

# On the Positive Pellian Equation $y^2 = 35x^2 + 29$

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## ABSTRACT

The binary quadratic equation represented by the Positive Pellian  $y^2 = 35x^2 + 29$  is analyzed for its distinct integer solutions. A few interesting relations among the solutions are given. Further, employing the solutions of the above hyperbola, we have obtained solutions of other choices of hyperbola and parabola. The formulation of second order Ramanujan numbers is illustrated.

**Keywords:** Binary quadratic, hyperbola, parabola, pell equation, integral solutions, second order Ramanujan numbers **2010 mathematics subject classification:** 11D09

## INTRODUCTION

A binary quadratic equation of the form  $y^2 = Dx^2 + 1$  where D is non-square positive integer has been studied by various mathematicians for its non-trivial integral solutions when D takes different integral values [1-2]. For an extensive review of various problems, one may refer [3-12]. In this communication, yet another interesting hyperbola given by  $y^2 = 35x^2 + 29$  is considered and infinitely many integer solutions are obtained. A few interesting properties among the solutions are obtained.

### Method of Analysis

Consider the positive pell equation

$$y^2 = 35x^2 + 29 \quad (1)$$

which is satisfied by

$$x_0 = 2, y_0 = 13$$

To obtain the other solutions of (1), consider the pellian equation

$$y^2 = 35x^2 + 1 \quad (2)$$

Initial solution is given by

$$\tilde{x}_0 = 1, \tilde{y}_0 = 6$$

The general solution  $(\tilde{x}_n, \tilde{y}_n)$  of (2) is obtained by

$$\tilde{x}_n = \frac{1}{2\sqrt{35}} g_n, \tilde{y}_n = \frac{1}{2} f_n$$

where

$$f_n = (6 + \sqrt{35})^{n+1} + (6 - \sqrt{35})^{n+1}$$

$$g_n = (6 + \sqrt{35})^{n+1} - (6 - \sqrt{35})^{n+1}$$

Using the lemma of Brahmagupta between  $(x_0, y_0)$  and  $(\tilde{x}_n, \tilde{y}_n)$ , the other integer solutions to (1) are given by

$$x_{n+1} = f_n + \frac{13}{2\sqrt{35}} g_n$$

$$y_{n+1} = \frac{13}{2} f_n + \frac{70}{2\sqrt{35}} g_n$$

Now,

$$2\sqrt{35}x_{n+1} = 2\sqrt{35}f_n + 13g_n \quad (3)$$

$$2\sqrt{35}y_{n+1} = 13\sqrt{35}f_n + 70g_n \quad (4)$$

Interchanging  $n$  by  $n+1$  in (3), we get

$$\begin{aligned} x_{n+2} &= f_{n+1} + \frac{13}{2\sqrt{35}} g_{n+1} \\ &= (6f_n + \sqrt{35}g_n) + \frac{13}{2\sqrt{35}} (6g_n + \sqrt{35}f_n) \\ 2\sqrt{35}x_{n+2} &= 25\sqrt{35}f_n + 148g_n \end{aligned} \quad (5)$$

Interchanging  $n$  by  $n+1$  in (5), we get

$$\begin{aligned} x_{n+3} &= \frac{25}{2}f_{n+1} + \frac{74}{\sqrt{35}}g_{n+1} \\ &= \frac{25}{2}(6f_n + \sqrt{35}g_n) + \frac{74}{\sqrt{35}}(6g_n + \sqrt{35}f_n) \\ x_{n+3} &= 149f_n + \frac{1763}{2\sqrt{35}}g_n \\ 2\sqrt{35}x_{n+3} &= 298\sqrt{35}f_n + 1763g_n \end{aligned} \quad (6)$$

Omitting  $f_n$  and  $g_n$  between (3), (5) and (6), we have

$$x_{n+1} - 12x_{n+2} + x_{n+3} = 0, n = 1, 2, 3 \dots \quad (7)$$

Proceeding the above steps, we have

$$2\sqrt{35}y_{n+2} = 148\sqrt{35}f_n + 875g_n \quad (8)$$

$$2\sqrt{35}y_{n+3} = 1763\sqrt{35}f_n + 10430g_n \quad (9)$$

Omitting  $f_n$  and  $g_n$  between (4), (8) and (9), we have

$$y_{n+1} - 12y_{n+2} + y_{n+3} = 0, n = 1, 2, 3 \dots \quad (10)$$

Thus (7) and (10) represent the recurrence relations satisfied by the values of  $X$  and  $Y$  respectively.

