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Study of G –function in relationship with Fractional Hilfer–Prabhakar Derivative

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ABSTRACT

The objective of this paper is to investigate a generalization of Hilfer derivatives of the $G-function$ defined and studied byCornels Simon Meijer (1936). We analyze and discuss its properties in terms of Mittag-Leffler functions. Further, we show some applications of these generalized Hilfer–Prabhakar derivatives in classical equations of mathematical physics, like the heat and the free electron laser equations, and in difference-differential equations governing the dynamics of generalized renewal stochastic processes. 2020 Mathematical Sciences Classification: Primary: 26A33, 33C99, Secondary: 33E12, 33E99.

Key Words: $G-function$, Hilfer–Prabhakar derivative, Riemann-Liouville fractional Integrals, Mittag-Leffler Function.

INTRODUCTION

In the recent years fractional calculus has gained much interest mainly thanks to the increasing presence of research works in the applied sciences considering models based on fractional operators. Besides that, the mathematical study of fractional calculus has proceeded, leading to intersections with other mathematical fields such as probability and the study of stochastic processes. Currently, in the literature, there are several different definitions of fractional integrals and derivatives. Some of them such as the Riemann–Liouville integral, the Caputo and the Riemann–Liouville derivatives are thoroughly studied and actually used in applied models. Other less-known definitions such as the Hadamard and Marchaud derivatives are mainly subject of mathematical investigation (the reader interested in fractional calculus in general can consult one of the classical reference texts such as [2, 5,7].

In this paper we introduce a generalization of derivatives of both Riemann–Liouville and Caputo types and show the effect of using them in equations of mathematical physics or related to probability. In order to do so, we start from the definition of generalized fractional derivatives given by Hilfer [1]. The so-called Hilfer fractional derivative is in fact a very convenient way to generalize both definitions of derivatives as it actually interpolates them by introducing only one more real parameter $\nu \in [0, 1]$. The further generalization that we are going to discuss in this paper is given by replacing Riemann–Liouville fractional integrals with Prabhakar integrals in the definition of Hilfer derivatives. We recall that the Prabhakar integral [6] is obtained by modifying the Riemann–Liouville integral operator by extending its kernel with a three-parameter Mittag– Leffler function. This modified Mittag–Leffler function was used by Tamarkin [9] in 1930 and later gained importance in treating problems of fractional relaxation and oscillation [1,3].

1. **Definitions and Preliminaries**

Riemann-Liouville fractional derivatives

$(I_{0+}^{\alpha}f)x = \frac{1}{\Gamma(\alpha)}\int_0^x \frac{f(t)}{(x-t)^{1-\alpha}}dt; \quad (x>0)(2.1)$

Caputo derivative

$(D_{a+}^{\alpha}f)x = \frac{1}{\Gamma(m-\alpha)}\int_a^t (t-s)^{m-1-\alpha}\frac{d^m}{ds^m} f(s)ds; \quad (x>0)(2.2)$

Hilfer fractional derivatives

Let $\mu \in (0,1), \vartheta \in [0,1],\ f \in L^1[0,1], -\infty \leq a < t < b \leq \infty$.

TheHilfer derivative is defined as

$$(\mathrm{D}_{a+}^{\mu,\vartheta} f)t = (\mathrm{I}_{a+}^{\vartheta(1-\mu)} \frac{d}{dt} (\mathrm{I}_{a+}^{(1-\mu)(1-\vartheta)} f)(t) \quad (2.3)$$

Here after and without loss of generality we set a = 0. The generalization (2.3), for ν = 0, coincideswith the Riemann–Liouville derivative (2.1) and for ν = 1 with the Caputo derivative (2.2). Hilfer studied applications of a generalized fractional operator having the Riemann–Liouville and the Caputo derivatives as specific cases (see also [8, 10]).

Meijer's G-Function

The Meijer's G-function is defined as [4]

$$G_{p,q}^{mn}\left[z\left|\begin{matrix}a_1, a_2, \dots, a_p\\ b_1, b_2, \dots, b_q\end{matrix}\right.\right] = G\begin{matrix}m,n\\p,q\end{matrix}\left[z\left|\begin{matrix}a_p\\ b_q\end{matrix}\right.\right] = G_{p,q}^{m,n}(z)$$

$$= \frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m} \Gamma(b_j - s) \prod_{j=1}^{n} \Gamma(1-a_j+s) z^s ds}{\prod_{j=m+1}^{q} \Gamma(1-b_j+s) \prod_{j=n+1}^{p} \Gamma(a_j - s)} \quad (2.4)$$

where an empty product is interpreted as 1. In above equation , 0< m < q, 0< n< p, and the parameters are such, that no pole of $\Gamma(b_j - s)$, j = 1, 2, 3, … , m coincides with any pole of

$\Gamma(1- a_k+s)$, k = 1, 2,3,… ,n.

