Chronological Background of Number Theory & Various Integer Solutions Within Algebraic Equations

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ABSTRACT

This thesis explores the historical evolution of number theory and the development of Diophantine equations, highlighting key milestones from ancient to modern times. Originating with the Babylonians and Greeks, early studies focused on arithmetic properties and primitive algebraic forms. Diophantus of Alexandria systematized the study of equations with integer solutions, laying the foundation for what would later be called Diophantine analysis.

The Islamic Golden Age and the European Renaissance introduced novel algebraic methods, with Fermat, Euler, and Lagrange making significant advances. The 19th century saw the formalization of number theory under Gauss and others, while the 20th century brought rigorous proof techniques, including Hilbert's 10th Problem and the rise of elliptic curves. Notably, Andrew Wiles' proof of Fermat's Last Theorem marked a watershed moment.

Today, Diophantine equations remain at the forefront of research, with connections to algebraic geometry, cryptography, and computational number theory. Despite centuries of progress, numerous conjectures and equations, such as the Birch and Swinnerton-Dyer Conjecture, continue to challenge mathematicians, ensuring the field's ongoing vitality.

INTRODUCTION

The study of integers, and more especially positive integers, is referred to as number theory in the branch of mathematics known as mathematics. Since quite some time ago, its relevance in mathematics has been well-established and well recognised. For the simple reason that this is a subject that has a great deal of historical relevance. Unlike the majority of other fields, this one allows for the observation of results. From the time of antiquity till the present day, humanity have been able to develop new and exciting insights into the nature of numbers in each and every century. The great majority of the world's most accomplished mathematicians have, throughout the course of their careers, made substantial contributions to the field of number theory. Because of the basic qualities that it has, number theory has captured the interest of the most prominent scientists.

A branch of mathematics known as number theory, which is often referred to as the "Queen of Mathematics," examines the characteristics and relationships of integers. Included in the list of possible predecessors are the first mathematical publications, which focused on basic concepts related to numbers. There is a set of equations known as Diophantine equations that are looking for integer solutions. Number theory has developed and changed over the course of thousands of years, and it has been related to these equations. This introduction aims to outline the historical progression of number theory and the central role played by Diophantine equations, setting the stage for contemporary research.

Many people regard Pythagoras, a Greek mathematician and physicist, to be the "father of number theory" because of the ground-breaking work he did in the domains of geometry and number theory. The Pythagorean theorem, which is concerned with the sides of a right triangle, is considered to be one of the most significant discoveries that Pythagoras and his pupils made in the field of number theory. In the discipline of number theory, Euclid, Fermat, and Diophantus are famous personalities who have made substantial contributions to the study of equations, number systems, and prime numbers. These individuals have also made significant contributions to the field. Even in this day and age, having a solid understanding of number theory in the context of mathematics is essential and useful.

Number theory is a topic of mathematics that encompasses everything that has to do with numbers, including their underlying characteristics, operations, and characteristics of their nature. For hundreds of years, mathematicians have been fascinated by this fascinating subject in order to satisfy their curiosity. In addition to the domains of engineering and general science, it may also be beneficial to the subject of computer science.

A fundamental focus of number theory is the study of numbers and the characteristics that distinguish them from one another. Congruences, prime numbers, diophantine equations, and the capacity to divide are a few instances of this kind of talent. A prime number is a positive integer that can only be divided by itself and by one. Prime numbers are an example of this. It is one of the many forward-thinking ideas in number theory, and it has a prominent position. The numbers two, three, five, seven, eleven, and thirteen are among the prime numbers that were created first.

In number theory, prime numbers are the topic of a significant amount of research because of the many fascinating properties that they possess. A well-known theory known as the "twin prime conjecture" asserts that there is an endless number of prime number pairs that are not identical to one another, such as three and five. In spite of the absence of evidence, mathematicians have made significant progress in their understanding of the distribution of prime numbers and the characteristics they possess.

However, this is yet another essential notion in the field of number theory. Specifically, the focus of this area of study is on the process of dividing numbers by other integers. In this area of research, the study of prime numbers is an essential component. The fact that the integer n can be divided by the integer m may be deduced from the fact that a number k can be found such that n = km. There are a number of domains, including computer science and security, that might potentially benefit from the use of this approach. There are also a great deal of mathematical applications that may be made use of it.

Number theory also includes congruences, which are an important mathematical concept. It is necessary for two numbers to have the same result when divided by a predefined integer in order for them to be termed congruent. The reason why 12 and 2 are comparable to 2 (mod 5) is due to the fact that when divided by 5, each of these numbers contain a residue of 2. Not only are congruences useful in the process of learning modular mathematics, but they are also very important in the fields of computer science and software security.

Diamond equations are also included in the field of number theory. These equations, which are based on integers, are presented as a tribute to the Greek scholar Diophantus. Despite the fact that Diophantine equations are notoriously difficult, some well-known scientists have spent years seeking to find a solution to them.

Number theory has a wide range of exciting applications, some of which include quantum physics, engineering, and computer science. In spite of the fact that the ideas and concepts that underpin number theory seem to be ambiguous and complicated, the area has a long and illustrious history, and it has been an indispensable contributor to the advancement of mathematics and science over the course of many centuries. Expanding one's knowledge in the fascinating topic of number theory would be beneficial for everyone, from scholars and scientists to just interested members of the general public.

