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Some Results on Autocentric Groups

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ABSTRACT

In this paper we define a new concept of Autocentric groups and prove some results for a group having cyclic center to be Autocentric.

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INTRODUCTION

Let G be a group and $Z(G)$, $Inn(G)$ and $Aut(G)$ denote the center ,group of all inner automorphisms and the group of all automorphism of G, respectively. An automorphism σ of a group is called central automorphism if it commutes with all inner automorphisms of the group.The set of all central automorphisms of a group G is a normal subgroup of $Aut(G)$ and is denoted by $Aut_c(G)$.We know that if G is cyclic group of order n, then $Aut(G)$ is also a cyclic group of order $\phi(n)$. It is alsoeasy to see that if $Z(G)$ is trivial, then $Z(Aut(G))$ is trivial. But we also have groups having center non-trivial whlile $Z(Aut(G))$ is trivial (for example $C_2 \times C_2$). It is natural to find out the conditionsunder which the groups $Z(Aut(G))$ and $Aut(Z(G))$ are isomorphic. In [2] Farrokhi and Moghaddam find the structure of $Z(Aut(G))$ for a group G having cyclic center. We intend to study finite groups and try to find out the conditions that make $Z(Aut(G))$ and $Aut(Z(G))$ isomorphic. We take all the groups to be finite without explicitly mentioned throughout this paper.

PRILIMINARIES

In this section we state some results and definitions that will be used in our main results.

Proposition 2.1. *Let $Aut_c(G)$ be the group of all central automorphisms of a group G. Then $g^{-1}\phi(g) \in Z(G)$ for each $\phi \in Aut_c(G)$ and $g \in G$. In particular $Z(Aut(G)) \subseteq Aut_c(G)$.*

Proof. It is easy to prove by using the the definition of central automorphism.

Now we define autocentric groups.

Definition 2.2. *A group G is called autocentric if $Z(Aut(G)) \cong Aut(Z(G)$.*

Clearly a cyclic group is autocentric. Example of non-cyclic autocentric group is S_3.

Remark 2.3. *A centerless group is autocentric. In general a complete group is autocentric.*

Lemma 2.4. *A finite autocentric group which is not cyclic must be non-abelian.*

Proof. Suppose G is abelian. Then $Aut(Z(G)) = Aut(G)$. Thus, we have $Aut(G) \cong Z(Aut(G))$. And this gives $Aut(G)$ is abelian. But we know if G is abelian, then $Aut(G)$ is abelian if and only if G is cyclic. A contradiction.

Definition 2.5. *A group G is called Miller group if $Aut(G)$ is abelian.*

Remark 2.6. *A non-abelian Miller group can not be autocentric.*

Lemma 2.7. *If* $G = H \times K$ *,where* $(|H|, |K| = 1)$, *then* $Aut(G) = Aut(H) \cong Aut(K)$.

Lemma 2.8. *If* $G = H \times K$ *,where* $(|H|, |K| = 1)$ *and both are autocentric, then* G *is autocentric.* *Proof.* It is easy to prove by using the above lemma.

Remark 2.9. *A group G can not be autocentric unless* $Z(G)$ *is non-trivial cyclic group. Thus, a necessary condition for a group to be autocentric is to have cyclic center. But it is not a sufficient condition. Thus, autocentric groups are purely non-abelian as we know the groups having centre as cyclic group is purely non-abelian.*

MAIN RESULTS

In this section we aim to present our main result.

Let C^* denote the group of all central automorphisms fixing the center elementwise.

Theorem 3.1. *Let G be a group with cyclic center. If* $Z(Aut(G)) \cap C^* =< 1 >$, *then* $Z(Aut(G))$ *is isomorphic to subgroup of* $Aut(Z(G))$.

Proof. To begin the proof , first we state the following lemmas and definitions, which have been statedin [2]

Definition 3.2. *Let G be a group with cyclic center* $Z(G) =< z >$ *and let* $\phi \in Aut_c(G)$. *Then* $\overline{\phi}$ *is the homomorphism from G to Z(G), which sends g to* $g^{-1}\phi(g)$, *for each* $g \in G$. *Define* α_ϕ *to be the smallest non-negative integer k such that* $\overline{\phi}(z) = z^k$.

Using the definition of α_ϕ, the following lemma can be proved:

Lemma 3.3. *Let G be a group with cyclic center of order n. Then for all* $\phi, \psi \in Z(Aut(G))$,
n
(1) $\alpha_{\phi\psi} + 1 \equiv (\alpha_\phi + 1)(\alpha_\psi + 1)$,
(2)$exp(Z(Aut(G)) \leq exp(Z(G))$.

Proof. see [2]

Proof of the main theorem is as under:

Define a map
$\alpha^* : Z(Aut(G)) \rightarrow Aut(Z(G) \cong U(Z_n)$ as:
$\alpha^*(\phi) = \widehat{\phi + 1}, \phi \in Z(Aut(G))$, where $\widehat{\phi + 1}$ is an automorphism of Z(G) which sends z to $z^{\alpha_\phi + 1}$.
Then α^* is a homomorphism with kernel $Z(Aut(G) \cap C^*$, hence the result.

Corollary 3.4. *Let G be a group with cyclic center. Then* $\frac{Aut_c(G)}{C^*}$ *isomorphic to subgroup of* $Aut(Z(G))$.

Proof. Proceed as above by taking map α^* from $Aut_c(G)$ to Aut(Z(G)).

Corollary 3.5. *If G is finite group of nilpotency class 2 with* $Z(G)$ *cyclic, then* $\frac{Aut_c(G)}{Inn(G)}$ *is isomorphic to subgroup of* $Aut(Z(G))$.

Proof. In this case $C^* \cong Inn(G)$, see[1].
Using the above theorem, we can state the following remark:

Remark 3.6. *If G is a p-group with* $|Z(G)| = p$ *and p- odd prime, then G can not be autocentric.*

The above remark is not true for p=2, for example, Q_8, the quaternion group of order 8, is auto-centric.

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