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Statistical properties of Exponential Rayleigh distribution and Its Applications to Medical Science and Engineering

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ABSTRACT

In this paper, we introduce a new two parameter Exponential-Rayleigh distribution and study its different statistical properties. The method of maximum likelihood is proposed for estimating the parameters and the observed Fisher's information matrix is derived. Finally, the procedure is illustrated by the two real data sets and it is shown that the introduced model is more competitive than other models.
Keywords: AIC, BIC, Exponential-Rayleigh distribution, Maximum likelihood estimation, and Moments.

I. INTRODUCTION

The Exponential and Rayleigh distributions are two main distributions in life testing and reliability theory. They possess some important structural properties and show large mathematical flexibility. Balakrishnan and Basu (1995) explored the history of exponential distribution while the exponential distribution was generalized by introducing a shape parameter and considered widely by Gupta and Kundu (1999, 2001). The Rayleigh distribution has been widely used in communication engineering, applied statistics, operation research, health and biology. The Rayleigh Distribution was introduced by Rayleigh (1980). The generalization of Rayleigh distribution was proposed by Merovci (2013).

A random variable X is said to have the Rayleigh distribution (RD) with parameter β if its distribution function is given by

$$G(x)=1-e^{-\frac{\beta}{2}x^2}, x>0, \beta>0, \tag{1}$$

while the probability density function is given by

$$g(x)=\beta x e^{-\frac{\beta}{2}x^2}, x>0, \beta>0\ , \tag{2}$$

where β denote the scale parameter.

Different techniques are used to generalize the well-established distributions in order to make the distribution more flexible so as to fit the real life data. Among the various techniques used in statistical literature for model construction one technique is the T-X family. Alzaatreh et al. (2013) introduced the T-X families of distributions. These families of distributions were used to generate a new class of distributions which offer more flexibility in modelling a variety of data sets. Many authors have obtained different classes of these distributions. Alzaatreh et al. (2013) introduced the Weibull-Pareto distribution and while then, a number of authors have proposed and generalized various standard distributions based on the T-X families. The Exponential Lomax distribution was studied and defined by Bassiouny et al. (2015) and the Weibull-Rayleigh distribution was proposed by Afaq et al. (2017). In this paper new distribution, Exponential- Rayleigh will be introduced by using the method of T-X family.

II. EXPONENTIAL-RAYLEIGH DISTRIBUTION

The c.d.f. of the Exponential-Rayleigh distribution is given by the form as

$$F(x)=\int_0^{\frac{G(x)}{1-G(x)}} \lambda e^{-\lambda t}\,dt \quad =1-e^{-\lambda\left(\frac{G(x)}{1-G(x)}\right)} \tag{3}$$

where $G(x)$ is a basline cdf. The corresponding family of pdf becomes

$$f(x)=\lambda\frac{g(x)}{(1-G(x))^2}e^{-\lambda\left\{\frac{G(x)}{1-G(x)}\right\}}, x\in R; \lambda>0\ . \tag{4}$$

So, the cdf of the Exponential-Rayleigh (ER) distribution is given by

$$F(x)=1-e^{-\lambda\left(e^{\frac{\beta}{2}x^2}-1\right)} \tag{5}$$

Therefore, the corresponding pdf of Exponential-Rayleigh (ER) distribution, is given by

$$f(x)=\lambda \beta x e^{\frac{\beta}{2}x^2} e^{-\lambda\left(e^{\frac{\beta}{2}x^2}-1\right)}, x \in R; \lambda, \beta > 0 \; . \qquad (6)$$

By choosing various values for parameters λ and β, we provide the different possible shapes for the pdf of the Exponential-Rayleigh distribution as shown in figure1 as below:

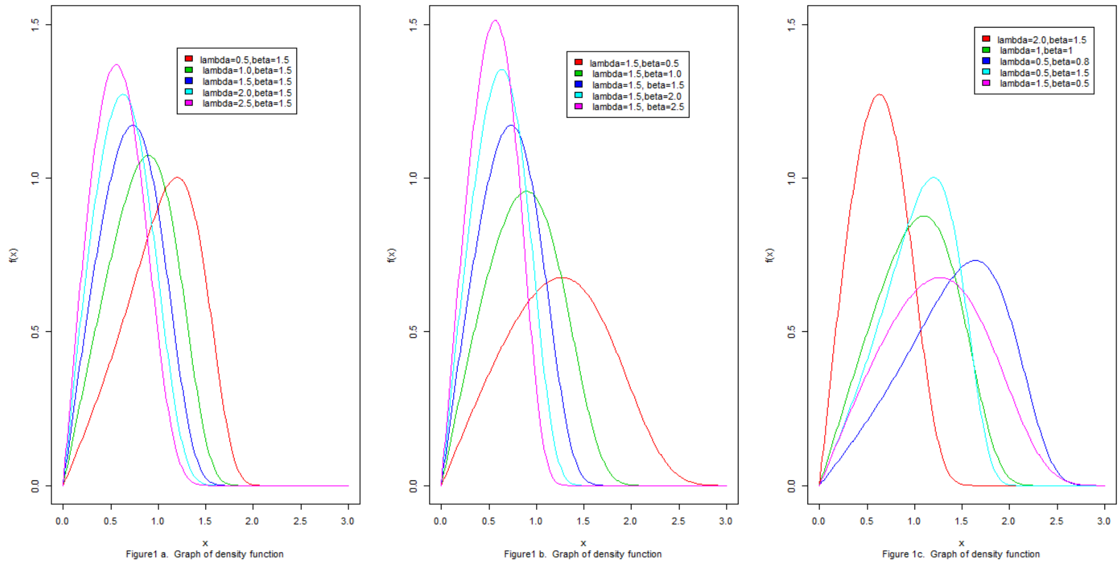

Figure1. The graph of density function

Fig. 1 shows that the proposed distribution is quite flexible and can take a variety of shapes such as positively skewed, reversed-J and tends to be symmetric. We further observe that the density function is positively skewed and tends to be more and more peaked as the value of each parameter is increased keeping other fixed.