The importance of the number has always been quite high for human beings because of this reason. The idea that "the number is the essence of things" was a significant philosophical subject for Pythagoras to consider. In the middle of the sixth century B.C., the Pythagoreans [1] started looking into positive fractions, natural numbers, and the disparities between them. All of the Pythagoreans were particularly interested in the concept of divisibility. To be more specific, they investigated odd and even numbers and produced a thorough theory of "even and odd," which Euclid later used in his own work titled "Elements" [2–5]. According to the Pythagoreans, the natural numbers may be divided into three distinct categories: deficient, perfect, and ample [6]. If the sum of all divisors was lower than the number that was supplied, then we would state that the number is considered to be inadequate. If the total of all the divisors of the given number is equal to the number itself, then we say that the number is perfect. In situations when the total of all divisors is more than the number that is being offered, we state that the number is readily available. This concept was used, which states that a number may be represented as the sum of all of its divisors. Additionally, the Pythagoreans devised a system of acceptable numbers, which might be defined as sets of integers in which the product of the divisors of two numbers is equal to one and the other, and the opposite is also true.

The publishing of "Elements" took place in the latter part of the fourth century B.C., at the commencement of the Alexandrian era. As far as mathematics is concerned, it continues to be the highest point of artistic excellence. Written by Euclid himself. He was one of the very best in the world when it comes to scientific knowledge. Approximately 250 years after the common era, the Greek scientist Diophantus of Alexandria died dead. Despite the fact that Elements was unquestionably the most important tool for teaching geometry, he nevertheless managed to make substantial advances to the understanding of number theory. Next, he wrote a book on the theory of algebraic numbers and equations, which was a massive work. It was referred to as "Arithmetica," which literally translates to "the science of numbers." from seven to 10. By producing this ground-breaking book, Diophantus, who is known as the "father of algebra," made an indelible influence on the field of mathematics. A total of thirteen works were formerly considered to be part of Diophantus' output; however, only six of those books have been found. Shortly after that, the 130 equations that were included in Arithmetica were given the name "Diophantine." As was the case with a great number of other equations, the exponential Diophantine equation remained unresolved for hundreds of years. During the Middle Ages, Arabian academics were the ones who looked into a significant number of the topics that are now considered to be part of contemporary number theory. Divisibility, congruences, and problems that cannot be solved were the topics that were discussed in the bulk of these topics. The preservation of an Arabian translation of Abū Kāmil's "Kitāb al-ṭarā'if fi'l-ḥisāb" [11–12] occurs between the years 1211 and 1218. This translation is considered to be among the first works that explore these topics. Throughout the course of the book, the fundamental concept that was given was that linear indeterminate equation systems might be solved by locating roots that are ordered groups of natural numbers.

Non-linear indeterminate equations were studied by the Persian engineer and scientist Abū Bakr Muḥammad ibn al Ḥasan al-Karajī in his book titled "Al-Fakhri." The book known as "Arithmetica" was written by Diophantus in the third century AD, and it contains a number of the questions and answers that were asked. At the beginning of the first century after the common era, Abū Maḥmūd Ḥāmid ibn al-Khiḍr al-Khujandī attempted to demonstrate that the equation $x3+y3=w3$ did not possess a solution that was an integer. It is possible that this may be considered a new addition to number theory as well, despite the fact that the proof was not satisfactory.

During the same century that the rest of Western Europe was absorbed with Diophantus' "Arithmetica," the French translation of the work by Claude-Gaspard Bachet de Méziriac (1581–1638) had an enormous amount of success. In addition to being a lawyer and advisor to the Toulouse Parliament, Pierre de Fermat was one among the persons who expressed interest in translating "Arithmetica." Among the individuals who were a part of this group were Bernard Frenicle de Bessy, Jacques de Billy, Marin Mersenne, René Descartes, Blaise Pascal, and a number of other thinkers and doers in the disciplines of philosophy, mathematics, and physics. Huygens or Schooten of the Netherlands and Wallis or Brouncker of England ultimately became interested in number theory. Both of these individuals were from the Netherlands. Through his work during this time period, Pierre de Fermat made substantial advances to the field of number theory. Additionally, he was the one who came up with "the method of infinite descent," which is considered to be one of the most consequential ways of establishing conclusions in number theory. Fermat put out a variety of hypotheses concerning prime numbers. Fermat's Little Theorem, Fermat primes, and Fermat numbers are all names that have been given to these notions.

Additionally, Fermat was interested in the possibility of displaying prime numbers using other quadratic forms, such as 2+2, which he found intriguing. There were some of the facts that Diophantus had already been aware of before he learnt them. Prime numbers that are similar to 4_e+1, where $\boldsymbol{S}$ is a natural integer, may be represented as $x2+y2$, for example, since he was aware of these prime numbers. A estimate was made by Bachet de Méziriac in 1621, when he was commenting on Diophantus' "Arithmetica," that any natural number might be represented as the sum of no more than four natural numbers squared. The hypothesis that all natural numbers are either $_e$-gonals or the sum of numerous $_e$-gonals was improved further by Fermat, who was the one who first proposed this theory. Euler, Lagrange, Gauss, and Cauchy were among the most prominent scientists of the 18th and 19th centuries who worked hard to find a solution to this problem [11, 12].In addition, subjects such as mathematical systems and number theory were addressed throughout. Additionally, Euclid's Elements contains a number of allusions to the Euclidean method as well as the concept of divisibility.

