \left[f\left(s, x_m(s, x_0), h(s) x_m(s, x_0), \int_{-\infty}^{s} G(s,\tau) P\left(\tau, x_m(\tau, x_0), h(\tau) x_m(\tau, x_0)\right) d\tau \right) \right] (t-s)^{\alpha-1} ds - \right.$$

$$-\frac{1}{\Gamma(\alpha)}\int_0^t\left[f\left(s,x(s,x_0),h(s)x(s,x_0),\int_{-\infty}^{s}G(s,\tau)P\left(\tau,x(\tau,x_0),h(\tau)x(\tau,x_0)\right)d\tau\right)\right](t-s)^{\alpha-1}ds\Bigg\|$$

$$\le\frac{\Psi}{\Gamma(\alpha)}\int_0^t\left\|x_m(\tau,x_0)-x(\tau,x_0)\right\|(t-s)^{\alpha-1}ds$$

Then

$$\lim_{m\to\infty}\Bigg\|\frac{1}{\Gamma(\alpha)}\int_0^t\left[f\left(s,x_m(s,x_0),h(s)x_m(s,x_0),\int_{-\infty}^{s}G(s,\tau)P\left(\tau,x_m(\tau,x_0),h(\tau)x_m(\tau,x_0)\right)d\tau\right)\right](t-s)^{\alpha-1}ds-$$

$$-\frac{1}{\Gamma(\alpha)}\int_0^t\left[f\left(s,x(s,x_0),h(s)x(s,x_0),\int_{-\infty}^{s}G(s,\tau)P\left(\tau,x(\tau,x_0),h(\tau)x(\tau,x_0)\right)d\tau\right)\right](t-s)^{\alpha-1}ds\Bigg\|$$

$$\le\lim_{m\to\infty}\frac{\Psi}{\Gamma(\alpha)}\int_0^t\left\|x_m(\tau,x_0)-x(\tau,x_0)\right\|(t-s)^{\alpha-1}ds$$

Since the sequence $\{x_m(t,x_0)\}_{m=0}^{\infty}$ is uniformly convergent on $[0,T]$ to the function $x(t,x_0)$ on the same interval, then, we have

$$\lim_{m\to\infty}\Bigg\|\frac{1}{\Gamma(\alpha)}\int_0^t\left[f\left(s,x_m(s,x_0),h(s)x_m(s,x_0),\int_{-\infty}^{s}G(s,\tau)P\left(\tau,x_m(\tau,x_0),h(\tau)x_m(\tau,x_0)\right)d\tau\right)\right](t-s)^{\alpha-1}ds=$$

$$=\frac{1}{\Gamma(\alpha)}\int_0^t\left[f\left(s,x(s,x_0),h(s)x(s,x_0),\int_{-\infty}^{s}G(s,\tau)P\left(\tau,x(\tau,x_0),h(\tau)x(\tau,x_0)\right)d\tau\right)\right](t-s)^{\alpha-1}ds\Bigg\|$$

So $x(t,x_0)\in G_\alpha$, for all $x_0\in G_{\alpha f}$

III. UNIQUENESS OF SOLUTION

The study of the uniqueness solution of (1.1), will be introduced by the following:

Theorem 2.

Let all assumptions and conditions of theorem 1 be given then the problem (1.1), has a unique solution $x=x_\infty(t,x_0)$ on the domain (2.2).

Proof:

On the contrary, we suppose that there is another solution $\hat{x}(t,x_0)$ of the problem (1.1), which is defined by the following integral equation:

$$\hat{x}(t,x_0)=x_0+\frac{1}{\Gamma(\alpha)}\int_0^t\left[f\left(s,\hat{x}(s,x_0),h(s)\hat{x}(s,x_0),\int_{-\infty}^{s}G(s,\tau)P\left(\tau,\hat{x}(\tau,x_0),h(\tau)\hat{x}(\tau,x_0)\right)d\tau\right)\right](t-s)^{\alpha-1}ds \quad \ldots\ldots(3.1)$$