Similarly, for the different possible values of the parameters λ and β, the graphical plots for the cumulative distribution function are given as below:

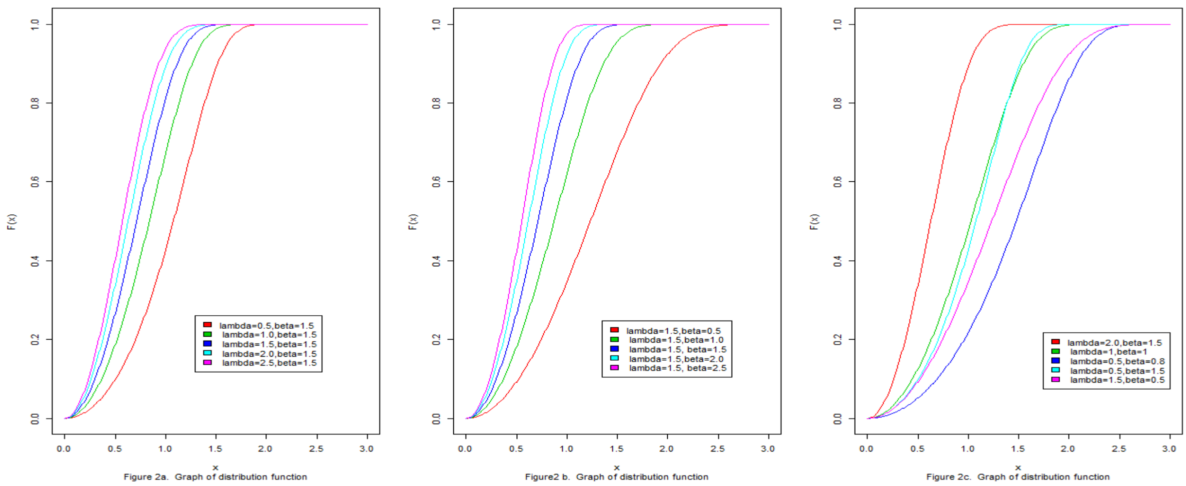

Figure2. The graph of distribution function

The graphical representation of the distribution function in Fig.2 shows that the cumulative density function is an increasing function with the different values of the parameters.

III. RELIABILITY ANALYSIS

The reliability (survival) function of Exponential-Rayleigh distribution is given by

$$R(x)=1-F(x)=e^{-\lambda\left(e^{\frac{\beta}{2}x^2}-1\right)}. \qquad (7)$$

The hazard function (failure rate) is given by

$$h(x) = \frac{f(x)}{R(x)} = \lambda \beta x e^{\frac{\beta}{2}x^2}. \quad (8)$$

The Reverse Hazard function of the ER distribution is obtained by

$$\phi(x) = \frac{f(x)}{F(x)} = \frac{\lambda \beta x e^{\frac{\beta}{2}x^2} e^{-\lambda\left(e^{\frac{\beta}{2}x^2}-1\right)}}{1-e^{-\lambda\left(e^{\frac{\beta}{2}x^2}-1\right)}}. \quad (9)$$

The graphical plotting of reliability function of ER distribution for different possible values of the parameters is given as follows:

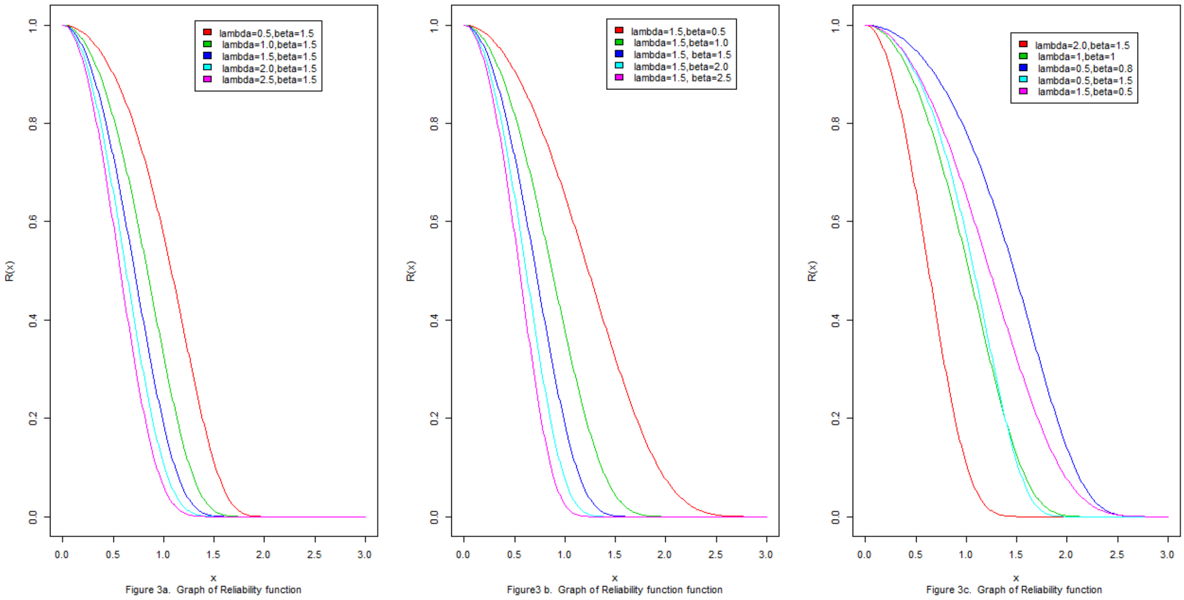

Figure3. The graph of Reliability function

The Fig. 3 indicates that the reliability function of a system is a decreasing function with the different possible values for parameters.

