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Grundy Colouring and B-Colouring for Shadow and Splitting Graph of Star and Comb Graph

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ABSTRACT

In this paper we have discussed about the different colourings of shadow graph of star graph . Also we have found the chromatic number $\varphi(G)$, b-chromatic number χ(G) and Grundy colouring Γ(G).

Keywords: Proper colouring, b-colouring, Grundy colouring, shadow graph, star graph, comb graph.

INTRODUCTION

Graph colouring is a colourful concept in graph theory which has many application in real life situations. In 1979 Grundy number was defined initially by Christen and Selkow[4]. Manounchehrzaker[6] explored the results on the Grundy chromatic number of graphs andobtained the inequalities.

Victor Campose,et.al[8] analyzed the bounds on the Grundy number of products like direct, strong and lexigographic of graphs. In the year of 1999 Irwing and Manlone[9] gave an idea about b-colouring and discussed the bounds for b-chromatic number of graphs. Alkhateeb[2] characterized the b-colouring of various graphs. S.K. Vaidhya[10] estimated the b-chromatic number of shadow and splitting graph of path graph. N. Parvathi.et.al[11] estimated the Grundy colouring and b-colouring of join of path and complete graph.

PRELIMINARIES

Definition 1: [1]
A Graph is an ordered pair G=(V,E) comprising V the set of vertices (also called nodes or points,E the set of edges (also called links or lines)

Definition 2: [3]
A Proper colouring of the graph assigns colours to the vertices, edges or both so that proximal elements are assigned distinct colours. The chromatic number $\boldsymbol{\varphi(G)}$ of G is the minimum K forwhich G is k-chromatic.

Definition 3: [2]
The b-chromatic number χ(G) of a Graph G is the largest positive integer K such that G admits a proper K-colouring in which every colour class has a representative adjacent to atleast one vertexin each of the other colour classes. Such a colouring is called b-colouring.

Definition 4: [6]

A Grundy colouring of order k of a graph G is a k-colouring of G with colours 1,2,….k such thatfor each vertet x the colour of x is the smallest positive integer not used as a colour on any neighbour of x in G. The Grundy number $\Gamma(\boldsymbol{G})$ is the largest integer k for which G has a Grundycolouring of order k.

Definition 5: [10]

The shadow graph of the connected Graph G is constructed by taking two copies of G say G”and G’. Join each vertex u’ in G’ to the neighbours of the corresponding vertex u” in G”.

Definition 6: [12]
The star graph s_n of order n is a tree on n nodes with one node having vertex degree (n-1) andthe other (n-1) node having vertex degree 1 . star is a complete biograph $k_{1,n}$

Definition 7: [5]
$\boldsymbol{P_n}$ be a path graph with n vertices. The comb graph is defined as $\boldsymbol{P_n} \odot \boldsymbol{K_1}$. It has 2n vertices and2n-1 edges.

Definition 8: [10]
The splitting graph of a graph G, is obtained by adding a new vertex u' corresponding to each vertex u of G such that N(u)=N(u') where N(u) and N(u') are the neighbourhood sets of u and u'.

Result 1: [6]
Let G be a graph set $\boldsymbol{\theta(G) = k}$ and $(\boldsymbol{G}) = m$, thenΓ(G) $\leq \frac{\boldsymbol{k+1}}{\boldsymbol{2}} m$.

Grundy number for shadow graph of star and comb graphsTheorem1:
For n$\geq \boldsymbol{2}$, chromatic number of G is 2 where G is shadow graph of star graph.

Proof:
Let G be a shadow graph of star graph. Vertex set V(G)= {u,$u_1, u_2 \ldots \ldots \ldots u_n, u' u_1'$ .... u_n'}
For proper coloring, let us consider the color set {1,2} Since u and $u'$are non-adjacent they are assigned by colour 1 Since $u_1, u_2 \ldots \ldots \ldots u_n$ are adjacent to u then it is coloured by 2 and u_1, u_2 .... $u_n$are adjacent to $u'$and not adjacent to u_1, u_2 .... u_n it is colored by 2.
Hence $\varphi(G)$=2

Theorem 2:
Let G be a shadow graph of star graph then χ(G)=2 for all n

Proof:
Let G be a shadow graph of star graph on 2n+2 vertices u,$u_1, u_2 \ldots \ldots \ldots u_n, u', u_1'$ .... u_n'
Here the indepensdent sets are {u,u' }..................................{u_n,u_n'}
Hence $\alpha(G)$= 2, ω(G) = 2, n(G) = 2n+2 , $\Delta(G) = 2n$

From reference 4,we know that, χ(G)$\leq \frac{\Delta(G)+\omega(G)+n(G)+2}{2}$

Let us prove this theorem by induction on n.

