International Journal of Enhanced Research in Science, Technology & Engineering
ISSN: 2319-7463, Vol. 5 Issue 5, May-2016

Feedback Linearized Model of DC Motor using Differential Geometry

Abhishek Bansal

M.S- Mathematics, M. Tech. (P) –Electrical Engg. (Power System & Control Engg.), Delhi, India

ABSTRACT

This research paper discusses the existing technique of obtaining an exact feedback linearization mathematical model of non linear model of a DC motor using differential geometry approach. The singularly perturbed system, input-state linearization and the input-output linearization has also been developed along with these tools and the results of geometric approach are compared and found similar. In this paper, all equations are developed for DC motor which is field controlled rather than armature controlled.

Keywords: Lyapunov stability, Lie derivative, DC motor, feedback, differential geometry, linearization, perturbation,field control.

1. INTRODUCTION

A DC motor is a DC machine which converts DC (Direct Current) electrical power into mechanical power.

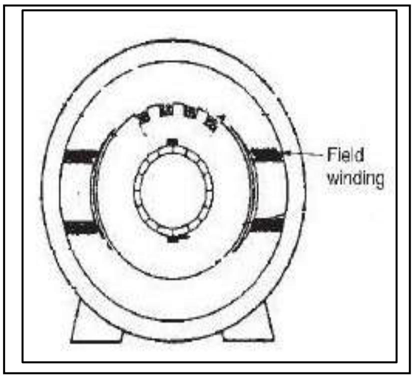

Figure 1. A DC machine

A DC motor contains a current carrying armature which is connected to the supply end through commutator segments and brushes and placed within the poles of electromagnet. Fig. 1 shows the basic part of a DC machine, and the field windings which are the windings (i.e. many turns of a conductor) wound round the pole core and the current passing through this conductor creates electromagnet. DC motors can be either shunt wound or series wound or a combination of both known as compound DC motor. The speed of these DC motors can be controlled by either armature control method or field control method.

The field controlled method affects the flux per pole. In the field control of DC series motor, either field diverter or tapper field method is used ,whereas in shunt and compound DC motors, generally field rheostat control method is employed. This paper is not of the speed controller designing but of the linearized mathematical modeling of a non-linear model.

In the field controlled, let the control input is the voltage of the field circuit, υ_f, then we have the following set of equations:

Field circuit Equation : $\upsilon_f = R_f i_f + L_f \frac{d\, i_f}{\mathrm{dx}}$, where υ_f is the voltage of the field circuit,R_f is the resistance of the field circuit, i_f is the current of the field circuit, L_f is the inductance of the field circuit.

Let the time constant of the field circuit, $T_f = L_f / R_f$

Armature circuit Equation : $v_a = c_1 i_f \omega + L_a \frac{d\, i_a}{\mathrm{dx}} + R_a\, i_a$,where v_a is the voltage of the armature circuit,R_a is the resistance of the armature circuit,i_a is the current of the armature circuit,L_a is the inductance of the armature circuit, $c_1 i_f \omega$ is the back e.m.f. induced in the armature circuit.
Let the time constant of the armature circuit, $T_a = L_a / R_a$

Torque Equation for the shaft: $J \frac{d\,\omega}{\mathrm{dx}} = c_2 i_f i_a - c_3 \omega$,where J is the rotor inertia, c_3 is the damping inertia and $c_2 i_f i_a$ is the torque produced by the interaction of the armature current with the field circuit flux.

Let v_a, v_f be constant such that $v_a = V_a$ and $v_f = U$.

