Study of Two State Queueing Model with Transition Probability

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ABSTRACT

In this model we analyses a single server queuing model assuming that the customer arrive in batches of variable sizes and also serve in random size. We have considered a bulk arrival, bulk service queueing model with removable server. We derive the time dependent solution of mean number of arrivals and departure. Such situations are not uncommon in our daily life like horses in their stable, students in their class rooms before starting the class and after ending the class. Therefore using different parameters Mean unit length in system and queue can be determined.

I. INTRODUCTION

In this chapter we discuss a Markovian Queueing system in which service is performed by a single server in batches. The server is removed from the system as soon as the system becomes empty for a duration which is exponentially distributed. Such situations are not uncommon in our daily life. For example –The situation in a Cinema Hall, the person taking ticket one by one and enter the Hall, after the end of the movie person left the hall with random groups. This type of situation can also observe in computer system; we enter the data one by one i.e. single service and gain the out put in batches.

The following assumptions describe the system:-

1. Arrivals occurs under Poisson, law with parameter λ.
2. The queue discipline is FCFS.
3. The service time distribution is exponential with parameter μ.
4. The various stochastic processes in the system are statistically independent.
5. Services occur in batches of variable size. Service times are exponentially distributed.
6. The server will be removed from its service as soon as it becomes empty.

$P_{i,j,0}(t)$ = Probability that there are exactly c arrivals & is departures by time 't' & no vacation has occurred
$P_{i,s,1}(t)$ = Probability that there are exactly l arrivals & j departures by time t & a vacation has occurred.
$P_{ij}(t)$ = The probability that there are exactly i arrivals & j departures by time 't'
Initially $P_{0,0,0}(0) = 0$
$P_{0,0,1}(0) = 1$

The difference-differential equation governing the model is

$$P'_{i,0k}(t) = (\lambda+\mu)\,P_{i,0k}(t) + \lambda\, q_1 \sum_{\ell=0}^{1} a\,\ell\, P_{1-\ell,ok}(t)$$

$$+\,\mu_2\,\theta_2\, b_0 \sum_{\ell=0}^{1} \frac{a_e}{(b_0^{\ell})}\, P_{i-\ell 0,1-k}(t) \qquad \ldots(1)$$

$$P'_{i,0,0}(t) = -(\lambda+\mu)\,P_{i,0,0}(t) + \lambda\theta_1 \sum_{\ell=0}^{1} a_\ell\, P_{1-\ell,0,0}(t)$$

$$+\,\mu\, b_0\,\theta_2 \sum_{\ell=0}^{1} \frac{a_\ell}{b_0^{\ell}}\, P_{i-\ell,0,1}(t) \qquad \ldots(2)$$

$$P'_{i,0,1}(t) = -(\lambda+\mu)\,P_{i,01}(t) + \lambda\,\theta_1 \sum_{\ell=0}^{1} a_\ell\, P_{1,\ell,0,1}(t)$$

$$+ \mu\theta_2 b_0 \sum_{\ell=0}^{1} \frac{a_0}{b_0^{\ell}} P_{i-\ell,0,0}(t) \quad \ldots(3)$$

$$P^{1}_{i,k}(t) = -(\lambda+\mu)\, P_{i,ik}(t) + \lambda\theta_1 a_0 P_{i,j,k}(t) +$$

$$+ \mu\, a_0\theta_2 \sum_{\ell=0}^{1} b_1^{\ell}(1-\delta_{i0})^{\ell}\; P_{ii-\ell,1-k}(t) \quad \ldots(4)$$

$$P'_{1,0}(t) = -(\lambda+\mu)\, P_{ii0}(t) + \lambda\theta_1 a_0\, P_{i,0}(t) + \mu\, a_0\, \theta_2$$

$$\sum_{\ell=0}^{1} b_1^{\ell}(1-\delta_{i0})^{\ell}\; P_{i,i-\ell,1}(t) \quad \ldots(5)$$

$$P'_{i,1}(t) = -(\lambda+\mu)P_{i,1}(t) + \lambda\theta_1 a_0 P_{i,1}(t) +$$

$$+ \mu\theta_2 a_0 \sum_{\ell=0}^{1} b_1^{\ell}(1-\delta_{i0})^{\ell} P_{i,i-\ell,0}(t) \quad \ldots(6)$$