In addition, throughout the 17th century, it became less difficult to discern answers to problems that were unclear. A comprehensive description of the linear indeterminate equation $ax-by=1$ was provided by Bachet de Méziriac. In this equation, f and b are both unknowns, and f and b are both positive integer factors. In order to determine the values of the roots ($_e$,), natural integers were necessary. This was accomplished by him via the use of cases that had particular values. In the year 1612, the city of Lyon produced the first book that included these riddles. The title of the book was "Problems plaisants et délectables qui se font according to the numbers." Immediately after that, it was printed several times all the way up to the end of the century. In 1657, Fermat provided us with a problem: within the set of natural numbers, where a is a non-square integer, discover the root (x,y) of the equation $ax2+1=y2$. The reason why it is now often referred to as "Pell's equation" is that, in the 18th century, it was incorrectly assigned to the English scientist John Pell by Leonhard Euler. There is no way that anybody living in the 1600s could have anticipated the relevance of this equation in terms of coming up with answers to unclear problems.

After some time had passed, Euler and Lagrange came up with a solution that was suitable to the problem. Peter Gustav Lejeune-Dirichlet (1805–1859) presented in 1846 a version of this solution that was applicable to all situations and could be used anywhere. During this time period, Fermat also contemplated a number of other, more general concerns that his contemporaries had been unable to find solutions for. The equation $xn+yn=yn$ where n is greater than or equal to three was very remarkable. Furthermore, it is now referred to as Fermat's Last Theorem [14] in Section 2. An English physicist by the name of Andrew Wiles provided evidence in 1994 that confirmed the veracity of this idea. "Modular Elliptic Curves and Fermat's Last Theorem" was the title of a work that he published in the period of 1995. These ideas served as the foundation for the whole subject of number theory, beginning with Fermat's methods, discoveries, and challenges in the middle of the 17th century and continuing until Gauss's time. Fermat's contributions to the area of mathematics are largely responsible for the development of a wide variety of subfields within the discipline, including algebra, geometry, and arithmetic.

LITERATURE REVIEW

The study of Diophantine equations has evolved from ancient algebraic roots to modern computational and geometric methods. Its cross-disciplinary applications, from cryptography to algebraic geometry, showcase its enduring significance. Despite many breakthroughs, numerous open problems continue to inspire contemporary research in number theory.

In 1978, Robinowitz addressed the Diophantine equation and its solution. $2^n + px^2 = y^p$ where $x, y, n \in N$. He found all solutions $(x,y,n)$for $p=3$. Maohua (1995) proved that the equation $2^n + px^2 = y^p$ has no solution (x,y,n) with gcd(x,y)=1 for $p > 3$. And he proved that there is no solution to the above equation if $p > 3$ and $p \neq 7 \pmod{8}$. Le (1989) discussed the Diophantine equation $x^2 + D^m = p^n$. In the same year he also carefully examined the Diophantine equation carefully examined the Diophantine equation carefully examined the Diophantine equation carefully examined the Diophantine equation $x^2 = 4q^n + 4q + 1$. In (1991) he discussed the Ramanujan-Nagell equations $x^2 - D = p^n$ and $x^2 - D = 2^{n+2}$. In (1993) he discussed the Diophantine equation $\left(x^m - 1\right) / \left(x - 1\right) = y^n$ and the equation $D_1 x^2 + D_2 = 2^{n+2}$. Maohua (1995) discussed the Diophantine equation $D_1 x^2 - D_2 y^2 = \lambda k^z$, gcd$(x,y)$=1, z>0. He then gave many equations for integer solutions to $\lambda = 1$ or 4 according as 2□*k* or 2□*k* . In his 1995 paper, Sakmar covered generalised Diophantine equations of the Ramanujan type.

$$B_n^2 + 7A_n^2 = 2^{n+2}$$

and managed to get all of the answers for it.

Guan Wei and Ming Guang Lee addressed the Diophantine equation in their paper. Lee (2003) $2x^2 + 1 = 3^n$. Not only that, but they proved that there are precisely three positive rational integral solutions to this Diophantine equation: $(x, n) = (1,1)$, $(2,2)$ and $(11,5)$.

In their 1996 paper, Harun and Adongo addressed the Diophantine equation. $3u^2 - 2 = v^6$ and shown that, in addition to the above equation, no integer solutions exist for $|u| = |v| = 1$. He established the following theorem:

If *d* is a cube free integer >1, then the equation $x^3 + dy^3 = 1$ has at most one solution in integers *x,y* different from zero. If x_1, y_1 is a solution , the number $x_1 + y_1\sqrt[3]{d}$ is either the fundamental unit of $K = Q\left(\sqrt[3]{d}\right)$ or its square, the latter can happen only for *d*=19,20,28.

The Diophntine equation was addressed by Michael A. Bennett and Gary Walsh (1999). $b^2x^4 - dy^2 = 1$. They proved that there is exactly one solution to this Diophantine problem. integral solution in *x, y* if *b* and *d* are positive integers and *b*>1. They also gave a precise description of this solution using the basic units of the related quadratic field.