Some numerical examples of  $X_n$  and  $Y_n$  satisfying (1) are given in the Table 1 below :

**Table: 1 Numerical Examples**

$n$	$x_{n+1}$	$y_{n+1}$
-1	2	13
0	25	148
1	298	1763
2	3551	21008
3	42314	250333

### **OBSERVATIONS**

- $x_n$  and  $y_n$  values are alternatively odd and even.
- One can generate second order Ramanujan numbers by choosing  $x$  and  $y$  values suitably .For illustration, consider

$$\begin{aligned}
 y_3 &= 21008 \\
 &= 2 * 10504 = 4 * 5252 = 8 * 2626 \\
 &= 5253^2 - 5251^2 = 2628^2 - 2624^2 = 1317^2 - 1309^2
 \end{aligned}$$

Now

$$\begin{aligned}
 5253^2 - 5251^2 &= 2628^2 - 2624^2 \\
 \Rightarrow 5253^2 + 2624^2 &= 2628^2 + 5251^2 = 34479385 \\
 5253^2 - 5251^2 &= 1317^2 - 1309^2 \\
 \Rightarrow 5253^2 + 1309^2 &= 1317^2 + 5251^2 = 29307490 \\
 2628^2 - 2624^2 &= 1317^2 - 1309^2 \\
 \Rightarrow 2628^2 + 1309^2 &= 1317^2 + 2624^2 = 8619865
 \end{aligned}$$

Thus, 34479385,29307490,8619865 are second order Ramanujan numbers.

### **1. Relations among the solutions :**

- ❖  $35x_{n+3} + y_{n+2} - 6y_{n+3} = 0$
- ❖  $35x_{n+2} + 6y_{n+2} - y_{n+3} = 0$
- ❖  $35x_{n+1} + 71y_{n+2} - 6y_{n+3} = 0$
- ❖  $420x_{n+3} + y_{n+1} - 71y_{n+3} = 0$
- ❖  $70x_{n+2} + y_{n+1} - y_{n+3} = 0$
- ❖  $420x_{n+1} + 71y_{n+1} - y_{n+3} = 0$
- ❖  $35x_{n+3} + 6y_{n+1} - 71y_{n+2} = 0$
- ❖  $35x_{n+2} + y_{n+1} - 6y_{n+2} = 0$
- ❖  $35x_{n+1} + 6y_{n+1} - y_{n+2} = 0$
- ❖  $x_{n+2} + y_{n+3} - 6x_{n+3} = 0$

- ❖  $x_{n+1} + 12y_{n+3} - 71x_{n+3} = 0$
- ❖  $6x_{n+2} + y_{n+2} - x_{n+3} = 0$
- ❖  $71x_{n+2} + y_{n+1} - 6x_{n+3} = 0$
- ❖  $71x_{n+1} + 12y_{n+1} - x_{n+3} = 0$
- ❖  $6x_{n+1} + y_{n+3} - 71x_{n+2} = 0$
- ❖  $x_{n+1} + y_{n+2} - 6x_{n+2} = 0$
- ❖  $6x_{n+1} + y_{n+1} - x_{n+2} = 0$
- ❖  $x_{n+1} + 2y_{n+2} - x_{n+3} = 0$

## 2. Expressions representing nasty numbers:

Solving (3) and (5), we get

$$f_n = \frac{2}{29} [13x_{n+2} - 148x_{n+1}] \quad (11)$$

$$g_n = \frac{2\sqrt{35}}{29} [25x_{n+1} - 2x_{n+2}] \quad (12)$$

Interchanging  $n$  by  $2n+1$  in (11) we get

$$f_{2n+1} = \frac{2}{29} [13x_{2n+3} - 148x_{2n+2}]$$

Now  $f_{2n+1} + 2 = f_n^2$

$$\Rightarrow \frac{12}{29} [13x_{2n+3} - 148x_{2n+2} + 29] = 6f_n^2, \text{ a nasty number} \quad (13)$$

For simplicity and clear

understanding, the other choices of nasty numbers are presented below:

- ❖  $\frac{1}{29} [13x_{2n+4} - 1763x_{2n+2} + 348]$
- ❖  $\frac{12}{29} [13y_{2n+2} - 70x_{2n+2} + 29]$
- ❖  $\frac{2}{29} [13y_{2n+3} - 875x_{2n+2} + 174]$
- ❖  $\frac{12}{2059} [13y_{2n+4} - 10430x_{2n+2} + 2059]$
- ❖  $\frac{12}{29} [148x_{2n+4} - 1763x_{2n+3} + 29]$
- ❖  $\frac{4}{29} [74y_{2n+2} - 35x_{2n+3} + 87]$
- ❖  $\frac{12}{29} [148y_{2n+3} - 875x_{2n+3} + 29]$
- ❖  $\frac{4}{29} [74y_{2n+4} - 5215x_{2n+3} + 87]$