A singularly perturbed system can be modeled if we assume the voltages of the armature circuit and the field circuit be constant and choose (I_f, I_a, Ω)as a nominal operating point, thusthe system has a unique equilibrium point at

$$I_f = \frac{U}{R_f}\,,\ I_a = \frac{c_3 V_a}{c_3 R_a + c_1 c_2 U^2 / R_f^2}\ \ ,\ \Omega = \frac{c_2 V_a U / R_f}{c_3 R_a + c_1 c_2 U^2 / R_f^2}$$

Now,let the state equations have discontinuous dependence on a small perturbation parameter$\varepsilon$and as $T_f \gg T_a$,the above system can be modeled as a singularly perturbed system with slow variables ($i_f$andΩ) and fast variables(i_a),which can written as
$\dot{x}_1 = -x_1 + u$; $\dot{x}_2 = a(x_1 z - x_2)$; $\varepsilon \dot{z} = -z - b x_1 x_2 + c$
where, $x_1 = i_f / I_f$; $x_2 = \omega / \Omega$; $z = i_a / I_a$; $u = v_f / U$; $\varepsilon = T_a / T_f$; $t' = t / T_f$; $a = L_f c_3 / R_f J$;
$b = c_1 c_2 U^2 / c_3 R_a R_f^2$; $c = V_a / I_a R_a$

Now, suppose the field circuit would have been driven by a current source ,then the control input would have been the field current instead field voltage of the field circuit. In this case, this system can be modeled, if the domain of operation is restricted to $x_1 > \theta_3 / 2\theta_1$,as :
$\dot{x}_1 = -\theta_1 x_1 - \theta_2 x_2 u + \theta_3$; $\dot{x}_2 = -\theta_4 x_2 - \theta_5 x_1 u$; $y = x_2$
where,x_1 is the armature current ; x_2 is the speed; u is the field current ;$\theta_{1,}\ \theta_{2,}\ \theta_{3,}\ \theta_{4,}\ \theta_5$are positive constants
$y_R^2 < (\theta_3^2\, \theta_5) / (4\, \theta_{1,} \theta_{2,} \theta_4)$

2. FEEDBACK LINEARIZATION CONCEPT USING DIFFERENTIAL GEOMETRY APPROACH

Consider a MIMO (multiple-input,-output) system involving$n$integrators having r inputs $u_1(t), u_2(t), u_3(t), \ldots, u_r(t)$; m outputs $y_1(t), y_2(t), \ldots, y_m(t)$ and the outputs of these integrators are defined as state variables $x_1(t), x_2(t), \ldots, x_n(t)$. Then, we get state-space equations defined as

System State Equation : $\dot{x} = f(x, u, t)$ (2.1)

and Output Equation : $y = g(x, u, t)$. (2.2)

Now if the state equation (2.1) is represented without inputs u, it is called unforced state equation ,written as

$$\dot{x} = f(x, t); \qquad (2.3)$$

where now input can either be zero or input could be specified as a function of time or a given feedback function of the state.

And if the equation (2.3) is invariant of time t, it makes the system an autonomous system, written as

$$\dot{x} = f(x) \qquad (2.4)$$

where,$f : D \to R^n$ is a locally Lipchitz map from a domain $D \subset R^n$ into R^n.

A function f is said to satisfy a Lipchitz condition of order α at c if there exists a positive number $M$and a 1-ball $B(c)$ such that $| f(x) - f(c) | < M |x - c|^{\alpha}$, whenever $x \in B(c), x \neq c$.
And $f(x)$ will be 'locally Lipchitz' on open and connected set,if each point of D has a neighborhood D_0 such that $f$satisfies the Lipchitz condition for all points in $D_0$with some Lipchitz constant $L_{0.}$

Let $x = 0$ be the equilibrium point for (2.4) and let n-dimensional Euclidean space is denoted by R^n, and $D \subset R^n$ be a domain containing the origin, i.e. $x = 0$. Now using Lyapunov stability theorem, which states that $x = 0$ is stable, if $V : D \to R$ is a continuously differentiable function, such that $V(0) = 0$ and $V(x) > 0$ in $D - \{0\}$ (2.5)

and $\dot{V}(x) \le 0$ in D (2.6)

where, $\dot{V}(x)$, is the derivative of V along the trajectories of (2.4), given by

$$\dot{V}(x) = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} \dot{x}_i = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} f_i(x) = \frac{\partial V}{\partial x} f(x) \qquad (2.7)$$