An equation involving Diophantine

$$A^4 + hB^4 = C^4 + hD^4$$

seems to have been discussed first by Gerardin (quoted by Dickson, pp. 647-48) although numerical solutions for the particular cases h=2 and h=5 were discussed by Grigorief and Werebrusow (quoted by Dickson, p. 647). Ajai Choudhry (1995) obtained the non-trivial solution of the above Diophantine equation for 75 positive integral values of . Ajai Choudhry (1998) obtained two parametric solutions of the Diophantine equation and indicated for the existence of more non-parametric solutions.

$$A^4 + 4B^4 = C^4 + 4D^4$$

Ajai Chaudhry (1999) discussed the quartic Diophantine equationAjai Chaudhry (1999) discussed the quartic Diophantine equation $f(x, y) = f(u, v)$ where $f(x, y) = ax^4 + bx^3y + cx^2y^2 + dxy^3 + ey^4$ and obtained a necessary and sufficient condition for the existence of non-trivial solution of this equation. He obtained the integer solution of the equation

$$x^4+x^3y+x^2y^2+xy^3+y^4=u^4+u^3v+u^2v^2+uv^3+v^4.$$

Ajai Chaudhry (1999) provided an elementary method for obtaining the complete solution of certain homogeneous Diophantine equations of the type $f(x_1, x_2, \ldots x_n)=cy^k$ where $f(x_i)$ is an integral form of degree k in the variables x_i, *I*=1, 2, …*n*. For $f(x_i)$ in three variables, he obtained non-trivial solutions of fourth and fifth degree equations. Ajai Chaudhry (2001) discussed the quartic Diophantine equation $X^4+Y^4+4Z^4=W^4$. Two new parametric solutions of equations of degree 8 and 16 have also been obtained. In the same year he showed that the system of simultaneous equations k=1, 2 and 5 has no non-trivial solution in integers. Ajai Choudhry (2001) showed that the system of simultaneousequations $\sum_{i=1}^{3} x_i^k=\sum_{i=1}^{3} y_i^k$, *k*=1, 2 and 5 has no non-trivial solutions in integers. Ajai Choudhry (2001) obtained several parametric solutions to the problem of finding two triads of cubes with equal sums and equal products. He also provided the method of obtaining such triads. Ajai Choudhry (2001) represented 1 as the sum or difference of kth powers of integers. Taking the minimum number of kth powers required to express 1 in infinitely many ways as the sum or difference of kth powers,he showed that *m*(2)=3, *m*(3)=3 and an upper bound is obtained for *m*(*k*) when $4\le k\le 8$. In the same year (2001) he also showed that for infinitely many integers N the number of representations of*N* as the sum of four integral fifth powers exceeds $2^{(0.2-\varepsilon)\log N/\log\log N}$ where $\varepsilon>0$. Ajai Chaudhry (2001) obtained necessary and sufficient condition for the solvability of the Diophantine equations $f(x,y)=f(u,v)$ where $f(x,y)$ is an arbitrary binary quintic or sextic form. He also obtained numerical or parametric solutions of certain specific quintic and sextic equations.

Ajai Choudhry (2005) obtained the complete solution of the equation $\sum_{i=1}^{s} x_i^4=\sum_{i=1}^{s} y_i^4$ when $s\le 13$. Ajai Choudhry (2005) discussed the problem of finding n integers such that their pair wise sums are cubes. He obtained 8 integers, expressed in parametric terms, such that all the 6 pair wise sums of four of these integers are cubes, Cohn (1997) discussed the Diophantine equation $x^4-Dy^2=1$. Michael, A. Bennelt and Gary Walsh (1999) extended the results of Cohn to the Diophantine equationextended the results of Cohn to the Diophantine equation $b^2x^4-dy^2=1$, where b and d are given integers. They showed that it possesses at most one solution in positive integers.

Chen Jianhua (1994) discussed the Diophantine equation $x^2+1=2y^4$ and $x^2+1=dy^4$. Chen Jianhua & Paul Voutier (1997) obtained the complete solution of the Diophantine equation and discussed a related family of quartic Thue equation. They showed that this equation has at most one solution in positive integers when $d\ge 3$. They showed that the solution is of the form $(u,\sqrt{v})$ when (u,v) is the fundamental solution of $x^2+1=2y^2$.

Zhenfu Cao (1986) discussed the Diophantine equationZhenfu Cao (1986) discussed the Diophantine equation $x^p-y^p=Dz^2$. In 1990, he discussed the equation $ax^m-by^n=2$. In 1991, he discussed the Diophantine equation $(ax^m-4c)/(abx-4c)=by^2$. In (2000) he discussed the Diophantine equation $x^p+2^{2m}=py^2$ and proved that if $p\equiv 1(\mathrm{mod}\,4)$ and p □ $B_{(P-1)/2}$ then the equation $x^p+1=py^2, y\ne 0$ and the equation $x^p+2^{2m}=py^2, m\in N$, gcd (*x*,*y*)=1, *p/y* has no solution. Here $B_{(P-1)/2}$ is (*p*-1)/2 the Bernaulli number.

Cao (2000) showed that subject to a certain condition on the odd prime p theequation $x^p+1=py^2$ has no solution in *x* and *y* provided also that $p\equiv 1(\mathrm{mod}\,4)$. Cohn (2002) discussed the same Diophantine equation $x^p+1=py^2$ and remove the restriction of Cao and provided a simple proof.

Sander (1996) discussed $a^3+b^3+c^3=d^3$. Rachel Gar-el & Leonid Vaserstein (2002) discussed the Diophantine equation $a^3+b^3+c^3+d^3=0$. They obtained rational solutions of this equation.

Sankar Sitaraman (2000) discussed Fermat-type Diophantine equation. He showedthat if *p*>3 is a regular prime and c is an integer divisible only by primes of the form*kp*-1, (*k*,*p*)=1 then the equation $x^p+y^p=pcz^p$ has no non-trivial solutions. He further showed that if *p*>3, is irregular and p/B_{P-3}, c is of the form $kp-1$, *q* is an odd prime such that $q\equiv 1(\mathrm{mod}\, p)$ and $x^p+y^p=pcz^p$ has non-trivial integer solution then *q*□ $pcz^p/(x+y)$.