$$P'_{i,k}(t) = -(\lambda+\mu)P_{ijk}(t) + \lambda\theta_1 \sum_{\ell=0}^{1} a_\ell\, P_{i-\ell,k}(t) + \mu\theta_2 \sum_{\substack{\ell=0\\ m=0}}^{1} \frac{a_\ell b_m}{b_0 \delta_{1\ell}\delta_{0m}}$$

$$P_{i-\ell,j\text{-}m,1-k}(t) \quad \ldots(7)$$

$$P'_{i,j0}(t) = -(\lambda+\mu)P_{ij0}(t) + \lambda\theta_1 \sum_{\ell=0}^{1} a_\ell P_{i-\ell,D}(t) + \mu\theta_2 \sum_{\substack{\ell=0\\ m=0}}^{1} \frac{a_\ell b_m}{b_0 \delta_{i\ell}\delta_{0m}} P_{i-\ell,j-m,1}(t)$$

$$\ldots(8)$$

$$P'_{i,j1}(t) = -(\lambda+\mu)P_{i,j1}(t) + > \theta_1 \sum_{\ell=0}^{1} a_\ell P_{i-\ell,j1}(t)$$

$$+ \mu\theta_2 \sum_{\substack{\ell=0\\ m=0}}^{1} \frac{a_\ell b_m}{\delta_{1\ell} S_{0m}} P_{i-\ell,j-m,0}(t) \quad \ldots(9)$$

$$\overline{P}_{0,0,k}(s) = \frac{k}{S+(\lambda+\mu)-\lambda a_0\theta_1} + \frac{\mu a_0 \theta_2 \overline{P}_{0,0,1-k}(s)}{S+(\lambda+\mu)-\lambda a_0 Q_1} \quad \ldots(10)$$

$$\overline{P}_{i,0,k}(s) - P_{i0k}(0) = -(\lambda+\mu)\overline{P}_{i,0k}(s) + \lambda\theta_1 a_0 \overline{P}_{i,ok}(s) + \lambda\theta_1 a_1 \overline{P}_{i-1,0,k}(s)$$

$$+ \mu b_0\, \theta_2 \frac{a_0}{b_0} \overline{P}_{i,0,1-k}(s) + \mu\delta_0 \theta_2 \frac{a_1}{b_0} \overline{P}_{i-1,0,1-k}(s)$$

$$(S+(\lambda+\mu) - \lambda\theta_1 a_0)\overline{P}_{i,0,k}(s) = \lambda\theta_1 a_1 \overline{P}_{i-1,0,k}(s) + \mu a_0 \theta_2 a_1 \overline{P}_{i-1,0,1-k}(s)$$

$$a_1\; \overline{P}_{i-k,01-k}(s)$$

$$\Rightarrow \quad \overline{P}_{i,0,k}(s) = \frac{\lambda a_1 \theta_1}{S+(\lambda+\mu)-\lambda a_0\theta_1} \overline{P}_{ij,0k}(s) + \frac{\mu a_0 \theta_2}{S+(\lambda+\mu)+\lambda a_0\theta_1} \quad \ldots(11)$$

From equation (4)

$$S\,\overline{P}_{iik}(s) - P_{i,k}(0) = -(\lambda+\mu)\overline{P}_{i,k}(s) + \lambda\; a_0\theta_1 \overline{P}_{iik}(s)$$

$$+ \mu a_0\theta_2 \sum_{\ell=0}^{1} b_1^{\ell}(1-\delta_{i0})^{\ell}(1-\delta_{i0})^{\ell}\,\overline{P}_{i,i-\ell,1-k}(s)$$

$$(S+(\lambda+\mu) - \lambda a_0\theta_1)\overline{P}_{i,k}(s) \mu a_0 \theta_2 \sum_{\ell=0}^{1} b_1^{\ell}(1-\delta_{i0})^{\ell}\, \overline{P}_{i,i-\ell,1-k}(s)$$

$$\overline{P}_{iik}(s) = \frac{\mu a_0 \theta_2}{s+(\lambda+\mu)-\lambda a_0\theta_1} \sum_{\ell=0}^{1} b_1^{\ell}\, \overline{P}_{i,i-\ell,1-k}(s) \quad i \ge 1,\, k = 0, 1 \quad \ldots(12)$$