Thus, the equilibrium point is stable if, for each $\varepsilon > 0$, there is $\delta = \delta(\varepsilon) > 0$ such that for all $t \ge 0$, we have $\|x(0)\| < \delta \Rightarrow \|x(0)\| < \varepsilon$ (2.8)

We know that if we have a map $M : U \subset V \to W$ such that there is bijection M between open sets $U_\alpha \subset V$ and $U_\beta \subset W$, then it is called a C^r- diffeomorphism if and only if M and M^{-1} are both C^r-differentiable.
If $r = \infty$ then M is called diffeomorphism.

In general, to obtain a linear mathematical model for a nonlinear system, it is assumed that the variables deviate only slightly from some operating point. To get feedback linearization model, nonlinear system should be of the form

$$\dot{x} = f(x) + g(x)u \qquad (2.9)$$
$$y = h(x) \qquad (2.10)$$

where, $f(x)$ is a locally Lipchitz ; $g(x), h(x)$ are continuous over R^n ; $x \in R^n$ is the state vector ;
$u \in R^r$ is the vector of inputs ; $y \in R^m$ is the vector of outputs

Then, the equivalent linearized system is obtained using the change of variables and a control input, which then presents a linear input-output map between the new input and the output.
Thus, we a state feedback control, given as

$$u = \alpha(x) + \beta(x)v \qquad (2.11)$$

and the change of variables by $z = T(x)$ (2.12)

From the theory of "Lie derivatives", we know that
Let there be a scalar field $p(\xi)$ in an n-dimensional space X_n and an infinitesimal point transformation such that, $T : {}'\xi^x = \xi^x + v^x\, dt$ and let the dragging along $(\chi) \to (\chi')$ of the coordinate system (χ) by the infinitesimal point transformation $T^{-1} : {}'\xi \to \xi$ inverse to T is given by

$$\xi^{x'} = \xi^x + v^x\, dt \qquad (2.13)$$

Suppose we have a covariant vector field $w_\lambda(\xi)$ in X_n and a new covariant vector field ${}'w_\lambda(\xi)$ at ξ as a field whose components ${}'w_{\lambda'}(\xi)$ w.r.t. (χ') at ξ such that ${}'w_{\lambda'}(\xi) \stackrel{\text{def}}{=} w_\lambda({}'\xi)$
Then, the Lie derivative of w_λ is given by $L_v w_\lambda = v^\mu\, \partial_\mu\, w_\lambda + w_\mu\, \partial_\lambda\, v^\mu$

Thus, Lie derivative evaluates the change of a field along the flow of another vector field. Using this concept in (2.9) and (2.10), we can write the Lie derivative of h along f as

$$L_f\, h(x) = \frac{\partial h}{\partial x} f(x) \qquad (2.14)$$

Similarly, we can define other Lie derivatives as

$$L_g\, L_f\, h(x) = \frac{\partial (L_f h)}{\partial x} g(x) \qquad (2.15)$$
$$L_f^k\, h(x) = \frac{\partial L_f^{k-1} h}{\partial x} f(x) \qquad (2.16)$$
$$L_f^0\, h(x) = h(x) \qquad (2.17)$$

The system in (2.9) is input-state linearizable if and only if $\exists\, h(x)$ satisfies (2.18) and (2.19)

$$L_g\, L_f^i\, h(x) = 0\,, 0 \le i \le n-2 \qquad (2.18)$$

and $L_g\, L_f^{n-i}\, h(x) \neq 0\,, \forall\, x \in D_x$ (2.19)

Thus, the system in (2.9) and (2.10) has a relative degree n in D_x and the change of variables (2.12) can be applied as

$$z = T(x) = \left[\, h(x), L_f h(x), \ldots, L_f^{n-1} h(x)\, \right]^T \qquad (2.20)$$

Now, since for all$x \in D$,the row vectors and column vectors are linearly independent, thus the Jacobian matrix $\left[\frac{\partial T}{\partial x}\right](x)$ is non-singular $\forall\, x \in D_x$.Hence, for each $x_0 \in D_x$,there is neighborhood N of x_0 such that $T(x)$, restricted to N,is a diffeomorphism.