IV. MIXTURE REPRESENTATION

In this section, we obtain different mathematical properties of the given model. We first examine that the analysis related to the ER (λ, β) distribution can also be performed using the following representation. By expanding the exponential term in equation (4), we have

$$f(x) = \lambda \frac{g(x)}{(1-G(x))^2} e^{-\lambda\left\{\frac{G(x)}{1-G(x)}\right\}}.$$

By using the power series for the exponential function and the generalized binomial theorem we have we obtain

$$e^{-\lambda\left\{\frac{G(x)}{1-G(x)}\right\}} = \sum_{i=0}^{\infty} \frac{(-1)^i \lambda^i}{i!} \left\{\frac{G(x)}{1-G(x)}\right\}^i \quad \text{and} \quad (1-G(x))^{-(i+1)} = \sum_{j=0}^{\infty} \frac{\Gamma(i+1+j)}{j!\,\Gamma(i+1)} (G(x))^j$$

$$f(x) = \beta \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \omega_{i,j}\, x (G(x))^{i+j}, \text{ where } \omega_{i,j} = \frac{(-1)^i \lambda^{i+1} \Gamma(i+1+j)}{i!\, j!\, \Gamma(i+1)}. \quad (10)$$

V. STATISTICAL PROPERTIES OF THE ER DISTRIBUTION

This section provides some basic statistical properties of the Exponential Rayleigh (ER) distribution, particularly moments, skewness, kurtosis, harmonic mean, moment generating function, characteristic function, quantile function, median and random number generation.

5.1Moments of the ER Distribution

Theorem 5.1: If $X \sim \mathrm{ER}(\lambda,\beta)$, then r[th] moment of a continuous random variable X is given as follow:

$$\mu_r = E(x^r) == \sum_{i,j,k=0}^{\infty} \delta_{i,j,k} \frac{\beta 2^{r/2}}{(k\beta)^{1+r/2}} \Gamma(1+r/2).$$

Proof: Let X is an absolutely continuous non-negative random variable with PDF $f(x)$, then the r[th] moment of X can be obtained by:

$$\mu_r = E(x^r) = \int_0^{\infty} x^r f(x)\,dx.$$

From the PDF of the ER distribution in (10), then shows that $E(X^r)$ can be written as:

$$E(x^r) = \beta \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\int_0^{\infty} x^r\, \omega_{i,j}\, x \left\{1 - e^{-\frac{\beta}{2}x^2}\right\}^{(i+j+1)-1} dx..$$

Using the expansion of $\left\{1 - e^{-\frac{\beta}{2}x^2}\right\}^{(i+j+1)-1} = \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(i+j+1)}{\Gamma(i+j+1-k)k!} e^{-\frac{k\beta}{2}x^2}$

The above expression takes the following form:

$$E(x^r) = \beta \sum_{i,j,k=0}^{\infty} \int_0^{\infty} x^r\, \delta_{i,j,k}\, x e^{-\frac{k\beta}{2}x^2}\, dx.$$

Putting $\frac{k\beta}{2}x^2 = t$, we get

$$E(x^r) = \sum_{i,j,k=0}^{\infty} \delta_{i,j,k} \frac{\beta 2^{r/2}}{(k\beta)^{1+r/2}} \int_0^{\infty} t^{r/2} e^{-t}\, dt.$$

$$= \sum_{i,j,k=0}^{\infty} \delta_{i,j,k} \frac{\beta 2^{r/2}}{(k\beta)^{1+r/2}} \Gamma(1+r/2)\ , \tag{11}$$

where $\delta_{i,j,k} = \frac{(-1)^{i+k} \lambda^{i+1} \Gamma(i+j+1)\Gamma(i+j+1)}{i!\, j!\, k!\, \Gamma(i+1)\Gamma(i+j+1-k)}$.

which completes the proof.

If we put r=1,2,3,4 in equation (11), we get the first four central moments of Exponential Rayleigh distribution which is given by

$$\mu_1' = \sum_{i,j,k=0}^{\infty} \delta_{i,j,k} \frac{1}{k^{3/2}} \sqrt{\left(\frac{\pi}{2\beta}\right)}. \tag{12}$$

Thus the variance of Exponential Rayleigh distribution is given by

$$\mu_2 = \sum_{i,j,k=0}^{\infty} \delta_{i,j,k} \frac{2}{\beta k^2} - \left(\sum_{i,j,k=0}^{\infty} \delta_{i,j,k} \frac{1}{k^{3/2}} \sqrt{\left(\frac{\pi}{2\beta}\right)}\right)^2. \tag{13}$$

If we put r=3 in equation (11), we have

$$\mu_3' = \sum_{i,j,k=0}^{\infty} \delta_{i,j,k} \frac{3}{\beta^{3/2} k^{5/2}} \sqrt{\frac{\pi}{2}}\ .$$

Thus $\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3$

After substituting the values of μ_3', μ_2' and μ_1' in μ_3, we have

$$\mu_3 = \sum_{i,j,k=0}^{\infty} \delta_{i,j,k} \frac{1}{k^{3/2}} \left\{ \frac{3}{\beta^{3/2} k} \sqrt{\frac{\pi}{2}} - \sum_{i,j,k=0}^{\infty} \delta_{i,j,k} \frac{2}{\beta k^2} \sqrt{\left(\frac{\pi}{2\beta}\right)} + 2\left(\sum_{i=1}^{n} \delta_{i,j,k} \frac{1}{k^{3/2}}\right)^2 \left(\frac{\pi}{2\beta}\right)^{3/2} \right\} \tag{14}$$