Sankar Sitaraman (2003) discussed the unsolvability condition of the equations ofthe form $x^p+y^p=pcz^p$ which were discussed by him in 2000. He presented the following theorem.

For any fixed odd positive integer *n<p-4* and any integer *c* divisible only by primes of the form *kp-1* where (*k*,*p*)=1, assume

(i) At least one of $C_p^{(3)}, C_p^{(5)}, \ldots, C_p^{(n)}$ is non-trivial.

(ii) $C_p^{(i)}$ =0 for $p-n-1\leq i\leq p-2$.

(iii) $2^i\neq 1\,\mathrm{mod}(p)$ for $1\leq i\leq n+1$.

Let *q* be an odd prime such that $q\equiv 1\,\mathrm{mod}(p)$, and such that there is a prime ideal Q over q in Q(□) whose ideal class is of the form I^pJ where *J* is non-trivial, not a *p*th power and $J\in C_p^{(3)}\oplus C_p^{(5)}\oplus\ldots\oplus C_p^{n}$.

For such *p* and *q*, if $x^p+y^p=pcz^p$ has a non-trivial solution $x,y,z\in Z$ and $(x,y,z)=1$, then q □ $\frac{pcz^p}{x+y}$.

Weger (1996) discussed the binomial Diophantine equation $\binom{n}{2}=\binom{m}{4}$. He showed that this equation has only three solutions given by(*n*,*m*)=(2,4), (6,6) and (21,10).

Florian Luca (1999) discussed the Diophantine equation $\varphi(|x^m+y^m|)=|x^n+y^n|$. He showed that the only non-trivial solutions of the given Diophantine equation are given by

$$(x,y,m,n)=\begin{cases}(2,0,n+1,n), or\\(2,2,n+1,n), or\\(3,3,n+1,n), or\\(3,1,2,1)\end{cases}$$

for some positive integer n. Florian Luca (2002) discussed the Diophantine equationFlorian Luca (2002) discussed the Diophantine equation $x^2=4q^m-4q^n+1.$ He showed that the only non-trivial solution of the given equation with*q* a prime power and $m>n\geq 0$ but $(m,n)\neq(1,0)$ are given by

$(x,q,m,n)=(37,7,3,0),(5,2,3,1),(11,2,5.1),(18,2,13,1),(31,3,5,1)$ and $(559,5,7,1)$.

Siva Rama Prasad & Srinivasa Rao (2002) discussed the Diophantine equation $x^2(x+1)^2=8y^2\pm 4$. They showed that the solution set of this Diophantine equation is $[(1,0),(-2,0),(2,\pm 2),(-3,\pm 2)]$. They also showed that the set of the Diophantine equation $x^2(x+1)^2=8y^2-4$ is $[(1,\pm 1),(-2,\pm 1)]$.

Arif & Abu Muriefah (1998) discussed the Diophantine equation $x^2+3^{2k+1}=y^n$ and solved it completely. In (1999) they discussed the Diophantine equation $x^2+5^{2k+1}=y^n$. They showed that the equation $x^2+5^{2k+1}=y^n$, $n\geq 3$ has no solution in integers *x*, *y* for all $k\geq 0$.

Arif, Fadwa and Abu Muriefah (2002) discussed the Diophantine equation $x^2+q^{2k+1}=y^n$. They showed that the equation $x^2+q^{2k+1}=y^n$ when *q* is an odd prime, *q* is not congruent to 7(*mod*8), *n* is an odd integer ¿5, *n* is not a multiple of 3 and (*n*,*h*)=1 where *h*is the class number of the field $Q(-q)$ has exactly two families of solutions given by

$q=19$, $n=5$, $k=5M$, $x=22434.19^{5M}$, $y=55.19^{2M}$,

$q=341$, $n=5$, $k=5M$, $x=2759646.341^{5M}$, $y=377.19^{2M}$.

They further showed that the equation $x^2+q^{2k+1}=y^n$, when *n* and *q* satisfy the above conditions, has no solution for (*q*,*x*)=1, except when *q*=19.341.

Kh. Hessami Pilehrood & Hessami Pilehrood (2005) discussed the Diophantineequation $x^2+3=py^n$. They proved the following results:

If *p* is an odd prime and $p-3$ is not a perfect square then the equation

$$x^2+3=py^{p-1}$$

has no solution in rational numbers *x*, *y*.

If prime $p\equiv 1(\mathrm{mod}\,4)$ then the equation

$$x^2+3=py^{\frac{p-1}{2}}$$

has no solution in rational numbers *x*, *y*.

If the equation

$$x^2+3=py^6,\ p>3$$

is solvable in rational numbers x, y then there exists positive integers A, B such that $p=A^2+3B^2$, *B* is a cubic residue modulo *p* and either $B\equiv 0(\mathrm{mod}\,9)$ or $B\equiv\pm 1(\mathrm{mod}\,9)$.

Jena (2008) obtained the solution of the Diophantine equation $mA^6+nB^3=C^2$ with m, n, A, B and C as integers and A, B and C pair-wise coprime.

Florian Luca (2008) find the solutions of the Diophantine equation $x^2+2^a.5^b=y^n$.