Step 1:
When n =2

$$\chi(G) \leq \frac{\Delta(G)+\omega(G)+n(G)+2}{2} \leq \frac{2n+2+2n+2+2}{2} \leq \frac{4n+6}{2} \leq 7$$

Here V(G) = {u,u_1, u_2,u', u_1', u_2'}
For b-coloring, let us consider the color set {1,2} such that cdv(1)= u,cdv(2) = u_1
Hence χ (G) = 2

Step 2:

When n=3

$$\chi(G) \leq \frac{\Delta(G)+\omega(G)+n(G)+2}{2} \leq \frac{2n+2+2n+2+2}{2} \leq \frac{4n+6}{2} \leq 9$$

Here V(G) = {u,u_1, u_2,u_3,$u'u_1'$, u_2', u_3'}
For b-coloring, let us consider the color set {1,2} such that cdv(1)= u, cdv(2) = u_1
Hence χ (G) = 2

Step 3 :

$$\chi(G) \leq \frac{\Delta(G)+\omega(G)+n(G)+2}{2} \leq \frac{4n+6}{2}$$

Now we consider the vertex set

V(G) = {u,u_1, u_2 ... ... u_n,u_1'................................u_n'}
Now we assign the existing colors to the additional vertices u_n and u_n'
Hence b-chromatic number of shadow graph of star graph is 2.

Theorem 3:
Grundy chromatic number of G is 2n+1 where G is a shadow graph of star graph for all n.

Proof:
Let G be a shadow graph of star graph with V(G) = {u,u_1, u_2 ... ... u_n,u_1' ....u_n'}
n(G) = 2n+2,$\Delta(G) = 2n$, ω(G) = 2

By result 1,Let G be a graph set $\theta(G) = k$ and $\omega(G) = \omega$, thenΓ(G) ≤ $\frac{k+1}{2}\omega$.

By considering following cases the proof can be obtain.

Step 1: When n =2
Here V(G) = {u,u_1, u_2,u', u_1', u_2'} then ω(G) = 2 ,$\theta(G) = 5$, n(G) = 6
Γ(G) ≤ $\left(\frac{5+1}{2}\right) 2 \leq 6$

Since there are 6 vertices in the graph, by the definition of Grundy coloring Γ(G) = 5

For Grundy coloring, let us consider the color set {1,2,3,4,5} such that
c(u) =1 , c(u') = 1, c(u_1) = 2 ,c(u_1') = 4,c(u_2) =3, c(u_2') = 5Hence Γ(G)= 5

Step 2: When n =3
Here V(G) = {u,u_1, u_2,u_3,$u'u_1'$, u_2', u_3'} where ω(G) = 2 ,$\theta(G) = 7$, n(G) = 8Γ(G) ≤ $\left(\frac{7+1}{2}\right) 2 \leq 8$

Since there ~~are 6~~ vertices in the graph, by the definition of Grundy coloringΓ(G) = 5

For Grundy coloring, let us consider the color set {1,2,3,4,5,6,7} such that

c(u) =1, c(u') = 1,c(u_1) = 2, c(u_1') = 5,c(u_2) =3 ,c(u_2') = 6,c(u_3) =4, c(u_3') = 7Hence Γ(G)= 7. Hence Grundy chromatic number of G is 2n+1

Theorem 4:

For n≥ 2 chromatic number of G is 2 where G is shadow graph of comb graph.

Proof:
Let G be a shadow graph of comb graph .The vertex set V(G) = {u_1, u_2 ... ... u_n,u_1'..u_n'}
For Proper coloring, let us consider the color set {1,2}

Therefore, u_2, u_4, u_6, u_8&u_2', u_4', u_6', u_8' is not adjacent to u_1, u_3, u_5, u_7&u_1', u_3', u_5', u_7'
Then $u_2, u_4, u_6, u_8, u_2', u_4', u_6', u_8'$ are coloured by 1.
$u_1, u_3, u_5, u_7, u_1', u_3', u_5', u_7'$are coloured by 2. Hence $\varphi(G) = 2$

Theorem 5:

Let G be a shadow graph of comb graph then χG)=2 for all n≥ **2** vertices.