If we put r=4 in equation (11), we have

$$\mu_4' = \sum_{i,j,k=0}^{\infty} \delta_{i,j,k} \frac{12}{\beta^2 k^3}\ .$$

Thus, $\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1'^2 - 3\mu_1'^4$

After substituting the values of μ_4', μ_3', μ_2' and μ_1' in μ_4, we have

$$\mu_4 = 3\sum_{i=0}^{\infty}\delta_{i,j,k}\left\{\frac{6}{\beta^2 k^3} - \sum_{i,j,k=0}^{\infty}\delta_{i,j,k}\frac{2\pi}{\beta^2 k^4} + \sum_{i,j,k=0}^{\infty}\delta_{i,j,k}\frac{4}{\beta k^2}\left(\frac{1}{k^{3/2}}\sqrt{\left(\frac{\pi}{2\beta}\right)}\right)^2 - \left(\sum_{i,j,k=0}^{\infty}\delta_{i,j,k}\frac{1}{k^{3/2}}\sqrt{\left(\frac{\pi}{2\beta}\right)}\right)^4\right\}. \quad (15)$$

5.4 Coefficient of Variation

It is the ratio of standard deviation and mean. Usually, it is denoted by C.V. and is given by

$$C.V = \frac{\sqrt{\sum_{i,j,k=0}^{\infty}\delta_{i,j,k}\frac{2}{\beta k^2} - \left(\sum_{i,j,k=0}^{\infty}\delta_{i,j,k}\frac{1}{k^{3/2}}\sqrt{\left(\frac{\pi}{2\beta}\right)}\right)^2}}{\sum_{i,j,k=0}^{\infty}\delta_{i,j,k}\frac{1}{k^{3/2}}\sqrt{\left(\frac{\pi}{2\beta}\right)}} \quad (16)$$

5.5 Coefficient of Skewness and Kurtosis

Coefficient of skewness is given by:

$$\gamma_1 = \sqrt{\beta_1} = \frac{\sum_{i,j,k=0}^{\infty}\delta_{i,j,k}\frac{1}{k^{3/2}}\left\{\frac{3}{\beta^{3/2}k}\sqrt{\frac{\pi}{2}} - \sum_{i,j,k=0}^{\infty}\delta_{i,j,k}\frac{2}{\beta k^2}\sqrt{\left(\frac{\pi}{2\beta}\right)} + 2\left(\sum_{i=1}^{n}\delta_{i,j,k}\frac{1}{k^{3/2}}\right)^2\left(\frac{\pi}{2\beta}\right)^{3/2}\right\}}{\left\{\sum_{i,j,k=0}^{\infty}\delta_{i,j,k}\frac{2}{\beta k^2} - \left(\sum_{i,j,k=0}^{\infty}\delta_{i,j,k}\frac{1}{k^{3/2}}\sqrt{\left(\frac{\pi}{2\beta}\right)}\right)^2\right\}^{3/2}}. \quad (17)$$

Coefficient of kurtosis is given by:

$$\gamma_2 = \frac{3\sum_{i=0}^{\infty}\delta_{i,j,k}\left\{\frac{6}{\beta^2 k^3} - \sum_{i,j,k=0}^{\infty}\delta_{i,j,k}\frac{2\pi}{\beta^2 k^4} + \sum_{i,j,k=0}^{\infty}\delta_{i,j,k}\frac{4}{\beta k^2}\left(\frac{1}{k^{3/2}}\sqrt{\left(\frac{\pi}{2\beta}\right)}\right)^2 - \left(\sum_{i,j,k=0}^{\infty}\delta_{i,j,k}\frac{1}{k^{3/2}}\sqrt{\left(\frac{\pi}{2\beta}\right)}\right)^4\right\}}{\left\{\sum_{i,j,k=0}^{\infty}\delta_{i,j,k}\frac{2}{\beta k^2} - \left(\sum_{i,j,k=0}^{\infty}\delta_{i,j,k}\frac{1}{k^{3/2}}\sqrt{\left(\frac{\pi}{2\beta}\right)}\right)^2\right\}^2} - 3. \quad (18)$$

5.6 Harmonic Mean

The harmonic mean (H) is given by:

$$\frac{1}{H} = E\left(\frac{1}{x}\right) = \beta\sum_{i,j,k=0}^{\infty}\int_0^{\infty} x^{-1}\,\delta_{i,j,k}\, x e^{-\frac{k\beta}{2}x^2}\,dx$$

After some calculations,

$$\frac{1}{H} = \sum_{i,j,k=0}^{\infty}\delta_{i,j,k}\frac{\beta^{3/2}k^{1/2}}{2^{1/2}}\Gamma(1/2)\ , \quad (19)$$

5.7 Moment Generating Function (MGF)

In this sub section, we derive the moment generating function of ER distribution.

Theorem 5.2: Let X have a ER distribution. Then moment generating function of X denoted by $M_X(t)$ is given by:

$$M_X(t) = \sum_{r=0}^{\infty}\sum_{i,j,k=0}^{\infty}\frac{t^r}{r!}\delta_{i,j,k}\frac{\beta 2^{r/2}}{(k\beta)^{1+r/2}}\Gamma(1+r/2). \quad (20)$$

Proof: -By definition

$$M_X(t) = E\left(e^{tx}\right) = \int_0^{\infty} e^{tx} f(x)\,dx = \sum_{r=0}^{\infty}\frac{t^r}{r!}\int_0^{\infty} x^r f(x)\,dx = \sum_{i=0}^{\infty}\frac{t^r}{r!}E\left(X^r\right)$$

$$\Rightarrow \quad M_X(t) = \sum_{r=0}^{\infty}\sum_{i,j,k=0}^{\infty}\frac{t^r}{r!}\delta_{i,j,k}\frac{\beta 2^{r/2}}{(k\beta)^{1+r/2}}\Gamma(1+r/2).$$

This completes the proof.

5.8 Characteristic Function

In this sub section, we derive the Characteristic function of ER distribution.