Now we shall prove that $\hat{x}(t,x_0)=x(t,x_0)$ for all $x_0\in D_{\alpha f}$, and to do this we need to prove the following inequality:

$$\left\|\hat{x}(t,x_0)-x_m(t,x_0)\right\|\le\left(\frac{T^\alpha}{\Gamma(\alpha+1)}\right)^{m+1}M^*\Psi^m \quad \ldots\ldots(3.2)$$

where $\left\| f\left(t, x(t), h(t)x(t), Q(t)\right)\right\| \le M^*$, $\Psi = \left(1 - \frac{\delta}{\lambda}\right)(1+N)L$.

Let when m=0 in (2.1) and from (3.1) we find:

$$\left\|\hat{x}(t,x_0) - x_0\right\| = \left\| x_0 + \frac{1}{\Gamma(\alpha)}\int_0^t \left[f\left(s, \hat{x}(s,x_0), h(s)\hat{x}(s,x_0), \int_{-\infty}^{s} G(s,\tau)P\left(\tau, \hat{x}(\tau,x_0), h(\tau)\hat{x}(\tau,x_0)\right)d\tau\right)\right](t-s)^{\alpha-1}ds - x_0 \right\|$$

$$\le \frac{1}{\Gamma(\alpha)}\int_0^t \left\| f\left(s, \hat{x}(s,x_0), h(s)\hat{x}(s,x_0), \int_{-\infty}^{s} G(s,\tau)P\left(\tau, \hat{x}(\tau,x_0), h(\tau)\hat{x}(\tau,x_0)\right)d\tau\right)\right\|(t-s)^{\alpha-1}ds$$

$$\le \frac{1}{\Gamma(\alpha)}\int_0^t M^*(t-s)^{\alpha-1}ds$$

$$\le \frac{t^\alpha}{\Gamma(\alpha+1)}M^* \quad , \quad t \in [0,T]$$

$$\left\|\hat{x}(t,x_0) - x_0\right\| \le \frac{T^\alpha}{\Gamma(\alpha+1)}M^*$$

and when m=1 in (2.1) and from (3.1) we find:

$$\left\|\hat{x}(t,x_0) - x_1(t,x_0)\right\| = \left\| x_0 + \frac{1}{\Gamma(\alpha)}\int_0^t \left[f\left(s, \hat{x}(s,x_0), h(s)\hat{x}(s,x_0), \int_{-\infty}^{s} G(s,\tau)P\left(\tau, \hat{x}(\tau,x_0), h(\tau)\hat{x}(\tau,x_0)\right)d\tau\right)\right](t-s)^{\alpha-1}ds - \right.$$

$$\left. - x_0 - \frac{1}{\Gamma(\alpha)}\int_0^t \left[f\left(s, x_0(s,x_0), h(s)x_0(s,x_0), \int_{-\infty}^{s} G(s,\tau)P\left(\tau, x_0(\tau,x_0), h(\tau)x_0(\tau,x_0)\right)d\tau\right)\right](t-s)^{\alpha-1}ds \right\|$$

$$\le \frac{1}{\Gamma(\alpha)}\int_0^t \left[(1+N)L\left\|\hat{x}(\tau,x_0) - x_0(\tau,x_0)\right\| - \frac{\delta}{\lambda}(1+N)L\left\|\hat{x}(\tau,x_0) - x_0(\tau,x_0)\right\|\right](t-s)^{\alpha-1}ds$$

$$= \frac{1}{\Gamma(\alpha)}\left(1 - \frac{\delta}{\lambda}\right)(1+N)\int_0^t \left[L\left\|\hat{x}(\tau,x_0) - x_0(\tau,x_0)\right\|\right](t-s)^{\alpha-1}ds$$

$$\le \left(1 - \frac{\delta}{\lambda}\right)(1+N)L\left(\frac{T^\alpha}{\Gamma(\alpha+1)}\right)^2 M^*$$

$$= \Psi\left(\frac{T^\alpha}{\Gamma(\alpha+1)}\right)^2 M^*$$

Therefore:

$$\left\|\hat{x}(t,x_0) - x_1(t,x_0)\right\| \le \Psi\left(\frac{T^\alpha}{\Gamma(\alpha+1)}\right)^2 M^*$$

We find that the inequality (3.2) is satisfying when m=0, 1, 2.