Proof:

Let G be a shadow graph of comb graph with 4n vertices. Here independent sets are $\{u_2, u_4, u_6, u_8, u_2', u_4', u_6', u_8'\}$ and
$\{u_1', u_3', u_5', u_7', u_1', u_3', u_5', u_7'\}$
Hence $\alpha(G)$= 2,ω(G) = 2 , n(G) = 4n ,$\Delta(G) = 6$

From (ref 4), χ (G)≤ $\frac{\Delta(\boldsymbol{G})+m(\boldsymbol{G})+\boldsymbol{n}(\boldsymbol{G})+\boldsymbol{2}}{\boldsymbol{2}}$

Let us prove this theorem by induction on n

Step 1: When n=2

χ (G)≤ $\frac{\Delta(G)+\omega(G)+n(G)+2}{2} \leq \frac{6+2+4n+2}{2} \leq \frac{10+4n}{2} \leq 9$

Here V(G) = $\{u_1, u_2, u_3, u_4, u_1', u_2', u_3', u_4'\}$
For b-colouring, let us consider the colour set {1,2} such that cdv(1)= u_2,cdv(2) = u_3
Hence χ(G) = 2

Step 2: When n=

χ(G)≤ $\frac{\Delta(G)+\omega(G)+n(G)+2}{2} \leq \frac{6+2+4n+2}{2} \leq \frac{10+4n}{2} \leq 11$

Here V(G) = $\{u_1, u_2, u_3, u_4, u_5, u_6 u_1', u_2', u_3' u_4' u_5' u_6'\}$
cdv(1)=u_1,cdv(2) = u_2

Step 3

χ (G)≤ $\frac{\Delta(G)+\omega(G)+n(G)+2}{2} \leq \frac{6+2+4n+2}{2} \leq \frac{10+4n}{2}$

Consider the shadow graph of comb graph in which the vertex setV(G) = $\{u_1, u_2 \ldots\ldots\ldots u_n, u_1' \quad u_n'\}$
Now we assign existing colors to $\boldsymbol{u_n}$and $\boldsymbol{u_n}'$
Hence the b-chromatic number of shadow graph of comb graph is 2.

Theorem 6

Grundy chromatic number of G is 2n+1 where G is a shadow graph of comb graph for all n≥ **2**

Proof:

Let G be a shadow graph of comb graph with V(G) = $\{u_1, u_2 \ldots\ldots\ldots u_n, u_1' \ldots\ldots\ldots u_n'\}$
Where n(G) = 4n, $\Delta(G) = 6$, ω(G) = 2
By result 1 Γ(G) ≤ $\frac{k+1}{2}\omega$ where k = $\theta(G)$ and ω = ω(G)
By considering following cases the proof can be obtain.

Step 1: When n =2

Here V(G) = $\{u_1, u_2, u_3, u_4, u_1', u_2', u_3', u_4'\}$
ω(G) = 2 ,$\theta(G) = 6 = k$,n(G) = 8
Γ(G) ≤ ($\frac{\boldsymbol{6+1}}{}$ 2)

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By the definition of Grundy coloring, Γ(G) =5

For Grundy coloring, let us consider the color set {1,2,3,4,5} such that

C(u_1) = 1,c(u_1') = 1,c(u_2) =4,c(u_2') = 5,c(u_3) = 1,c(u_3')=1,c(u_4)=2 ,c(u_4')=3Hence Γ(G)= 5
Step 2:When n =3

Here V(G) = {$u_1, u_2, u_3,, u_4, u_5,, u_6$, $u_1', u_2', u_3', u_4', u_5', u_6'$}
ω(G) = 2 ,$\theta(G) = 10$, n(G) = 12
$\Gamma(G) \leq (\frac{10+1}{2}) 2 \leq 11$

For Grundy coloring, let us consider the color set {1,2,3,4,5,6,7} such that c(u_1) = c(u_5) = c(u_1') = c(u_5') =1, c(u_2) =6 , c(u_2') = 7, c(u_3) =1, c(u_3') = 1c(u_4) =4, c(u_4') = 5, c(u_6) =2, c(u_6') = 3 ∴Γ(G)=7
Hence Grundy chromatic number of G is 2n+1 for all n≥ 2 verticesHence Γ(G)= 7

Grundy number for Splitting graph of star and comb graphsTheorem 7:
If G is a splitting graph of star graph then φ(G) = 2 for all n≥ 2

Proof :
Let G be a splitting graph of star graph vertex set V(G) = { u, u_1,…..u_n, $u'u_1'$,…… u_n'}For proper coloring, let us consider the color set {1,2}
∴ u and u' are non-adjacent they are assigned by colour 1