Thus, it can be said that the system is input-state linearizable if and only if it satisfies (2.21) and (2.22) and $\exists$ domain $D_x \subset D$, such that matrix $G(x)$ has a rank $n \quad \forall\, x \in D_x$ (2.21)
and distribution, Δ is involutive in D_x (2.22)
,where matrix$G(x) = \left[g(x), ad_f g(x), \dots, ad_f^{n-1} g(x)\right]$ and distribution, $\Delta = span\{g, ad_f g, \dots, ad_f^{n-2} g\}$

A non-linear system (2.9),where $D_x \in R^n, f: D_x \to R^n, G : D_x \to R^{n\times p}$, is said to be input-state linearizable, if there exists a diffeomorphism $T: D_x \to R^n$, such that $D_z = T(D_x)$,contains the origin and (2.12) transforms (2.9) into
$\dot{z} = Az + B\beta^{-1}(x)[\,u - \alpha(x)\,]$ (2.23)
with(A, B)controllable and $\beta(x)$non-singular for all,$x \in D_x$.

Thus, we get $\dot{z} = \frac{\partial T}{\partial x}\dot{x} = \frac{\partial T}{\partial x}[f(x) + G(x)u]$ (2.24)
and $\dot{z} = AT(x) + B\beta^{-1}(x)[\,u - \alpha(x)\,]$ (2.25)

From (2.24) and (2.25), we get $\frac{\partial T}{\partial x}[f(x) + G(x)u] = AT(x) + B\beta^{-1}(x)[\,u - \alpha(x)\,]$ (2.26)

Now if we put $u = 0$ in (2.26), we get
$\frac{\partial T}{\partial x} f(x) = AT(x) - B\beta^{-1}(x)\,\alpha(x)$ (2.27)
$\frac{\partial T}{\partial x} G(x) = B\beta^{-1}(x)$ (2.28)

Thus, if there is a map $T(\cdot)$,which satisfies (2.27) and (2.28) for some $\alpha, \beta, A, B,$ then (2.12) transforms (2.9) into (2.23).

Now, if apply linear state transformation in (2.23) with $\varsigma = M\,z$ with a singular M, then the state equation in the $\varsigma -$ coordinates are given by
$\dot{\varsigma} = M\,A\,M^{-1}\varsigma + M\,B\beta^{-1}(x)[\,u - \alpha(x)\,]$ (2.29)

Now if we have single input system instead of multiple-input, thus, now $p = 1$. In this case, for any controllable pair (A, B), we can find a nonsingular matrix M that transforms (A, B) into canonical form
$M\,A\,M^{-1} = A_c + B_c\,\lambda^T$ (2.30)
and $M\,B = B_c$ (2.31)
where

$$A_c = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & 0 & 1 \\ 0 & \dots & \dots & 0 & 0 \end{bmatrix}_{n\times n} \qquad (2.32)$$

and $B_c = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$ (2.33)

Let $T(x) = \begin{bmatrix} T_1(x) \\ T_2(x) \\ \vdots \\ T_{n-1}(x) \\ T_n(x) \end{bmatrix}$ (2.34)

Thus,

$$A_c T(x) - B_c \beta^{-1}(x)\,\alpha(x) = \begin{bmatrix} T_2(x) \\ T_3(x) \\ \vdots \\ T_n(x) \\ -\alpha(x)/\beta(x) \end{bmatrix} \qquad (2.35)$$

and $$B_c \beta^{-1}(x) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1/\beta(x) \end{bmatrix} \qquad (2.36)$$