Theorem 5.3: Let X have a ER distribution. Then characteristic function of X denoted by $\phi_X(t)$ is given by:

$$\phi_X(t)=\sum_{r=0}^{\infty}\sum_{i,j,k=0}^{\infty}\frac{(it)^r}{r!}\delta_{i,j,k}\frac{\beta 2^{r/2}}{(k\beta)^{1+r/2}}\Gamma(1+r/2). \quad (21)$$

Proof: -By definition

$$\phi_X(t)=E\left(e^{itx}\right)=\int_0^{\infty}e^{itx}f(x)dx=\sum_{r=0}^{\infty}\frac{(it)^r}{r!}\int_0^{\infty}x^r f(x)dx=\sum_{r=0}^{\infty}\frac{(it)^r}{r!}E(X^r)$$

$$\Rightarrow \quad \phi_X(t)=\sum_{r=0}^{\infty}\sum_{i,j,k=0}^{\infty}\frac{(it)^r}{r!}\delta_{i,j,k}\frac{\beta 2^{r/2}}{(k\beta)^{1+r/2}}\Gamma(1+r/2).$$

This completes the proof.

5.9 Quantile Median and Random Number Generation

The Inverse CDF method is used for generating random numbers from a particular distribution. In this method, random numbers from a particular distribution are generated by solving the equation obtained on equating CDF of a distribution to a number u. The number u is itself being generated from $U \sim (0,1)$. In case of Exponential Rayleigh distribution, we define it by the following equation:

$$F(x)=u, \text{where}\, u\sim U(0,1).$$

$$\Rightarrow \quad 1-e^{-\lambda\left(e^{\frac{\beta}{2}x^2}-1\right)}=u. \quad (22)$$

On solving equation (22) for x, at fixed values of parameters (λ,β), we will obtain the random number from the ER distribution as:

$$x=\sqrt{\frac{2}{\beta}\ln\left[1+\left(\frac{-\ln(1-u)}{\lambda}\right)\right]} \quad (23)$$

For u=1/4, 1/2 and ¾ in equation (23), we get the resulting first quartile, second quartile (Median) and third quartile of the proposed model respectively.

VI. RENYI ENTROPY

The entropy of a random variable X with probability density ER $(x;\lambda,\beta)$ is a measure of the variation of the uncertainty. The large value of entropy is an indicator of the greater uncertainty in the data. The Renyi entropy (1960) denoted by $I_R(\rho)$ for X is a measure of variation of uncertainty and is defined as:

$$I_R(\rho)=\frac{1}{1-\rho}\log\left\{\int_{-\infty}^{\infty}f(x)^{\rho}dx\right\}, \text{ where } \rho>0 \text{ and } \rho\neq 1. \quad (24)$$

By using the power series for the exponential function and the generalized binomial expansion, we obtain

$$I_R(\rho)=\frac{1}{1-\rho}\log\sum_{i,j=0}^{\infty}\rho_{i,j}\int_0^{\infty}g(x)^{\rho}G(x)^{i+j} \quad \text{,where } \rho_{i,j}=\frac{(-1)^i(\lambda)^{i+\rho}\rho^i\Gamma(2\rho+i)+j)}{i!\,j!\,\Gamma(2\rho+i)}.$$

Now inserting (1) and (2) in above we get

$$I_R(\rho)=\frac{1}{1-\rho}\log\sum_{i,j=0}^{\infty}\rho_{i,j}\int_0^{\infty}\left(\beta xe^{-\frac{\beta}{2}x^2}\right)^{\rho}\left(1-e^{-\frac{\beta}{2}x^2}\right)^{(i+j+1)-1}dx .$$

The above expression takes the following form:

$$I_R(\rho)=\frac{\beta^{\rho}}{1-\rho}\log\sum_{i,j,k=0}^{\infty}\rho_{i,j,k}\int_0^{\infty}x^{\rho}e^{-\frac{\beta}{2}x^2\{\rho+k\}}dx \quad \text{, where } \rho_{i,j,k}=\frac{(-1)^{i+k}(\lambda)^{i+\rho}\rho^i\Gamma(2\rho+i)+j)\Gamma(i+j+1)}{i!\,j!\,k!\,\Gamma(2\rho+i)\Gamma(i+j+1-k)}$$

On solving the above integral we get,

$$I_R(\rho)=\frac{1}{1-\rho}\log\sum_{i,j,k=0}^{\infty}\rho_{i,j,k}\,\frac{2^{\frac{1}{2}(\rho-1)}\beta^{\frac{1}{2}(\rho-1)}}{(\rho+k)^{\frac{1}{2}(\rho+1)}}\Gamma\left(\frac{1}{2}(\rho+1)\right), \quad (25)$$

The β or q-entropy introduced by Havrda and Charvat (1967) is denoted by $I_H(q)$ and is defined as

$$I_H(q)=\frac{1}{q-1}\left\{1-\int_{-\infty}^{\infty}f(x)^q\,dx\right\},$$

where $q>0$ and $q\neq 1$. From equation (25), we can easily obtain

$$I_H(q)=\frac{1}{1-q}\log\left[1-\left(\sum_{i,j,k=0}^{\infty}\rho_{i,j,k}\,\frac{2^{\frac{1}{2}(\rho-1)}\beta^{\frac{1}{2}(\rho-1)}}{(\rho+k)^{\frac{1}{2}(\rho+1)}}\Gamma\left(\frac{1}{2}(\rho+1)\right)\right)\right]. \quad (26)$$

VII. ORDER STATISTICS

Order statistics finds many applications in statistical theory and modelling. It can be applied in studying the reliability of a system and life testing. If $X_{(1)}, X_{(2)},\ldots\ldots, X_{(n)}$ denote the order statistics obtained from the random sample $X_1, X_2,\ldots\ldots, X_n$ drawn from ER distribution(β,λ) with cumulative density function and probability density function given in the equations (5) and (6) respectively, then the probability density function of the order statistics is given as below:

$$f_r(x)=\frac{n!}{(r-1)!(n-r)!}[F(x)]^{r-1}[1-F(x)]^{n-r}f(x) \quad for\ 1\le r\le n. \quad (27)$$