∴ u_1,…..u_n are adjacent to u then it is colored by 2 and u_1',…… u_n' are adjacent to u then theyare assigned by colour 2

Hence φ(G) = 2

Theorem 8:
Let G be a splitting graph of star graph χ (G) = for all n

Proof:
Let G be a splitting graph of star graph 2n+2 vertices { u, u_1,…..u_n, $u'u_1'$,…… u_n'}Here the independent sets are {u, , u'} {, u_1, u_1'}… ....{u_n, u_n'}
Hence α(G) = 2, ω(G) = 2, n(G) = 2n+2, Δ(G) = 2n

From reference (4) we know that $\chi(G) \leq \frac{\Delta(G)+\omega(G)+n(G)+2}{2}$

Let us prove this theorem by induction on n

Step 1: When n =2

$\chi(G) \leq \frac{\Delta(G)+\omega(G)+n(G)+2}{2} \leq \frac{2n+2+2n+2+2}{2} \leq \frac{4n+6}{2} \leq 7$

Here = { u, u_1,u_2, $u'u_1'$,u_2'}

For b-coloring let us consider the colour set such that cdv(1) = u , cdv(2) = u_1
Hence χ(G)=2

Step 2 : When n=3

χ(G)≤ $\frac{\Delta(G)+\omega(G)+n(G)+2}{2} \leq \frac{2n+2+2n+2+2}{2} \leq \frac{4n+6}{2} \leq 9$

Here = { u, u_1,u_2, u_3, $u'u_1'$,u_2', u_3'}}
For b-colouring let us consider the colour set {1,2} such that cdv(1) = u , cdv(2) = u_1'
Hence χ(G)=2

Step 3 : When n=4

G)≤ $\frac{\Delta(G)+\omega(G)+n(G)+2}{2} \leq \frac{2n+2+2n+2+2}{2} \leq \frac{4n+6}{2}$

Now consider the vertex set

V(G) = { u, u_1,…..u_n, $u'u_1'$,…… u_n'}
Now we assign the exisiting colours to the additional vertices ,u_n and u_n'
Hence b-chromatic number of splitting graph of star graph is 2.

Theorem 9:
Grundy chromatic number of G is 2n+1 where G is the splitting graph of star graph for all n≥2

Proof:

Let G be a splitting graph of star graphV(G) = { u, u_1,…..u_n, $u'u_1'$,…… u_n'}n(G) = 2n+2 , $\Delta(G) = 2n$, ω(G) = 2
By result 1,Γ(G) ≤ $\frac{k+1}{2}\,\varpi$ where

K = θ (G),ω= ω(G)

By considering the following cases the proof can be obtained.

Case (i) When n =2

Here V(G) = { u, u_1,u_2, u', u_1',u_2'}n(G) = 6 , $\Delta(G) = 4$, ω(G) = 2
By theorem Γ(G) ≤ $\frac{4+1}{2}\, 2 \leq 5$

Since there are 6 vertices in the graph by the defn of Grundy colouring Γ(G)=5For Grundy colouring let us consider the colour set {1,2,3,4,5} such that
C(u) = 1, c(u')=1,C(u_1) =2, c(u_1') = 4,C(u_2) =3, c(u_2') =5Hence Γ(G)=5
Case 2: when n = 3

Here V(G) = { u, u_1,u_2, u_3, u', u_1',u_2', u_3'} $\theta(G) = 6$,ω(G) = 2By theorem Γ(G) ≤ $\frac{6+1}{2}\, 2 \leq 7$

Since there are 8 vertices in the graph by the defn of Grundy colouring Γ(G)=7For Grundy colouring let us consider the colour set {1,2,3,4,5,6,7} such that C(u) = 1,c(u')=1,C(u_1) =2,c(u_1') = 4,C(u_2) =3,c(u_2') =5,C(u_3) = 4, c(u_3')=7
Hence Γ(G)=7

Hence Grundy chromatic number is 2n+1 for all vertices n≥ 2 vertices.

Theorem 10:
For n ≥2 chromatic number of G is 2 where G is splitting graph of comb graph.

Proof:

Let G be a splitting graph of comb graph

The vertex set V(G) = $\{u_1, \ldots u_n, u_1', \ldots u_n'\}$
For proper colouring let us consider the colour set {1,2}

Since u_2, u_4, u_6, u_8 and u_2', u_4', u_6', u_8' is not adjacent to u_1, u_3, u_5, u_7 and u_1', u_3', u_5', u_7'
$u_2, u_4, u_6, u_8, u_2', u_4', u_6', u_8'$ are coloured by 1 and
u_1, u_3, u_5, u_7 and u_1', u_3', u_5', u_7' are coloured by 2Hence φ(G) = 2

Theorem : 11

Let G be a splitting graph of comb graph then χ(G)=2 for all n≥ 2 vertices.