Suppose that the inequality (3.2) is satisfying when m=p as the following inequality:

$$\left\|\hat{x}(t,x_0) - x_p(t,x_0)\right\| \le \left(\frac{T^\alpha}{\Gamma(\alpha+1)}\right)^{p+1} M^*\Psi^p \qquad \ldots\ldots (3.3)$$

Now:

$$\left\|\hat{x}(t,x_0)-x_{p+1}(t,x_0)\right\|=\left\|x_0+\frac{1}{\Gamma(\alpha)}\int_0^t\left[f\left(s,\hat{x}(s,x_0),h(s)\hat{x}(s,x_0),\int_{-\infty}^{s}G(s,\tau)P\left(\tau,\hat{x}(\tau,x_0),h(\tau)\hat{x}(\tau,x_0)\right)d\tau\right)\right](t-s)^{\alpha-1}ds-\right.$$

$$\left.-x_0-\frac{1}{\Gamma(\alpha)}\int_0^t\left[f\left(s,x_p(s,x_0),h(s)x_p(s,x_0),\int_{-\infty}^{s}G(s,\tau)P\left(\tau,x_p(\tau,x_0),h(\tau)x_p(\tau,x_0)\right)d\tau\right)\right](t-s)^{\alpha-1}ds\right\|$$

$$\le\frac{1}{\Gamma(\alpha)}\int_0^t\left[(1+N)L\left\|\hat{x}(\tau,x_0)-x_p(\tau,x_0)\right\|-\frac{\delta}{\lambda}(1+N)L\left\|\hat{x}(\tau,x_0)-x_p(\tau,x_0)\right\|\right](t-s)^{\alpha-1}ds$$

$$=\frac{1}{\Gamma(\alpha)}\left(1-\frac{\delta}{\lambda}\right)(1+N)\int_0^t\left[L\left\|\hat{x}(\tau,x_0)-x_p(\tau,x_0)\right\|\right](t-s)^{\alpha-1}ds$$

$$\le\left(1-\frac{\delta}{\lambda}\right)(1+N)L\Psi^{p}\left(\frac{T^{\alpha}}{\Gamma(\alpha+1)}\right)^{p+2}M^{*}$$

$$=\Psi^{p+1}\left(\frac{T^{\alpha}}{\Gamma(\alpha+1)}\right)^{p+2}M^{*}$$

Then:

$$\left\|\hat{x}(t,x_0)-x_{p+1}(t,x_0)\right\|\le\left(\frac{T^{\alpha}}{\Gamma(\alpha+1)}\right)^{p+2}M^{*}\Psi^{p+1}$$

Thus we find that the inequality (3.2) is satisfying when m=0,1,2,… .
Then by a condition (2.8) we get:

$$\hat{x}(t,x_0)=\lim_{m\to\infty}x_m(t,x_0)=x(t,x_0)$$

and this proves that the two solutions are congruent in the domain (2.2).

V. Banach method[1].

In this section, we use the Banach fixed point theorem to prove the existence and uniqueness solution.

Theorem 3.

Let $(S,\|.\|)$ be a Banach space which define on a mapping T^* on D_α by:

$$T^{*}x(t,x_0)=x_0+\frac{1}{\Gamma(\alpha)}\int_0^t\left[f\left(s,x(s,x_0),h(s)x(s,x_0),\int_{-\infty}^{s}G(s,\tau)P\left(\tau,x(\tau,x_0),h(\tau)x(\tau,x_0)\right)d\tau\right)\right](t-s)^{\alpha-1}ds \quad \dots\dots$$

(4.1)

Since the equation (4.1) defined in the domain (2.2), continuous in t, x and satisfies the inequalities (1.4), (1.5), and (1.6), then $T^{*}\in D_\alpha$, and hence $T^{*}:D_\alpha\to D_\alpha$, Next we claim that T^* is a contraction mapping on S.

