Mathematical Modeling of Glucose Insulin Dynamics

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ABSTRACT

The glucose-insulin dynamics model describes the interaction between blood glucose levels and insulin activity in the human body. This study simulates these dynamics using coupled differential equations, solved numerically with the 4th-order Runge-Kutta method. Key parameters include glucose utilization and insulin decay rates with external events such as food intake and insulin injection modeled as time-dependent exponential inputs. The simulation demonstrates a glucose spike following food intake and a delayed insulin peak after injection, reflecting their physiological roles. These results provide insights into glucose regulation and have applications in diabetes management and treatment optimization.

Keywords: Mathematical Model, Runge-Kutta Method, Numerical Simulation, Insulin Activity, Diabetes Management, Food Intake Model, Insulin Injection Simulation

INTRODUCTION

The regulation of blood glucose levels is a critical physiological process that ensures metabolic stability in the human body. Glucose serves as the primary energy source while insulin, a hormone secreted by the pancreas, facilitates glucose uptake by cells and maintains homeostasis. Disruptions in this balance such as those observed in diabetes can lead to severe health complications necessitating effective management strategies.

Mathematical modeling of glucose-insulin dynamics provides a framework for understanding the interaction between these two key variables. By representing the system with differential equations researchers can simulate real-life events such as food intake, which raises blood glucose levels, and insulin injections, which help reduce them. These models are essential for studying physiological responses and optimizing treatments in diabetes management.

Previous studies have extensively utilized mathematical approaches for simulating glucose-insulin behavior. For example, Bergman et al. [11] introduced the minimal model for glucose and insulin dynamics, which remains a cornerstone in this domain. Meanwhile Li et al. [12] extended the model to explore the effects of external insulin injections. These studies underscore the importance of using computational techniques to bridge theoretical models with clinical applications.

This study aims to simulate glucose-insulin dynamics using a numerical approach based on the 4th-order Runge-Kutta method. By incorporating events like food intake and insulin injections into the model, the simulation replicates their effects on blood glucose and insulin levels over time. The results offer insights into the physiological interactions between glucose and insulin, providing a basis for improving therapeutic interventions and dietary planning. Diabetes mellitus commonly known as diabetes is a collection of symptoms it indicating irregularities in metabolic function. This disorder is often caused by a combination of genetic, environmental and lifestyle factors that lead to abnormal blood sugar levels a condition known as hyperglycemia [1].

In all organism glucose is an essential source of energy, derived primarily from carbohydrate intake. During digestion carbohydrate are broken down into glucose which enters the blood stream. Insulin, a hormone produced by the pancreas, regulates glucose levels by facilitating its absorption into body cells. When this process is disrupted either due to insufficient insulin production or reduced insulin sensitivity diabetes may develop, leading to an accumulation of glucose in the blood [4].

Diabetes is primarily categorized into two types. Type 1 diabetes is characterized by the pancreas's inability to produce insulin, often requiring patients to rely on external insulin administration. In contrast, type II diabetes is linked to the body reduced responsiveness to insulin production. For healthy individuals blood glucose concentration typically range between 70 to 150 mg/dL. Deviations from this range can lead severe complications. Prolonged hyperglycemia in diabetic patients may result in long term health issues, such as cardiovascular disease, kidney damage, retinal disorders, limb amputations [8] etc. On the other hand, improper insulin administration can lead to hypoglycemia, with glucose levels dropping below 50 mg/dL resulting in potentially causing life-threatening consequences such as coma or sudden death [2]. Patient suffering from type 1 diabetes mellitus are required to take insulin multiple times a day by artificial method such as injection or pills [9]. On other hand type 2 diabetes turns out when secreted insulin is not capable of attaching with body cells to take in and hold the additional sugar from the blood stream [3]. Clearly to control glucose levels simple and effective interventional measures like regular exercise, eating healthy, eating meals on time can play a significant role in reducing the risk of developing the chronic disorder diabetes mellitus.

The objective of this study is to simulate and analyze the glucose –insulin dynamics in the human body using a mathematical model. The goals included understanding the impact of food and insulin injections on blood glucose and insulin levels over time, evaluating the physiological interaction between glucose utilization and insulin activity, developing a numerical framework using the 4th order Runge-Kutta method to accurately solve the coupled differential equations governing the system, providing insights for potential applications in diabetes management, including optimizing insulin therapy and dietary interventions. For developing a **mathematical model** for blood sugar detection that incorporates relevant dynamics, we considered key aspects of glucose-insulin interaction and system dynamics. This study provides a simplified approach, building from basic glucose-insulin dynamics, which could indirectly use Dirac delta functions in specific scenarios.