Proof:

Let G be a splitting graph of comb graph on 4n vertices.

Here independent sets are $\{u_1, u_3, u_5, u_7, u_1', u_3', u_5', u_7'\}$and
$\{u_2, u_4, u_6, u_8, u_2', u_4', u_6', u_8'\}$ therefore $\alpha(G) = 2, \omega(G) = 2, n(G) = 4n, \Delta(G) = 6$

From reference 4 we have , χ(G)≤ $\frac{\Delta(\boldsymbol{G})+m(\boldsymbol{G})+\boldsymbol{n}(\boldsymbol{G})+\mathbf{2}}{\mathbf{2}}$

Let us prove this theorem on induction on n.

Step:1 when n=2

$$\chi(G) \leq \frac{\Delta(\boldsymbol{G})+m(\boldsymbol{G})+\boldsymbol{n}(\boldsymbol{G})+\mathbf{2}}{\mathbf{2}} \leq \frac{4n+10}{2} \leq 9$$

here $V(G) = \{u_1, u_2, u_3, u_4, u_1', u_2', u_3', u_4'\}$for b coloring consider the color set {1,2}
such that cdv(1)= u_1,cdv(2)= u_2
hence χ(G)=2

step:2 when n=3

$$\chi(G) \leq \frac{\Delta(\boldsymbol{G})+m(\boldsymbol{G})+\boldsymbol{n}(\boldsymbol{G})+\mathbf{2}}{\mathbf{2}} \leq \frac{4n+10}{2} \leq 11$$

here $V(G) = \{u_1, u_2, u_3, u_4, u_5, u_6, u_1', u_2', u_3', u_4', u_5', u_6'\}$cdv(1)= u_1,cdv(2)= u_2
hence χ(G)=2

step:3

$$\chi(G) \leq \frac{\Delta(\boldsymbol{G})+m(\boldsymbol{G})+\boldsymbol{n}(\boldsymbol{G})+\mathbf{2}}{\mathbf{2}} \leq \frac{4n+10}{2}$$

consider the splitting graph of comb graph in which the vertex set V(G) = $\{u_1, \ldots u_n, u_1', \ldots u_n'\}$. Now we assign the existing colors to the vertices $u_n$and u_n'.Hence the b chromatic number of splitting graph of comb graph is 2.

Theorem 12:

The Grundy chromatic number of G is 2n+1 where G is a splitting graph of comb graph for alln≥ 2 vertices.

Proof:

Let G be a splitting graph of comb graph with V(G) = $\{u_1, \ldots u_n, u_1', \ldots u_n'\}$.Where n(G)=4n, $\Delta(\boldsymbol{G}) = 6, \omega(G) = 2$

By the result 1 we know that Γ (G) $\leq \frac{k+1}{2}\omega$ where ,K = θ(G),ω= ω(G)

By considering following cases the proof can be obtained

Step 1: when n =2
here $V(G) = \{u_1, u_2, u_3, u_4, u_1', u_2', u_3', u_4'\}$n(G)=8,$\omega(G) = 2$, θ(G)=6=k
$\Gamma(G) \leq \frac{k+1}{2}\omega \leq 7$
By the definition of Grundy coloring Γ(G)=5

For Grundy coloring consider the color set {1,2,3,4,5}$c(u_1) = 1, c(u_2) = 4, c(u_3) = 1, c(u_4) = 5$
$c(u_1') = 1, c(u_2') = 2, c(u_3') = 1, c(u_4') = 3$
therefore Γ(G)=5.

Step 2: when n=3

Here $V(G) = \{u_1, u_2, u_3, u_4, u_5, u_6, u_1', u_2', u_3', u_4', u_5', u_6'\}$n(G)=12,$\omega(G) = 2$, θ(G)=10=k
$\Gamma(G) \leq \frac{k+1}{2}\omega \leq 11$

For Grundy coloring let us consider the color set {1,2,3,4,5,6,7} $c(u_1), c(u_5), c(u_1'), c(u_5'), c(u_3), (u_3') = 1$,$c(u_2) = 6$, $c(u_2') = 7c(u_4) = 4, c(u_4') = 5, c(u_6) = 2, c(u_6') = 3$
Therefore Γ(G)=7

Hence, Grundy chromatic number of G is 2n+1 for all n$\geq$ 2 vertices

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