Proof:

Let $x(t,x_0),\ z(t,x_0)\in D_\alpha$, then:

$$\left\|T^{*}x(t,x_0)-T^{*}z(t,x_0)\right\|=\left|x_0+\frac{1}{\Gamma(\alpha)}\int_0^t\left[f\left(s,x(s,x_0),h(s)x(s,x_0),\int_{-\infty}^{s}G(s,\tau)P\left(\tau,x(\tau,x_0),h(\tau)x(\tau,x_0)\right)d\tau\right)\right](t-s)^{\alpha-1}ds-\right.$$

$$\left.-z_0-\frac{1}{\Gamma(\alpha)}\int_0^t\left[f\left(s,z(s,x_0),h(s)z(s,x_0),\int_{-\infty}^{s}G(s,\tau)P\left(\tau,z(\tau,x_0),h(\tau)z(\tau,x_0)\right)d\tau\right)\right](t-s)^{\alpha-1}ds\right|$$

$$\leq |x_0 - z_0| + \frac{1}{\Gamma(\alpha)} \int_0^t \left[(1+N)L|x(\tau,x_0) - z(\tau,x_0)| - \frac{\delta}{\lambda}(1+N)L|x(\tau,x_0) - z(\tau,x_0)| \right] (t-s)^{\alpha-1} ds$$

$$= |x_0 - z_0| + \frac{1}{\Gamma(\alpha)} \left(1 - \frac{\delta}{\lambda}\right)(1+N) \int_0^t \left[L|x(\tau,x_0) - z(\tau,x_0)| \right] (t-s)^{\alpha-1} ds$$

$$= |x_0 - z_0| + \frac{1}{\Gamma(\alpha)} \Psi \int_0^t |x(\tau,x_0) - z(\tau,x_0)| (t-s)^{\alpha-1} ds$$

$$\leq |x_0 - z_0| + \frac{t^\alpha}{\Gamma(\alpha+1)} \Psi |x(t,x_0) - z(t,x_0)|$$

Let

$|x_0 - z_0| \leq \sigma |x(t,x_0) - z(t,x_0)|$, $\sigma < 1$.

We get:

$$\|T^* x(t,x_0) - T^* z(t,x_0)\| \leq \sigma |x(t,x_0) - z(t,x_0)| + \frac{t^\alpha}{\Gamma(\alpha+1)} \Psi |x(t,x_0) - z(t,x_0)|$$

So

$$\|T^* x(t,x_0) - T^* z(t,x_0)\| = \max_t |T^* x(t,x_0) - T^* z(t,x_0)|$$

Therefore:

$$\max_t |T^* x(t,x_0) - T^* z(t,x_0)| \leq \max_t \left[\sigma |x(t,x_0) - z(t,x_0)| + \frac{t^\alpha}{\Gamma(\alpha+1)} \Psi |x(t,x_0) - z(t,x_0)| \right]$$

$$= \max_t \left(\sigma + \frac{t^\alpha}{\Gamma(\alpha+1)} \Psi \right) |x(t,x_0) - z(t,x_0)|$$

From (1.8), then

$$\|T^* x(t,x_0) - T^* z(t,x_0)\| \leq \gamma \|x(t,x_0) - z(t,x_0)\|$$

and hence T^* is a contraction mapping on S. From theorem (1.8) [1] that is T^* has a unique continuous solution $x(t,x_0)$, $T^* x(t,x_0) = x(t,x_0)$ and so:

$$x(t,x_0) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t \left[f\left(s, x(s,x_0), h(s)x(s,x_0), \int_{-\infty}^s G(s,\tau) P(\tau, x(\tau,x_0), h(\tau)x(\tau,x_0)) d\tau \right) \right] (t-s)^{\alpha-1} ds$$

is a solution of (1.1)

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