Bennett & Ellenberg (2010) discussed the Diophantine equation $A^4+2^5B^2=C^n$. Gopalan & Janaki (2010) discussed the Diophantine equation $x^3+y^3+z^3=(x+y+z)w^4$ and presented different observations.

Both "Lilavati" by Bhaskara II and "Ganita Sara Sangraha" by Mahavira are guaranteed to attract and challenge contemporary readers with their breathtaking imagery.
There is a widespread misconception that the author of "Ganita Kaumudi," Narayana Pandita (AD 1357), and the Bhaskara scholar and collaborator with "Kriyakramakari" are the same person. This is a frequent misconception.

3. DIOPHANTINE EQUATIONS RELATED TO SOME STANDARD GEOMETRICAL FIGURES

Pythagorean triangles attracted many mathematicians. **Gopalan & Devibala,** (2005) discussed a special type of Pythagorean triangles in which the area of the triangle, with sides (*x*,*y*,*z*) such that $x^2+y^2=z^2$, is equal to the perimeter of the triangle numerically i.e. $\frac{1}{2}xy=x+y+z$. They have shown that there exist one primitive integral solution and one non-primitive integral solution of such triangles.

In this chapter, the Pythagorean triangles have been discussed for other relations concerning their area and perimeter. Besides Pythagorean triangles, the Diophantine equations relating area and perimeter of rectangle, circle and cardioid and surface area and volumes of rectangular parallelepiped, sphere, cylinder, and cone have been discussed. The integral solutions have been obtained under different conditions.

Special Pythagorean Tringles:

Let (*x,y,z)* be the Pythagorean triplet such that $x^2+y^2=z^2$. Most common solution of such triangle is given by $x=m^2-n^2, y=2mn$ and $z=m^2+n^2$. The area of this triangle is given by $A=\frac{1}{2}xy$ and its perimeter is given by $P=x+y+z$. Now we consider the following cases:

(i) $A=2P$

Substitution of values of *A* and *P* gives

$$\frac{1}{2}xy = 2(x+y+z) \qquad \ldots(2.1)$$

The above values of *x*,*y* and *z* reduce the equation (2.1) to

$$mn(m^2 - n^2) = 4m(m+n)$$

or $$n(m-n) = 4$$

or $$m = n + \frac{4}{n}. \qquad \ldots(2.2)$$

Now n=1,2,4 gives the integral values of m=5,4,5 respectively. Thus (5,1), (4,2) and (5,4) are the only integral solutions of (2.2). These values of *m* and *n* give

(a) x=24, y=10 and z=26,

(a) x=12, y=16 and z=20,

(c) x=9, y=40 and z=41.

These are the required integral solutions of Pythagorean triangle. Solutions (a) and (b) are non-primitive solutions while (c) is the primitive solution.

(i) *2A=P.*

Substitution of values of *A* and *P* gives

$$xy = (x+y+z). \qquad \ldots(2.3)$$

The values of *x*,*y* and *z* reduce (2.3) to

$$mn(m^2 - n^2) = m(m+n),$$

or $$n(m-n) = 1,$$

or $$m = n + \frac{1}{n}. \qquad \ldots(2.4)$$

Now n=1 is the only value which gives integral value of m=2. Thus (m,n)=(2,1) is the only solution of (2.4). These values of *m* and *n* give x=3, y=4 and z=5.

Thus (3,4,5) is the required solution of the given Pythagorean triangle.

(iii) *A=3P*

Substitution of values of *A* and *P* gives

$$\frac{1}{2}xy = 3(x+y+z). \qquad \ldots(2.5)$$

The values of *x*,*y* and *z* reduce (2.5) to

$$mn(m^2 - n^2) = 6m(m+n),$$

or $$n(m-n) = 6,$$

or $$m = n + \frac{6}{n}. \qquad \ldots(2.6)$$

Now n=1,2,3 and 6 gives the integral values of m=7,5,5 and 6 respectively. Thus (7,1), (5,2), (5,3) and (7,6) are the only integral solutions of (2.6). These values of *m* and *n* give

(a) *x=48, y=14* and *z=50,*

(a) *x=21, y=20* and *z=29,*

(b) *x=16, y=30* and *z=34* and

(c) *x=13,y=84* and *z=85.*

These are the required integral solutions of the given Pythagorean triangle. Solutions (a) and (c) are non-primitive solutions while (b) and (d) are primitive solutions.

(iv) *A=pP* where *p* is prime.

Substitution of values of *A* and *P* gives

$$\frac{1}{2}xy = p(x+y+z). \qquad \ldots(2.7)$$

The values of *x*,*y* and *z* reduce (2.7) to

$$mn(m^2 - n^2) = 2pm(m+n),$$

or $$n(m-n) = 2p,$$

or $m = n + \frac{2p}{n}$. …(2.8)

Now $n=2$ and p gives the integral values of $m=p+2$ and $p+2$ respectively. Thus $(p+2, 2)$ and $(p+2, 2)$ are the only integral solutions of (2.8). These values of m and n give

(a) $x = p^2 + 2p,\ y = 4(p+2)$ and $z = p^2 + 4p + 8$,

(b) $x = 4(p+1),\ y = 2p(p+2)$ and $z = 2p^2 + 4p + 4$.

These are the required integral solutions of the given Pythagorean triangle.

(v) 2 $A=pP$ where p is prime.