Lemma2.1

$${}^kD^{\mu,\nu}\left[(t-a)^{\frac{\lambda}{k}-1}\right](x) = \frac{\Gamma_k(\lambda)}{k\Gamma_k(1-\mu+\lambda-k)}(x-a)^{\frac{1-\mu+\lambda-k}{k}-1}.$$

Lemma2.2

$$I_k^{(1-\nu)(1-\mu)}\left[(t-a)^{\frac{\lambda}{k}-1}\right](x) = \frac{\Gamma_k(\lambda)}{\Gamma_k((1-\nu)(1-\mu)+\lambda)}(x-a)^{\frac{(1-\nu)(1-\mu)}{k}+\frac{\lambda}{k}-1}$$

(1

2. Main Result

In this section, we consider the Hilfer fractional derivative of the Meijer's G-function and making use of the given lemma to derive following useful results.

Theorem 3.1 Let $\mu \in (0,1), \vartheta \in [0,1],\ f \in L^1[0,1], -\infty \leq a < t < b \leq \infty$, and $(\mathrm{D}_{a+}^{\mu,\vartheta} f)t$ be the Hilfer fractional derivative associated with Meijer's G-function . Then there holds the following relationship

$$D_{a+}^{\mu,\vartheta}\left\{G_{p,q}^{mn}\left[x\left|\begin{matrix}a_1, a_2, \dots, a_p\\ b_1, b_2, \dots, b_q\end{matrix}\right.\right]\right\} = (x)^{\mu-1} G_{p,q+1}^{mn}\left[z\left|\begin{matrix}a_1, a_2, \dots, a_p & \\ b_1, b_2, \dots, b_q & (1-\mu, 1)\end{matrix}\right.\right]$$

Provided each member of the equation exists.

Proof By using the definition of the generalized function of fractional calculus and the fractional integral operator, we get

$$D_{a+}^{\mu,\vartheta}\left\{G_{p,q}^{mn}\left[x\left|\begin{matrix}a_1,a_2,\dots,a_p\\ b_1,b_2,\dots,b_q\end{matrix}\right.\right]\right\} = D_{a+}^{\mu,\vartheta}\left\{\frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma(b_j-s)\prod_{j=1}^{n}\Gamma(1-a_j+s)x^s ds}{\prod_{j=m+1}^{q}\Gamma(1-b_j+s)\prod_{j=n+1}^{p}\Gamma(a_j-s)}\right\}$$

$$=\frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma(b_j-s)\prod_{j=1}^{n}\Gamma(1-a_j+s)ds}{\prod_{j=m+1}^{q}\Gamma(1-b_j+s)\prod_{j=n+1}^{p}\Gamma(a_j-s)} D_{a+}^{\mu,\vartheta}\{x^s\} \quad (3.1)$$

By making use of lemma 2.1 in above equation, we get

$$D_{a+}^{\mu,\vartheta}\left\{G_{p,q}^{mn}\left[x\left|\begin{matrix}a_1,a_2,\dots,a_p\\ b_1,b_2,\dots,b_q\end{matrix}\right.\right]\right\} =$$

$$\frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma(b_j-s)\prod_{j=1}^{n}\Gamma(1-a_j+s)ds}{\prod_{j=m+1}^{q}\Gamma(1-b_j+s)\prod_{j=n+1}^{p}\Gamma(a_j-s)}\frac{\Gamma(s)}{\Gamma(1-\mu+s)}x^{1-\mu+s}$$

Or

$$\frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma(b_j-s)\prod_{j=1}^{n}\Gamma(1-a_j+s)ds}{\prod_{j=m+1}^{q}\Gamma(1-b_j+s)\prod_{j=n+1}^{p}\Gamma(a_j-s)} = (x)^{\mu-1}G_{p,q}^{mn}\left[z\left|\begin{matrix}a_1,a_2,\dots,a_p\\ b_1,b_2,\dots,b_q \quad (1-\mu,1)\end{matrix}\right.\right]$$