$$\overline{P}_{i,j,k}(s) = \frac{\lambda a_1 \theta_1}{S + (\lambda + \mu) - \lambda a_0 \theta_1} \overline{P}_{i-1,j,k}(s) + \frac{\mu}{S + (\lambda + \mu) - \lambda a_0 \theta_1} \sum_{\substack{m=0 \\ \ell=0}}^{1} \frac{a_\ell b_m}{b_0 S_{i\ell} \delta_{0n}} \overline{P}_{i-\ell, j-m, 1-k}(s) \quad \ldots(13)$$

$$\overline{P}_{ij}(s) = \overline{P}_{i,j,0}(s) + \overline{P}_{i,j,1}(s) \; j \le 1$$

$$\Rightarrow \quad \overline{P}_{ij}(s) = \sum_{k=0}^{1} \overline{P}_{i,jk}(s)$$

Solution of equation 10, 11, 12 & 13

$$\sum_{k=0}^{P} \sum_{i=0}^{\infty} \sum_{j=0}^{i} \overline{P}_{ijk}(s) = \frac{1}{S}$$

Taking Inverse transform of $\overline{P}_{ijk}(s)$ we get

$$P_{i,j,k}(t) = \frac{-(\lambda + \mu) a_1}{2} (1 - \delta_{i,j,1})(\theta_1^{\ell-1} + \theta_2 b_0) e^{-(T_p - \mu\theta_2 a_0 b_0)t} - (-1)^n (\theta_1^{p-1} - \theta_2 b_0) e^{-(T_p + \mu\theta_2 a_0 b_0)t} + \sum_{n=0}^{1} \frac{-(\lambda + \mu) a_1 (1 + (-1)^n \gamma}{2} e^{-(T_p + \mu\theta_2 a_0 b_0)t}$$

$$+ \frac{\mu\theta_2 b_1}{2} [e^{-(T_p + \mu\theta_2 a_0 b_0)t} - (-1)^\ell e^{-(T_p - \mu\theta_2 a_0 b_2)t} \times (a_1 P_{i-1,j+k}(t) + a_0 P_{ij-1,k}(t))] \quad k = 0, 1$$

$$\sum_{k=0}^{1} \sum_{i=0}^{\infty} \sum_{j=0}^{i} P_{i,jk}(t) = 1$$

Also $L^{-1}\left(\frac{1}{s}\right) = 1$

Hence a verification

$\Rightarrow$ $P_{i.}(t)$, the probability that exactly i, units arrive by time t is obtained by taking the Laplace inverse transform of $\overline{P}_{i.}(s)$

$$\overline{P}_{i.}(s) = \sum_{k=0}^{1} \sum_{j=0}^{i} \overline{P}_{i,j,k}(s)$$

$$\overline{P}_{i.}(s) = \frac{(\lambda a_1)^i}{(S + \lambda a_1)^{i+1}}$$

$$L^{-1} \overline{P}_{i.}(s) = L^{-1} \frac{(\lambda a_1)^i}{(S + \lambda a_1)^{i+1}}$$

$$\Rightarrow \quad P_{i.}(t) = \frac{(\lambda a_1 t)^i}{\lfloor i} e^{-\lambda a_1 t}$$

The arrivals follow a Poisson distribution with rate λa_1, the probability of the total number of arrivals is not effected by the vacation time of the server.

$\Rightarrow$ **Mean Number of arrivals**

The mean number of arrivals by time t is obtained by taking Laplace Transform of

$$\sum_{i=0}^{\infty} i \, \overline{P}_{i.}(s) = \frac{\lambda a_1}{S^2}$$

The inverse transform of the mean number of arrivals by time t is

$= va_1 t$

2. $P_{.j}(t)$, the probability of j departures by time t is obtained by taking the laplace inverse transform of

$$\overline{P}_{.j}(s) = \sum_{k=0}^{1} \sum_{i=j}^{\infty} \overline{P}_{ijk}(s)$$

$$\overline{P}_{.j}(s) = \frac{\mu\theta_2 b_1}{(S + \mu\theta_2 b_1)} \left[\overline{P}_{.j-1}(s) + \sum_{k=0}^{1} \overline{P}_{i,jk}(s) \right]$$

$$\overline{P}_{.j}(t) = \mu\theta_2 b_1 e^{-\mu\theta_2 b_1 t} \left[P_{.j-1}(t) + \sum_{k=0}^{1} P_{i,j,k}(t) \right]$$

3. $P_n(t)$, the probability of n units in the system by time t is obtained by taking the laplace inverse transform of

$$\overline{P}_n(s) = \sum_{k=0}^{1} \sum_{j=0}^{\infty} \overline{P}_{j+n,j,k}(s)$$

$$\Rightarrow \quad \overline{P}_n(s) = \sum_{j=0}^{\infty} \left[\overline{P}_{j+n,j0}(s) + \overline{P}_{j+n,j,1}(s) \right]$$

Substituting the value of $\overline{P}_{j+n,j,0}(s)$ and $\overline{P}_{j+n,j,1}(s)$ for all j from equation 10, 11, 12 and 13.