Using the equations (5) and (6), the pdf of the first order statistic $X_{(1)}=\min(X_1, X_2,\ldots X_n)$ is given by

$$f_1(x)=n\lambda\,\beta\,xe^{\frac{\beta}{2}x^2}e^{-n\lambda\left(e^{\frac{\beta}{2}x^2}-1\right)}. \quad (28)$$

Similarly, the pdf of the nth order statistic $X_{(n)}=\max(X_1, X_2,\ldots X_n)$ is given as follows:

$$f_n(x)=n\lambda\,\beta\,xe^{\frac{\beta}{2}x^2}e^{-\lambda\left(e^{\frac{\beta}{2}x^2}-1\right)}\left(1-e^{-\lambda\left(e^{\frac{\beta}{2}x^2}-1\right)}\right)^{n-1}. \quad (29)$$

VIII. JOINT DISTRIBUTION FUNCTION OF ITH AND JTH ORDER STATISTICS

The joint density function of (x_i, x_j) for $1\le i\le j\le n$ is given by

$$f_{i:j:n}(x_i,x_j)=C[F(x_i)]^{i-1}[F(x_j)-F(x_i)]^{j-i-1}[1-F(x_j)]^{n-j}f(x_i)\,f(x_j), \quad (30)$$

where $C=\frac{n!}{(i-1)!(j-i-1)!(n-j)!}$.

Then the joint distribution function of the ith and jth order statistics of Exponential Rayleigh distribution is as follows:

$$f_{i:j:n}(x)=C(1-h_{(i)})^{i-1}(h_{(i)}-h_{(j)})^{j-i-1}h_{(j)}{}^{n-j}\lambda\beta\,x_ie^{\frac{\beta}{2}x_i^2}h_{(i)}\lambda\beta\,x_je^{\frac{\beta}{2}x_i^2}h_{(j)}, \quad (31)$$

where $h_{(k)}=e^{-\lambda\left(e^{\frac{\beta}{2}x_k^2}-1\right)}$ for $k=i,j$.

For the special case $i=1$ and $j=n$, we get the joint distribution of minimum and maximum order statistics as follows:

$$f_{1n}(x)=n(n-1)[F(x_n)-F(x_1)]^{n-2}f(x_1)f(x_n)$$

$$f_{1n}(x) = n(n-1)\left(h_{(1)} - h_{(n)}\right)^{n-2} \lambda\beta\, x_1 e^{\frac{\beta}{2}x_1^2} h_{(1)} \lambda\beta\, x_n e^{\frac{\beta}{2}x_n^2} h_{(n)}, \qquad (32)$$

where $h_{(n)} = e^{-\lambda\left(e^{\frac{\beta}{2}x_n^2} - 1\right)}$ and $h_{(1)} = e^{-\lambda\left(e^{\frac{\beta}{2}x_1^2} - 1\right)}$.

IX. PARAMETER ESTIMATION

In this section, the unknown parameters β and λ of the Exponential- Rayleigh distribution are estimated by the maximum likelihood estimation procedure. The sample consisting of n observations $x_1, x_2, x_3, \ldots x_n$ is considered. The likelihood function of the proposed distribution is as follows: $L(x) = \lambda^n \beta^n \prod_{i=1}^{n} \left\{ x_i e^{\frac{\beta}{2}x_i^2} e^{-\lambda\left(e^{\frac{\beta}{2}x_i^2} - 1\right)} \right\}$

(33)

The corresponding log likelihood function of the equation (33) is given as under:

$$\log L(x) = n\log\lambda + n\log\beta + \sum_{i=1}^{n}\log x_i + \frac{\beta}{2}\sum_{i=1}^{n} x_i^2 - \lambda\sum_{i=1}^{n}\left(e^{\frac{\beta}{2}x_i^2} - 1\right).$$

Differentiating log likelihood function with respect to β and λ partially,

$$\frac{\partial}{\partial\beta}\log L(x) = \frac{n}{\beta} + \frac{1}{2}\sum_{i=1}^{n} x_i^2 - \frac{\lambda}{2}\sum_{i=1}^{n} x_i^2 e^{\frac{\beta}{2}x_i^2} = 0 \qquad (34)$$

$$\frac{\partial}{\partial\lambda}\log L(x) = \frac{n}{\lambda} - \sum_{i=1}^{n}\left(e^{\frac{\beta}{2}x_i^2} - 1\right) = 0 \qquad (35)$$

Setting these expressions to zero and solving them simultaneously, the maximum likelihood estimates of the two parameters is obtained.

X. FISHER INFORMATION MATRIX

For the two parameters of ER$(x;\lambda,\beta)$ the second order derivatives of the log- likelihood function exist. Thus, the inverse dispersion matrix is given by:

$$\begin{pmatrix} \hat{\beta} \\ \hat{\lambda} \end{pmatrix} \sim N\left[\begin{pmatrix} \beta \\ \lambda \end{pmatrix}, \begin{pmatrix} \hat{V}_{\beta\beta} & \hat{V}_{\beta\lambda} \\ \hat{V}_{\lambda\beta} & \hat{V}_{\lambda\lambda} \end{pmatrix}\right] \qquad (36)$$

$$V^{-1} = -E\begin{pmatrix} V_{\beta\beta} & V_{\beta\lambda} \\ V_{\lambda\beta} & V_{\lambda\lambda} \end{pmatrix}, \quad \text{where}$$

$$V_{\beta\beta} = \frac{\partial^2 L}{\partial\beta\partial\beta} = \frac{-n}{\beta^2} - \frac{\lambda}{4}\sum_{i=1}^{n} x_i^4 e^{\frac{\beta}{2}x_i^2}$$

$$V_{\beta\lambda} = V_{\lambda\beta} = \frac{\partial^2 L}{\partial\beta\partial\lambda} = -\frac{1}{2}\sum_{i=1}^{n} x_i^2 e^{\frac{\beta}{2}x_i^2}$$

$$V_{\lambda\lambda} = \frac{\partial^2 L}{\partial\lambda\partial\lambda} = -\frac{n}{\lambda^2}$$

By deriving the inverse dispersion matrix, the asymptotic variances and covariances of the ML estimators for β and λ are obtained.