- ❖  $\frac{12}{2059} [1763y_{2n+2} - 70x_{2n+4} + 2059]$
- ❖  $\frac{2}{29} [1763y_{2n+3} - 875x_{2n+4} + 174]$
- ❖  $\frac{12}{29} [1763y_{2n+4} - 10430x_{2n+4} + 29]$
- ❖  $\frac{12}{203} [175y_{2n+2} - 14y_{2n+3} + 203]$
- ❖  $\frac{2}{29} [149y_{2n+2} - y_{2n+4} + 174]$
- ❖  $\frac{12}{203} [203y_{2n+3} - 175y_{2n+4} + 203]$

### 3. Expressions representing cubical integers:

Interchanging  $n$  by  $3n+2$  in (11), we get

$$f_{3n+2} = \frac{2}{29} [13x_{3n+4} - 148x_{3n+3}]$$

$$\text{Now } f_{3n+2} = f_n^3 - 3f_n$$

$$\Rightarrow f_{3n+2} + 3f_n = f_n^3$$

$$\Rightarrow \frac{2}{29} [13x_{3n+4} - 148x_{3n+3} + 39x_{n+2} - 444x_{n+1}] = f_n^3 \text{ is a cubical integer.}$$

For simplicity and clear understanding, the other choices of cubical integers are presented below:

- ❖  $\frac{1}{174} [13x_{3n+5} - 1763x_{3n+3} + 39x_{n+3} - 5289x_{n+1}]$
- ❖  $\frac{2}{29} [13y_{3n+3} - 70x_{3n+3} + 39y_{n+1} - 210x_{n+1}]$
- ❖  $\frac{1}{87} [13y_{3n+4} - 875x_{3n+3} + 39y_{n+2} - 2625x_{n+1}]$
- ❖  $\frac{2}{2059} [13y_{3n+5} - 10430x_{3n+3} + 39y_{n+3} - 31290x_{n+1}]$
- ❖  $\frac{2}{29} [148x_{3n+5} - 1763x_{3n+4} + 444x_{n+3} - 5289x_{n+2}]$
- ❖  $\frac{2}{87} [74y_{3n+3} - 35x_{3n+4} + 222y_{n+1} - 105x_{n+2}]$
- ❖  $\frac{2}{29} [148y_{3n+4} - 875x_{3n+4} + 444y_{n+2} - 2625x_{n+2}]$
- ❖  $\frac{2}{87} [74y_{3n+5} - 5215x_{3n+4} + 222y_{n+3} - 15645x_{n+2}]$

- ❖  $\frac{2}{2059} [1763y_{3n+3} - 70x_{3n+5} + 5289y_{n+1} - 210x_{n+3}]$
- ❖  $\frac{1}{87} [176y_{3n+4} - 875x_{3n+5} + 5289y_{n+2} - 2625x_{n+3}]$
- ❖  $\frac{2}{29} [1763y_{3n+5} - 10430x_{3n+5} + 5289y_{n+3} - 31290x_{n+3}]$
- ❖  $\frac{2}{203} [175y_{3n+3} - 14y_{3n+4} + 525y_{n+1} - 42y_{n+2}]$
- ❖  $\frac{1}{87} [149y_{3n+3} - y_{3n+5} + 447y_{n+1} - 3y_{n+3}]$
- ❖  $\frac{2}{203} [2086y_{3n+4} - 174y_{3n+5} + 6258y_{n+2} - 525y_{n+3}]$

#### 4. Expressions representing bi-quadratic integers:

Interchanging  $n$  by  $4n+3$  in (11), we get

$$f_{4n+3} = \frac{2}{29} [13x_{4n+5} - 148x_{4n+4}]$$

Now  $f_{4n+3} + 4f_n^2 - 2 = f_n^4$

$$\Rightarrow \frac{2}{29} [13x_{4n+5} - 148x_{4n+4} + 52x_{2n+3} - 592x_{2n+2} + 87] = f_n^4$$

a bi-quadratic integer.