We get

$$\{(S + T_p) - \mu\theta_2 b_0 a_0 - \mu\theta_2 b_1 a_1\} \overline{P}_n(s)$$

$$= \left(\lambda\theta_1 a_1 + \frac{\mu\theta_2 a_1 b_0}{b_0 \delta_{n1}} \right) \overline{P}_{n-1}(s) + \mu\theta_2 a_0 b_1 \overline{P}_{n+1}(s) \; n > 0$$

$$\overline{P}_0(s)(S + \lambda a_1) = 1 + \mu\theta_2 a_0 b_1 \overline{P}_1(s)$$

$$\overline{P}(z,s) = \sum_{n=0}^{\infty} z^n \overline{P}_n(s)$$

Such that the summation's convergent in and on the unit circle for $1 \leq 1$.

$$\{(S + T_p) - \mu\theta_2 b_0 a_0 - \mu\theta_2 b_1 a\} (\overline{P}(z,S) - \overline{P}_0(s)) + (S + -(\lambda+\mu)a_1) \overline{P}_0(s)$$

$$= 1 + \lambda\theta_1 a_1 \sum_{n=1}^{\infty} \overline{P}_{n-1}(s) z^n + \mu\theta_2 a_1 b_0 \left(\sum_{n=1}^{\infty} \frac{\overline{P}_{n-1}(s) z^n}{b_0 S_{n1}} \right)$$

$$+ \frac{\mu\theta_2 a_0 b_1}{z} \sum_{n=0}^{\infty} \overline{P}_{n+1}(s) z^{n+1}$$

Hence

$\overline{P}(z,S)$ Converges in the region $|z| \leq 1$,

$$P(z) = \lim_{s \to 0} S\,\overline{P}(z,S)$$

$$P(z) = \frac{\mu\theta_2 b_1 (a_1 z^2 - z(a_1 - a_0) - a_0) \left\{ 1 - \frac{a_1}{\theta b_1} \right\}}{(\lambda + \mu) a_1 (\theta_1 + \theta_2 b_0) z^2 + \{-(\lambda + \mu) a_1 + \mu\theta_2 b_1 (a_0 - a_1)\} z - \mu\theta_2 a_0 b_1}$$

$$P(z) = \left(1 - \frac{a_1}{a_2 b_1} \right) (a_0 + z_1 a_1) \left\{ a_0 + z\left(a_1 - \frac{a_1}{\theta_2 b_1}\right) \right\}^{-1}$$

Selecting the coefficients of z^n

$$P_n = \left(1 - \frac{a_1}{\theta_2 b_1} \right) \left(\frac{a_1}{\theta_2 b_1} \right)^n a_0 \left(\frac{1 - \theta_2 b_2}{a_0} \right)^{n-1} \quad n > 0$$

The mean number of units in the system

$$L = \sum_{n=0}^{\infty} n \quad P_n$$

$$L = \frac{\left(1 - \frac{a_1}{\theta_2 b_1}\right)\left(\frac{a_1}{\theta_2 b_1}\right) a_0^3}{\left\{a_0 - \frac{a_1}{\theta_2 b_1}\left(1 - \theta_2 b_1\right)\right\}^2}$$

The mean number of units in the queue

$$L_q = L - \frac{a_1}{\theta_2 b_1}$$

$$L_q = \frac{\left(1 - \frac{a_1}{\theta_2 b_1}\right)\left(\frac{a_1}{\theta_2 b_1}\right) a_0^3}{\left\{a_0 - \frac{a_1}{\theta_2 b_1}(1 - \theta b_2)\right\}^2} - \frac{a_1}{\theta_2 b_1}$$

This shows the mean number of arrival in queue.

COCLUSION

It is clear from the equation that if the arrival rate is increases than queue length is also increases. Also**,** it is clear that if the service rate is increases than queue length decreases.

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