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XI. DATA ANALYSIS

In this section, we provide applications of the proposed Exponential-Rayleigh distribution to show the importance of the new model, where the Exponential-Rayleigh model is compared with other related models, namely Exponential-Lomax, Weibull-Exponential, Weibull-Rayleigh, Exponential, Rayleigh and Weibull distributions.
The pdf of distributions applied in the paper are given in the following

- The pdf of the Weibull distribution is given by

$$f(x) = \alpha\beta x^{\beta-1} e^{-\alpha x^{\beta}} .; x > 0,\ \alpha, \beta > 0.$$

- The pdf of Weibull Rayleigh distribution is defined and is given by

$$f(x) = \frac{\alpha x}{\lambda\beta^2}\left(\frac{x^2}{2\lambda\beta^2}\right)^{\alpha-1} \exp\left(-\left(\frac{x^2}{2\lambda\beta^2}\right)^{\alpha}\right) \quad ; x > 0,\ \alpha, \beta, \lambda > 0.$$

- The pdf of the Weibull exponential distribution is given by

$$f(x) = \alpha\beta\lambda\left(1-e^{-\lambda x}\right)^{\beta-1} \exp\left(\lambda\beta x - \alpha\left(e^{-\lambda x}-1\right)^{\beta}\right) \quad ; x > 0,\ \alpha, \beta, \lambda > 0.$$

- The pdf of the Rayleigh distribution is given by

$$f(x) = \beta x e^{-\frac{\beta}{2}x^2}, x > 0, \beta > 0.$$

- The pdf of the Exponential distribution is given by

$$f(x) = \lambda e^{-\lambda x}, x > 0, \lambda > 0.$$

The pdf of Exponential Lomax is defined and is given by

- $$f(x) = \frac{\alpha\lambda}{\beta}\left(\frac{\beta}{x+\beta}\right)^{-\alpha+1} e^{-\lambda\left(\frac{\beta}{x+\beta}\right)^{-\alpha}} \quad ; x \geq -\beta,\ \alpha, \beta, \lambda > 0.$$

In order to compare the distributions, we consider some criteria like AIC (Akaike information criterion) and BIC (Bayesian information criterion). The distribution which provides us lesser values of AIC and BIC is considered as best. The values of AIC and BIC can be computed as follows:

AIC=2k-2logL and BIC=klogn-2logL,

where k is the number of parameters in the statistical model, n is the sample size and -2logL is the maximized value of the log-likelihood function under the considered model. The analysis of both the data sets is performed through R software. The MLEs of the parameters are obtained with standard errors shown in parentheses. Further, the corresponding log-likelihood values, AIC and BIC are displayed in Table 1and 2.

Data set 1: The first data set is the failure times of 84 Aircraft Windshield. The windshield on a large aircraft is a complex piece of equipment, comprised basically of several layers of material, including a very strong outer skin with a heated layer just beneath it, all laminated under high temperature and pressure. Failures of these items are not structural failures. Instead, they typically involve damage or delamination of the non structural outer ply or failure of the heating system. These failures do not result in damage to the aircraft but do result in replacement of the windshield. These data on failure times are reported in the book "Weibull Models" by Murthy et al. (2004, page 297). This data set is previously studied by Bassiouny et al. (2015) to fit the Exponential Lomax (EL) distribution.

The failure times of 84 Aircraft Windshield is

0.040, 1.866, 2.385, 3.443, 0.301, 1.876, 2.481, 3.467, 0.309, 1.899, 2.610, 3.478, 0.557, 1.911, 2.625, 3.578, 0.943, 1.912, 2.632, 3.595, 1.070, 1.914, 2.646, 3.699, 1.124, 1.981, 2.661, 3.779,1.248, 2.010, 2.688, 3.924, 1.281, 2.038, 2.82,3, 4.035, 1.281, 2.085, 2.890, 4.121, 1.303, 2.089, 2.902, 4.167, 1.432, 2.097, 2.934, 4.240, 1.480, 2.135, 2.962, 4.255, 1.505, 2.154, 2.964, 4.278, 1.506, 2.190, 3.000, 4.305, 1.568, 2.194, 3.103, 4.376, 1.615, 2.223, 3.114, 4.449, 1.619, 2.224, 3.117, 4.485, 1.652, 2.229, 3.166, 4.570, 1.652, 2.300, 3.344, 4.602, 1.757, 2.324, 3.376, 4.663.The summary of the data is given in Table 1.

Data set 2: The second data set is from Abuammoh et al. (1994) and represents the lifetime in years of 40 patients suffering from blood cancer (Leukemia) from one of ministry of health hospital in Saudi Arabia and the ordered values in years are:
0.315, 0.496, 0.616, 1.145, 1.208, 1.263, 1.414, 2.025, 2.036,2.162,2.211, 2.370, 2.532, 2.693, 2.805, 2.910, 2.912, 3.192,3.263, 3.348, 3.348, 3.427, 3.499, 3.534, 3.767, 3.751, 3.858,3.986, 4.049, 4.244, 4.323, 4.381, 4.392, 4.397,4.647, 4.753,4.929, 4.973, 5.074, 5.381. The summary of the data is given in Table 3.