MATHEMATICAL MODEL

The system can be represented as:

1. **Glucose Level :**
   The change in glucose level with time cam be represented by a linear differential equation given by

   $$\frac{dG}{dt} = -k_1 G + \sum_{i=1}^{n} \alpha_i \delta (t - t_i) - I(t) \tag{1}$$

2. **Insulin Dynamics**:
   The variation in glucose level with time cam be represented by a linear differential equation given by

   $$\frac{dI}{dt} = -k_2 I + \sum_{j=1}^{m} \beta_j \delta(t - t_j) \tag{2}$$

We can see the impact food taken by human body and insulin I(t) activity with respect to time on blood glucose G(t) level.

MATHEMATICAL DESCRIPTION OF MODEL

Equation of glucose level:
Blood glucose G(t) level can be modeled by the ordinary differential equation

$$\frac{dG}{dt} = -K_1 G + R(t) - I(t) \tag{3}$$

Where,

k_1 is the rate of use the natural glucose.
R(t) is increment in glucose from food.
I(t) is lack in glucose due to insulin.

Intake of food:
Food causes a rapid increase in glucose levels. This can be modeled with a delta function given by

$$R(t) = \alpha \cdot e^{-\beta (t - t_m)} H(t - t_m) \tag{4}$$

Where,

α: maximum glucose from uptake from food.

β: The rate at which glucose is absorbed from food.

t_m : Meal time and

H (t - t_m): Heaviside step function is activated on t $\geq t_m$.

Response of Insulin I(t):

The insulin levels can be modeled by the following equation,

$$\frac{dI}{dt} = -k_2 \, I(t) + S(t) \tag{5}$$

With, k_2 : Decay rate of the insulin and

S(t): Insulin secretion or injection rate.

Initial conditions:

Initial glucose level: G(0) = G_0

Initial insulin level: I(0) = I_0

Simulation Process:

Step 1: Set the standard starting value.

$k_1 = 0.1 min^{-1}$: Natural glucose utilization.

$k_2 = 0.5 \, min^{-1}$: Insulin decay.

$\alpha = 1.00$ mg/dL: Glucose increases from food.

$\beta = 0.02 \, min^{-1}$: Decrease in the effect of the food.

S(t) = $10e^{0.1 \, (t-t_m)}$ H (t - t_i) : Insulin at t_i.

MATHEMATICAL MODEL OVERVIEW

Glucose Dynamics G(t)

The rate of change of blood glucose G(t) is given by the equation (3)

- k_1: Natural glucose utilization rate.
- R(t): Rate of glucose intake from food.
- I(t) I: Rate of glucose removal due to insulin.

Insulin Dynamics I(t)

The insulin level I(t) changes as the equation (5),

- k2 : Insulin decay rate.
- S(t): Rate of insulin secretion or injection.

Input Functions

- **Food Intake R(t)**: Food causes a sharp increase in glucose levels, modeled as the equation (4),

   α: Maximum glucose increase from food.
   β: Decay rate of glucose spike.
   tm: Time of food intake and
   H(t − tm): Heaviside step function to activate food intake at tm.

- **Insulin Injection (S(t))**: Insulin injection leads to a spike in insulin levels:

   S(t) = $\gamma e^{-\delta(t-t_i)}$ H(t - t_i)

With γ: Magnitude of insulin injection.

δ: Decay rate of insulin spike and

t_i: Time of insulin injection.

Simulation Setup

- Time: t = [0, 300] minutes.
- Initial conditions: G(0)=90 mg/dL, I(0)=5 units.
- Parameters: k_1=0.1, k_2=0.05, α=100, γ=10, etc.

Expected Output

1. **Glucose Curve G(t)**:
   Rises after food intake (t=50 min).
   Falls due to insulin activity after injection (t=70 min).
2. **Insulin Curve I(t)**:
   Peaks after injection and then decays over time.

4. **Numerical Integral (Runge - Kutta) Method**

The **Runge-Kutta method** is a numerical technique to solve differential equations like the glucose-insulin dynamics model. Below is the implementation of the model using the 4th-**order Runge-Kutta (RK4) method** for numerical integration. The algorithm to apply this method is as under-

Algorithm: RK4 Method

For a differential equation of the form:

$$\frac{dy}{dt} = f(t, y)$$

the RK4 method updates the solution y at each step $t_n \rightarrow t_{n+1}$ using:

$y_{n+1}, = y_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

Where:

- $k_1 = h f(t_n, y_n)$
- $k_2 = h f(t_n + \frac{h}{2}, y_n + k_1/2)$
- $k_3 = h f(t_n + \frac{h}{2}, y_n + k_2/2)$
- $k_4 = h f(t_n + h, y_n + k_3)$

IMPLEMENTATION: GLUCOSE-INSULIN DYNAMICS

Expected Output:

- **Glucose (G(t))**:
  - Peaks after food intake (t = 50 min).
  - Falls due to insulin activity after injection (t = 70 min).
- **Insulin (I(t))**:
  - Peaks after injection and gradually decay

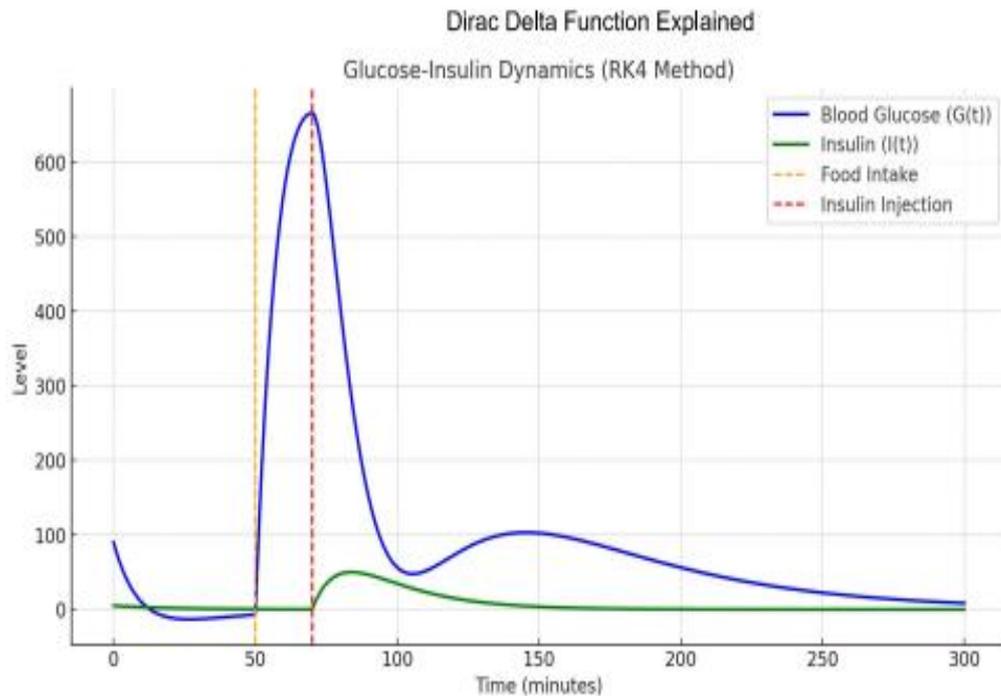The graph of glucose-insulin dynamics simulated using the Runge-Kutta (RK4) method is shown in Fig. (1).

Fig. (1)

- **Blood Glucose (Blue Line)**: Peaks after food intake at t=50 minutes and decreases due to insulin activity.
- **Insulin (Green Line)**: Peaks after the injection at t=70 minutes and gradually decays.

The orange and red dashed lines indicate the times of food intake and insulin injection, respectively.

RESULTS SUMMARY OF THE GLUCOSE-INSULIN DYNAMICS MODEL SOLUTION

1. Model Overview:

- **Glucose (G)** and **Insulin (I)** dynamics are governed by a set of coupled differential equations:

$$\frac{dG}{dt} = -k_1 G + R(t) - I \quad (6)$$

$$\frac{dI}{dt} = -k_2 I + S(t) \quad (7)$$

Where:

- k_1: Glucose utilization rate.
- k_2: Insulin decay rate.
- R(t): Glucose input from food.
- S(t): Insulin secretion from injections.

Events in the Simulation

- **Food intake** at t=50 minutes introduces glucose into the bloodstream, modeled as an exponential decay **R(t)** $\propto e^{-\beta t}$
- **Insulin injection** at t=70 minutes provides insulin into the bloodstream, also modeled as an exponential decay **S(t)** $\propto e^{-\gamma t}$

Numerical Method:

- The **4th-order Runge-Kutta (RK4)** method was used to numerically solve the coupled differential equations.
- The solution provides glucose (G(t)) and insulin (I(t)) levels over time with high accuracy.

Results:

o **Glucose (G(t)):**

▪ Peaks shortly after the food intake event, representing a spike in blood sugar levels.
▪ Gradually declines due to the combined effects of natural glucose utilization ($-k_1 G$) and insulin activity ($-I$).

o **Insulin (I(t)):**

▪ Peaks after the injection event, representing the presence of active insulin in the bloodstream.
▪ Gradually decays due to natural insulin breakdown ($-k_2$) I.

Graph Insights:

o The **glucose spike** occurs right after food intake and declines following the insulin injection.
o The **insulin curve** shows a delayed peak and a gradual decay as the body processes the injected insulin.
o Together, these curves demonstrate the interaction between glucose levels and insulin activity.

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