For simplicity and clear understanding ,the other choices of bi-quadratic integers are presented below:

- ❖  $\frac{1}{174} [13x_{4n+6} - 1763x_{4n+4} + 52x_{2n+4} - 7052x_{2n+2} + 1044]$
- ❖  $\frac{2}{29} [13y_{4n+4} - 70x_{4n+4} + 52y_{2n+2} - 280x_{2n+2} + 87]$
- ❖  $\frac{1}{87} [13y_{4n+5} - 875x_{4n+4} + 52y_{2n+3} - 3500x_{2n+2} + 522]$
- ❖  $\frac{2}{2059} [13y_{4n+6} - 10430x_{4n+4} + 52y_{2n+4} - 41720x_{2n+2} + 6177]$
- ❖  $\frac{2}{29} [148x_{4n+6} - 1763x_{4n+5} + 592x_{2n+4} - 7052x_{2n+3} + 87]$
- ❖  $\frac{2}{87} [74y_{4n+4} - 35x_{4n+5} + 296y_{2n+2} - 140x_{2n+3} + 261]$
- ❖  $\frac{2}{29} [148y_{4n+5} - 875x_{4n+5} + 592y_{2n+3} - 3500x_{2n+3} + 87]$
- ❖  $\frac{2}{87} [74y_{4n+6} - 5215x_{4n+5} + 296y_{2n+4} - 20860x_{2n+3} + 261]$

- ❖  $\frac{2}{2059} [1763y_{4n+4} - 70x_{4n+6} + 7052y_{2n+2} - 280x_{2n+4} + 6177]$
- ❖  $\frac{1}{87} [1763y_{4n+5} - 875x_{4n+6} + 7052y_{2n+3} - 3500x_{2n+4} + 522]$
- ❖  $\frac{2}{29} [1763y_{4n+6} - 10430x_{4n+6} + 7052y_{2n+4} - 41720x_{2n+4} + 87]$
- ❖  $\frac{2}{203} [175y_{4n+4} - 14y_{4n+5} + 700y_{2n+2} - 56y_{2n+3} + 609]$
- ❖  $\frac{1}{87} [149y_{4n+4} - y_{4n+6} + 596y_{2n+2} - 4y_{2n+4} + 522]$
- ❖  $\frac{2}{203} [2086y_{4n+5} - 175y_{4n+6} + 8344y_{2n+3} - 700y_{2n+4} + 609]$

##### **5. Expressions representing quintic integers:**

Interchanging  $n$  by  $5n+4$  in (11), we have

$$f_{5n+4} = \frac{2}{29} [13x_{5n+6} - 148x_{5n+5}]$$

Now  $f_{5n+5} = f_n^5 - 5f_n^3 + 5f_n$

$$f_n^5 = f_{5n+5} + 5f_n^3 - 5f_n$$

$$\Rightarrow \frac{2}{29} [13x_{5n+6} - 148x_{5n+5} + 65x_{3n+4} - 740x_{3n+3} + 130x_{n+2} - 1480x_{n+1}] = f_n^5$$

is a quintic integer.

For simplicity and clear understanding, the other choices of quintic integers are presented below:

- ❖  $\frac{1}{174} [13x_{5n+7} - 1763x_{5n+5} + 65x_{3n+5} - 8815x_{3n+3} + 130x_{n+3} - 17630x_{n+1}]$
- ❖  $\frac{2}{29} [13y_{5n+5} - 70x_{5n+5} + 65y_{3n+3} - 350x_{3n+3} + 130y_{n+1} - 700x_{n+1}]$
- ❖  $\frac{1}{87} [13y_{5n+6} - 875x_{5n+5} + 65y_{3n+4} - 4375x_{3n+3} + 130y_{n+2} - 8750x_{n+1}]$
- ❖  $\frac{2}{2059} [13y_{5n+7} - 10430x_{5n+5} + 65y_{3n+5} - 52150x_{3n+3} + 130y_{n+3} - 104300x_{n+1}]$
- ❖  $\frac{2}{29} [148x_{5n+7} - 1736x_{5n+6} + 740x_{3n+5} - 8815x_{3n+4} + 1480x_{n+3} - 17630x_{n+2}]$
- ❖  $\frac{2}{87} [74y_{5n+5} - 35x_{5n+6} + 370y_{3n+3} - 175x_{3n+4} + 740y_{n+1} - 350x_{n+2}]$
- ❖  $\frac{2}{29} [148y_{5n+6} - 875x_{5n+6} + 740y_{3n+4} - 4375x_{3n+4} + 1480y_{n+2} - 8750x_{n+2}]$
- ❖  $\frac{2}{87} [74y_{5n+7} - 5215x_{5n+6} + 370y_{3n+5} - 26075x_{3n+4} + 740y_{n+3} - 52150x_{n+2}]$