Table 1: Data summary for the Data set first

Minimum	1st Qu.	Median	Mean	3rd Qu.	Max.	Variance	Skewness	kurtosis
0.040	1.866	2.385	2.563	3.376	4.663	1.239	0.0865	2.365

Table 2: Estimates and performance of the distributions for the data set first

	Distribution	α	λ	β	Log-likelihood	AIC	BIC
Data Set I	Exponential-Rayleigh	–	1.48615 (0.74415)	0.11558 (0.03896)	-129.0849	**262.1698**	**267.0551**
	Exponential-Lomax	3.19093 (0.52005)	0.00567 (0.00228)	0.72097 (0.30318)	-128.6771	263.3542	270.6822
	Weibull-Exponential	0.17059 (0.16947)	0.52515 (0.29217)	1.31278 (0.40795)	-128.3865	262.7731	270.101
	Weibull-Rayleigh	1.19671 (0.10507)	1.41451 (72.51471)	1.70521 (43.70852)	-131.2884	268.5769	275.9048
	Exponential	–	0.39023 (0.04233)	–	-164.9877	331.9754	334.418
	Rayleigh	–	–	0.25668 (0.02784)	-133.2132	268.4265	270.8691
	Weibull	0.08033 (0.02219)	–	2.39318 (0.20997)	-131.2884	266.5769	271.4622

Table 3: Data summary for the data set second

Minimum	1st Qu.	Median	Mean	3rd Qu.	Max.	Variance	Skewness	Kurtosis
0.315	2.199	3.348	3.141	4.264	5.381	1.8465	-0.41672	2.2738

Table 4: Estimates and performance of the distributions for the data set second

	Distribution	α	λ	β	Log-likelihood	AIC	BIC
Data Set II	Exponential-Rayleigh	–	0.61936 (0.37313)	0.13836 (0.04451)	-66.26984	**136.5397**	**139.9174**
	Exponential-Lomax	5.14975 (2.52971)	0.01149 (0.00973)	2.64865 (2.49316)	-67.40827	140.8165	145.8832
	Weibull-Exponential	0.05143 (0.03588)	0.90351 (0.58146)	0.88965 (0.46751)	-65.71648	137.4330	142.4996
	Weibull-Rayleigh	1.24974 (0.16853)	1.82712 (156.92381)	1.84057 (79.03964)	-69.55796	145.1159	150.1826
	Exponential	–	0.31839 (0.05034)	–	-85.77815	173.5563	175.2452
	Rayleigh	–	–	0.17146 (0.027109)	-70.8058	143.6116	145.3005
	Weibull	0.04317 (0.02128)	–	2.49837 (0.33559)	-69.55797	143.1159	146.4937

From table 2 and table 4 compares the Exponential-Rayleigh model with the Exponential-Lomax, Weibull-Exponential, Weibull-Rayleigh, Exponential, Rayleigh and Weibull models. We note that the Exponential-Rayleigh model gives the lowest values for the AIC and BIC statistics among all fitted models. So, the Exponential-Rayleigh model could be chosen as the best model.

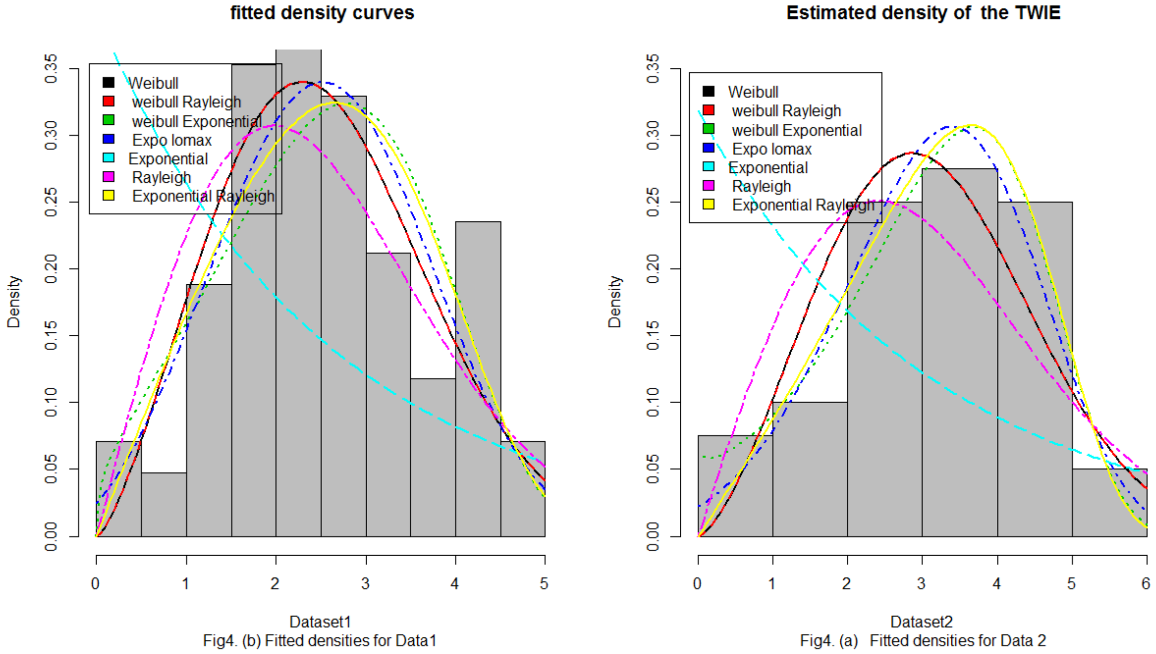

Figure 4: Graphs of the fitted Weibull, Weibull- Rayleigh, Weibull-Exponential, Exponential Lomax, Exponential, Rayleigh and Exponential-Rayleigh distributions for data sets 1 and 2.

XII. CONCLUSION

In this paper, we introduce a new class of distribution referred to as Exponential Rayleigh distribution by taking Rayleigh distribution as the base distribution and the Exponential distribution as the generator distribution by using generator technique. We derive the expressions for the moments, harmonic mean, survival function, hazard rate, Renyi entropy and order statistics. Also, the estimation of parameters is obtained by the method of maximum likelihood estimation. The Exponential Rayleigh distribution is useful as a life testing model. The model is applied to two real life data sets and it can be said that the Exponential Rayleigh distribution is more flexible than other some related models. So we conclude that the introduced model is highly competitive in the sense of fitting these two real data sets.

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