Substitution of values of A and P gives

$xy = p(x+y+z)$. …(2.9)

The values of x,y and z reduce (2.9) to

$mn(m^2 - n^2) = pm(m+n)$,

or $n(m-n) = p$,

or $m = n + \frac{p}{n}$. …(2.10)

Now $n=1$ and p gives the integral values of $m=p+1$ and $p+1$ respectively. Thus $(p+1,1)$ and $(p+1,p)$ are the only integral solutions of (2.10). These values of m and n give

(a) $x = p^2 + 2p,\ y = 2(p+1)$ and $z = p^2 + 2p + 2$,

(b) $x = 2(p+1),\ y = 2p(p+1)$ and $z = 2p^2 + 2p + 1$.

These are the required integral solutions of the given Pythagorean triangle.

4. NUMBER OF LATTICE POINTS WHITHIN AND ON SOME STANDARD TWO AND THREE DIMENSIONAL GEOMETRICAL FIGURES

Number Theory is classified into four categories: Elementary Number Theory (or Classical Number Theory), Analytic Number Theory, Algebraic Number Theory and Geometric Number Theory. Geometric Number Theory is an important class of Number Theory. In Geometric Number Theory, we deal with the problem of Number Theory by geometric methods. In a plane orthogonal coordinate system, a point (x,y) is called an integral part (or lattice point) if its coordinates x and y are positive integers. In this class of Number Theory we have to find lattice points in the given closed figure. A very famous and difficult unsolved problem in Number Theory is Gauss's integral point problem stated as: How many integral points are there inside the circle with center at the origin and radius r? If A(r) is the number of integral points within and on the circle then Gauss's problem is to seek the relationship between A(r) and r. Since the area of the circle with radius r is , it is conjectured that . Some results regarding this problem are as follows:

Result given by **Sierpinski** is $A(r) = \pi r^2 + o\left(r^{2/3} \log r\right)$.

Result given by **Lu-Keng Hua** is $A(r) = \pi r^2 + o\left(r^{13/20} (\log r)^{9/8}\right)$.

Result given by **Jing-Run Chen** is $A(r) = \pi r^2 + o\left(r^{\frac{3}{4}+8}\right)$. This is the best result known till now.

Minkowski showed that there must exist a non-zero integral point inside symmetric convex whose volume is greater than . In this chapter, an attempt will be made to obtain the number of lattice points within and on some two dimensional and three dimensional geometrical closed figures such as circle, ellipse, cardioid, square, rectangle, sphere, cuboid and rectangular parallelopiped etc. by considering their equations as Diophantine equations.

Lattice Points inside the Circle:

The equation of a circle whose center is at the origin and radius is r, is given by

$x^2 + y^2 = r^2$. …(3.1)

For different values of radius r, we have to find positive integral values of x and y lying inside and on the circle given by the equation (3.1).

(i) If $r < \sqrt{2}$ then there exist no positive integral values of x and y satisfying(3.1). Thus there exist no lattice point inside and on the circle $x^2 + y^2 = r^2$ when $r < \sqrt{2}$.

(ii) If $r=\sqrt{5}$ then there exist three sets of positive integral values of x and y lying in the region of the circle (3.1) given by $(x,y)=(1,1)$, (1,2) and (2,1). Thus there exist three lattice points (1,1), (1,2) and (2,1) inside and on the circle $x^2+y^2=r^2$ when $r=\sqrt{5}$.

(iii) If $r=2\sqrt{2}$ then there exist four sets of positive integral values of x and y lyingin the region of the circle (3.1) given by (*x,y*)= (1,1), (1,2), (2,1) and (2,2). Thus there exist four lattice points (1,1), (1,2), (2,1) and (2,2) inside and on the circle (3.1) when $r=2\sqrt{2}$.

(iv) If $r=\sqrt{10}$ then there exist six sets of positive integral values of x and y lying in the region of the circle (3.1) given by(*x,y*)= (1,1), (1,2), (2,1), (2,2), (1,3) and (3,1). Thus there exist six lattice points (1,1), (1,2), (2,1), (2,2), (1,3) and (3,1) inside and on the circle (3.1) when $r=\sqrt{10}$.

(v) If $r=\sqrt{13}$ then there exist eight sets of positive integral values of x and y lying in the region of the circle (1) given by(*x,y*)= (1,1), (1,2), (2,1), (2,2), (1,3), (3,1), (2,3) and (3,2). Thus there exist eight lattice points (1,1), (1,2), (2,1), (2,2), (1,3), (3,1), (2,3) and (3,2) inside and on the circle (3.1) when $r=\sqrt{13}$.

(vi) If $r=\sqrt{17}$ then there exist ten sets of positive integral values of *x* and *y* lying in the region of the circle (3.1) given by (*x,y*)= (1,1), (1,2), (2,1), (2,2), (1,3), (3,1), (2,3), (3,2), (1,4) and (4,1). Thus there exist ten lattice points (1,1), (1,2), (2,1), (2,2), (1,3), (3,1), (2,3), (3,2), (1,4) and (4,1) inside and on the circle (3.1) when $r=\sqrt{17}$.

(vii) If $r=3\sqrt{2}$ then there exist eleven sets of positive integral values of x and y lying in the region of the circle (3.1) givenby (*x,y*)= (1,1), (1,2), (2,1), (2,2), (1,3), (3,1), (2,3), (3,2), (3,3), (1,4) and (4,1). Thus there exist eleven lattice points (1,1), (1,2), (2,1), (2,2), (1,3), (3,1), (2,3), (3,2), (3,3), (1,4) and (4,1) inside and on the circle (3.1) when $r=3\sqrt{2}$.