Theorem 3.2 *Let* $\mu\in(0,1),\vartheta\in[0,1],\quad f\in L^1[0,1],-\infty\le a<t<b\le\infty,$ *and* $(I_{a+}^{\mu,\vartheta}f)t$ *be* the Hilfer fractional integral associated with Meijer's G-function. Then there holds the following relationship

$$I_{a+}^{(1-\mu)(1-\vartheta)}\left\{G_{p,q}^{mn}\left[x\left|\begin{matrix}a_1,a_2,\dots,a_p\\ b_1,b_2,\dots,b_q\end{matrix}\right.\right]\right\} =(x)^{\mu-1}G_{p,q}^{mn}\left[z\left|\begin{matrix}a_1,a_2,\dots,a_p\\ b_1,b_2,\dots,b_q(1-\mu,1)\,(1-\vartheta,1)\end{matrix}\right.\right]$$

Provided each member of the equation exists.

Proof By using the definition of the generalized function of fractional calculus and the fractional integral operator, we get

$$I_{a+}^{(1-\mu),(1-\vartheta)}\left\{G_{p,q}^{mn}\left[x\left|\begin{matrix}a_1,a_2,\dots,a_p\\ b_1,b_2,\dots,b_q\end{matrix}\right.\right]\right\} = I_{a+}^{(1-\mu),(1-\vartheta)}\frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma(b_j-s)\prod_{j=1}^{n}\Gamma(1-a_j+s)x^s ds}{\prod_{j=m+1}^{q}\Gamma(1-b_j+s)\prod_{j=n+1}^{p}\Gamma(a_j-s)}$$

$$I_{a+}^{(1-\mu),(1-\vartheta)}\left\{G_{p,q}^{mn}\left[x\left|\begin{matrix}a_1,a_2,\dots,a_p\\ b_1,b_2,\dots,b_q\end{matrix}\right.\right]\right\} =\frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma(b_j-s)\prod_{j=1}^{n}\Gamma(1-a_j+s)x^s ds}{\prod_{j=m+1}^{q}\Gamma(1-b_j+s)\prod_{j=n+1}^{p}\Gamma(a_j-s)} I_{a+}^{(1-\mu)(1-\vartheta)}\{x^s\}$$

By making use of lemma, 2.2 in above equation, we get

$$I_{a+}^{(1-\mu),(1-\vartheta)}\left\{G_{p,q}^{mn}\left[x\left|\begin{matrix}a_1,a_2,\dots,a_p\\ b_1,b_2,\dots,b_q\end{matrix}\right.\right]\right\}$$

$$=\frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma(b_j-s)\prod_{j=1}^{n}\Gamma(1-a_j+s)x^s ds}{\prod_{j=m+1}^{q}\Gamma(1-b_j+s)\prod_{j=n+1}^{p}\Gamma(a_j-s)}\frac{\Gamma(s)}{\Gamma(1-\mu+1-\vartheta+s)}x^{(1-\mu)(1-\vartheta)+s}$$

Or

$$(x)^{\mu-1}G_{p,q}^{mn}\left[z\left|\begin{matrix}a_1,a_2,\dots,\ a_p\\ b_1,b_2,\dots,b_q(1-\mu,1)\,(1-\vartheta,1)\end{matrix}\right.\right]$$

$$= (x)^{\mu-1}\,S_{p,q+1}^{(\alpha,\beta,\gamma,\tau)}\left[\begin{matrix}a_1,a_2,\dots a_p\\ b_1,b_2,\dots.b_q,(1-\mu)(1-\vartheta)\end{matrix}\ ,\ x\right]$$

APPLICATIONS

In the recent years more and more papers have been devoted to the mathematical analysis of versions like the ***heat and the free electron laser equations***, and in difference-differential equations governing the dynamics of generalized renewal stochastic processes. Here we study a generalization of the time-fractional heat equation involving Hilfer derivatives. We present analytical results for the time-fractional heat equation involving both regularized and non regularized Hilfer derivatives operator in order to highlight the main differences between the two cases.

CONCLUSION

The Mittag-Leffler function and its generalization the S-function are of fundamental importance in the fractional calculus. It has been shown that the solution of certain fundamental linear differential equations may be expressed in terms of these functions. These functions serve as generalization of the exponential function in the solution of fractional differential equation. Hence these functions play a central role in the fractional calculus. This paper explores various intra relationships of the S-function with Hilfer derivatives of the generalized fractional integration associated with Gauss-hypergeometric function, which will be useful in further analysis.

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