- ❖  $\frac{2}{2059} [1763y_{5n+5} - 70x_{5n+7} + 8815y_{3n+3} - 350x_{3n+5} + 1763y_{n+1} - 700x_{n+3}]$
- ❖  $\frac{2}{87} [1763y_{5n+6} - 875x_{5n+7} + 8815y_{3n+4} - 4375x_{3n+5} + 17630y_{n+2} - 8750x_{n+3}]$
- ❖  $\frac{2}{29} [1763y_{5n+7} - 10430x_{5n+7} + 8815y_{3n+5} - 5215x_{3n+5} + 17630y_{n+3} - 10430x_{n+3}]$
- ❖  $\frac{2}{203} [175y_{5n+5} - 140y_{5n+6} + 875y_{3n+3} - 70y_{3n+4} + 1750y_{n+1} - 140y_{n+2}]$
- ❖  $\frac{1}{87} [149y_{5n+5} - y_{5n+7} + 745y_{3n+3} - 5y_{3n+5} + 1490y_{n+1} - 10y_{n+3}]$
- ❖  $\frac{2}{203} [2086y_{5n+6} - 175y_{5n+7} + 10430y_{3n+4} - 875y_{3n+5} + 20860y_{n+2} - 1750y_{n+3}]$

**6 .** Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of hyperbola. For illustration ,let

$$X_n = 25x_{n+1} - 2x_{n+2}$$

$$Y_n = 13x_{n+2} - 148x_{n+1}$$

We know that  $f_n^2 - g_n^2 = 4$  (14)

By applying (11) and (12) in (14), we have

$$\begin{aligned} \frac{1}{841} Y_n^2 - \frac{35}{841} X_n^2 &= 1 \\ \Rightarrow Y_n^2 - 35X_n^2 &= 841 \end{aligned}$$

which represents a hyperbola.

**7.** Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of parabola .For illustration, let

$$\begin{aligned} X_n &= 25x_{n+1} - 2x_{n+2} \\ Y_n &= 13x_{2n+3} - 148x_{2n+2} + 29 \end{aligned}$$

From (13),

$$\frac{2}{29} [13x_{2n+3} - 148x_{2n+2} + 29] = f_n^2 (15)$$

By applying (15) and (12) in (14), we have

$$29Y_n - 70X_n^2 = 1682$$

which represents a parabola.

## CONCLUSION

The hyperbola represented by the positive pell equation  $y^2 = 35x^2 + 29$  is studied for finding its non-zero distinct integer solutions. A few interesting relations among the solutions are presented. As quadratic equations are rich in variety,one may attempt for determining the integer solutions to other choices of quadratic equations with two or more variables with suitable properties.

#### REFERENCES

- [1] L.E. Dickson, History of Theory of Numbers, Chelsea Publishing Company, Newyork, Vol.II, 1952
- [2] L.J. Mordell, Diophantine Equations, Academic Press , Newyork, 1969.
- [3] M.A. Gopalan et al., Integral points on the Hyperbola  $x^2 + 6xy + y^2 + 40x + 80y + 40 = 0$ , Bessel J Math., 2(3), 2012, 159-164.
- [4] M.A. Gopalan et al., Observation on the Hyperbola  $y^2 = 24x^2 + 1$ , Bessel J Math, 4, 2013, 21-25.
- [5] T. Geetha et al., Observation on the hyperbola,  $y^2 = 72x^2 + 1$ , Scholars Journal of Physics, Mathematics and Statistics, 1(1), 2014, 1-13.
- [6] M.A. Gopalan et al., Remarkable Observations on the hyperbola  $y^2 = 24x^2 + 1$ , Bulletin of Mathematics and Statistics Research, 1, 2014, 9-12.
- [7] M.A. Gopalan et al., Observation on the hyperbola,  $y^2 = 220x^2 + 9$ , Star Research Journal, Vol.4, Issue3(2), 2016, 12-16.
- [8] S. Ramya, A. Kavitha, On the positive pell equation,  $y^2 = 90x^2 + 31$ , Journal of Mathematics and informatics, 11(1), 2017, 11-17.
- [9] G. Sumathi, Observations on the hyperbola,  $y^2 = 150x^2 + 16$ , International Journal of Recent Trends in Engineering and Research, 3(9), 2017, 198-206.
- [10] K. Meena et al, On the positive pell equation  $y^2 = 102x^2 + 33$ , International Journal of Advanced Education and Research 2(1), 2017, 91-96
- [11] K. Meena et al, On the positive pell equation  $y^2 = 35x^2 + 14$ , International Journal for Research in Applied Science & Engineering Technology (IJRASET), Volume 7, Issue I, Jan 2019, 240-247.
- [12] K. Meena et al, On the positive pell equation  $y^2 = 32x^2 + 41$ , International Journal for Research in Applied Science & Engineering Technology (IJRASET), Volume 7, Issue I, Jan 2019, 258-265.