(viii) If $r=2\sqrt{5}$ then there exist thirteen sets of positive integral values of x and y lying in the region of the circle (3.1) given by(*x,y*)= (1,1), (1,2), (2,1), (2,2), (1,3), (3,1), (2,3), (3,2), (1,4), (4,1) and (3,3). Thus there exist thirteen lattice points (1,1), (1,2), (2,1), (2,2), (1,3), (3,1), (2,3), (3,2), (3,3), (1,4), (4,1), (2,4) and (4,2) inside and on the circle (3.1) when $r=2\sqrt{5}$.

(ix) If $r=5$ then there exist fifteen sets of positive integral values of x and y lying in the region of the circle (3.1) given bythen there exist fifteen sets of positive integral values of x and y lying in the region of the circle (3.1) given by(*x,y*)= (1,1), (1,2), (2,1), (2,2), (1,3), (3,1), (2,3), (3,2), (1,4), (4,1), (3,3), (2,4), (4,2), (3,4) and (4,3). Thus there exist fifteen lattice points (1,1), (1,2), (2,1), (2,2), (1,3), (3,1), (2,3), (3,2), (1,4), (4,1), (3,3), (2,4), (4,2), (3,4) and (4,3) inside and on the circle (3.1) when $r=5$.

(x)If $r=\sqrt{26}$ then there exist seventeen sets of positive integral values of x and y lying in the region of the circle (3.1) givenby (*x,y*)= (1,1), (1,2), (2,1), (2,2), (1,3), (3,1), (2,3), (3,2), (1,4), (4,1), (3,3), (2,4), (4,2), (3,4), (4,3), (1,5) and (5,1). Thus there exist seventeen lattice points (1,1), (1,2), (2,1), (2,2), (1,3), (3,1), (2,3), (3,2), (1,4), (4,1), (3,3), (2,4), (4,2), (3,4), (4,3), (1,5) and (5,1) inside and on the circle (3.1) when $r=\sqrt{26}$.

(xi)If $r=\sqrt{29}$ then there exist fifteen sets of positive integral values of x and y lying in the region of the circle (3.1) given by (*x,y*)= (1,1), (1,2), (2,1), (2,2), (1,3), (3,1), (2,3), (3,2), (1,4), (4,1), (3,3), (2,4), (4,2), (3,4) and (4,3). Thus there exist fifteen lattice points (1,1), (1,2), (2,1), (2,2), (1,3), (3,1), (2,3), (3,2), (1,4), (4,1), (3,3), (2,4), (4,2), (3,4), (4,3), (1,5), (5,1), (2,5) and (5,2) inside and on the circle (3.1) when $r=\sqrt{29}$.

(xii) If $r=4\sqrt{2}$ then there exist twenty sets of positive integral values of x and y lying in the region of the circle (3.1) given by (*x,y*)= (1,1), (1,2), (2,1), (2,2), (1,3), (3,1), (2,3), (3,2), (1,4), (4,1), (3,3), (2,4), (4,2), (3,4), (4,3) and (4,4). Thus there exist twenty lattice points (1,1), (1,2), (2,1), (2,2), (1,3), (3,1), (2,3), (3,2), (1,4), (4,1), (3,3), (2,4), (4,2), (3,4), (4,3), (1,5), (5,1), (2,5), (5,2) and (4,4) inside and on the circle (3.1) when $r=4\sqrt{2}$.

3.3 Lattice Points inside the Concentric Circles:

(i) If $r_1=1$ and $r_2=\sqrt{2}$ then the provided concentric circles do not include any lattice points..

(ii) If $r_1=1$ and $r_2=\sqrt{3}$ then there is a single point on the lattice (1,1) lying inside the given concentric circleslying inside the given concentric circles.

(iii) If $r_1 = \sqrt{2}$ and $r_2 = \sqrt{6}$ then there exists two lattice points(1,2) and (2,1) lying inside the given concentric circles.

(iv) If $r_1 = \sqrt{5}$ and $r_2 = 3$ then there is a single point on the lattice (2,2)lying inside the given concentric circles.

(v) If $r_1 = 3$ and $r_2 = \sqrt{11}$ then there exist two lattice points(1,3) and (3,1) lying inside the given concentric circles.

(vi) If $r_1 = \sqrt{10}$ and then there exist two lattice points (2,3) and (3,2) lying inside the given concentric circles.

(vii) If $r_1 = 4$ and $r_2 = 2\sqrt{3}$ then there exist two lattice points (1,4) and (4,1) lying inside the given concentric circles.

(viii) If $r_1 = \sqrt{17}$ and then there exists one lattice point (3,3) lying inside the given concentric circles.

(ix) If $r_1 = \sqrt{19}$ and $r_2 = \sqrt{21}$ then there exist two lattice points (2,4) and (4,2) lying inside the given concentric circles.

(x) If $r_1 = 2\sqrt{6}$ and $r_2 = \sqrt{26}$ then there exist two lattice points (3,4) and (4,3) lying inside the given concentric circles.

5. Plan of Work

This plan outlines a structured approach to completing a PhD focused on the chronological development of number theory and Diophantine equations. The plan includes coursework, research goals, milestones, and dissemination of findings.

CONCLUSION

The present work is devoted to the discussion of some Diophantine equations. Diophantine equation is an important part of Number Theory which is one of the oldest branches of Mathematics. Most of the great masters of Mathematical Sciences, at some point in their careers, have contributed to Number Theory. In this Thesis an attempt has been made to solve some Diophantine equations.

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