Using these expressions in (2.27),we get simplified partial differential equations as

$$\frac{\partial T_1}{\partial x} f(x) = T_2(x); \qquad (2.37)$$

$$\frac{\partial T_2}{\partial x} f(x) = T_3(x); \qquad (2.38)$$

$$\vdots$$

$$\frac{\partial T_{n-1}}{\partial x} f(x) = T_n(x); \qquad (2.39)$$

$$\frac{\partial T_n}{\partial x} f(x) = -\alpha(x)/\beta(x) \qquad (2.40)$$

Using these expressions in (2.28), we get simplified partial differential equations as

$$\frac{\partial T_1}{\partial x} g(x) = 0; \qquad (2.41)$$

$$\frac{\partial T_2}{\partial x} g(x) = 0; \qquad (2.42)$$

$$\vdots$$

$$\frac{\partial T_{n-1}}{\partial x} g(x) = 0; \qquad (2.43)$$

$$\frac{\partial T_n}{\partial x} g(x) = 1/\beta(x) \neq 0 \qquad (2.44)$$

Now, the function $T_1(x)$ has to be searched, which satisfies

$$\frac{\partial T_i}{\partial x} g(x) = 0;\ (n-1) \leq i \leq 1 \qquad (2.45)$$

where, $T_{i+1}(x) = \frac{\partial T_i}{\partial x} f(x), \quad (n-1) \leq i \leq 1$

$$\frac{\partial T_n}{\partial x} g(x) \neq 0 \qquad (2.46)$$

If there is a function which satisfies (2.45) and (2.46), then

$$\beta(x) = \frac{1}{(\partial T_n/\partial x) g(x)} \qquad (2.47)$$

$$\alpha(x) = -\frac{(\partial T_n/\partial x) f(x)}{(\partial T_n/\partial x) g(x)} \qquad (2.48)$$

From differential geometry approach, we also noticed that for all $x \in D,\ \ k \geq 0,\ \ 0 \leq j \leq r-1,$ we have

$$L_{ad_f{}^i g} L_f{}^k h(x) = 0, 0 \leq j + k < r - 1 \qquad (2.49)$$

$$L_{ad_f{}^i g} L_f{}^k h(x) = (-1)^j L_g L_f{}^{r-1} h(x) \neq 0,\ j + k = r - 1 \qquad (2.50)$$

and the row vectors $\left[dh(x), \dots, dL_f{}^{r-1}h(x)\right]$ and column vectors$\left[g(x) \quad \dots \quad ad_f{}^{r-1}g(x)\right]$ are linearly independent.

From Jacobian identity, we know that for any real-valued λ and any vector fields f and β,

$$L_{[f,\beta]}\lambda(x) = L_f L_\beta \lambda(x) - L_\beta L_f \lambda(x) \qquad (2.51)$$

Also the distribution, Δ generated by $f_{1,} \dots, f_r$ is said to be completely integrable if for each $x_0 \in D, \exists$ a neighborhood N of x_0 and $n - r$ real-valued smooth functions $h_1(x), \dots, h_{n-r}(x)$ satisfy the partial differential equations

$$\frac{\partial h_i}{\partial x} f_i(x) = 0, \forall\ 1 \leq i \leq r, 1 \leq j \leq n - r \qquad (2.52)$$

and the covector fields $dh_j(x)$are linearly independent for all $x \in D$.Thus, a nonsingular distribution is completely integrable if and only if it is involutive, and is called as Frobenius theorem.

3. DC MOTOR LINEARIZATION

Let us now apply all these concepts in obtaining the exact linearization model of the DC motor.Let the field controlled motor is represented by the system as

$$\dot{x} = f(x) + g\,u, \qquad (3.1)$$

$$\text{where} \quad f(x) = \begin{bmatrix} -ax_1 \\ -bx_2 + \rho - cx_1x_3 \\ \theta x_1 x_2 \end{bmatrix}; \; g = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad (3.2)$$

and a, b, c, θ, ρ are positive constants

Let$x_1 = 0, x_2 = \rho/b$,be the equilibrium point for the open loop system and the desired operating point be $\varkappa$ given as $\varkappa = (0, \rho/b\,, \omega_0)^T$, where ω_0 is a desired set point for the angular velocity x_3.

Let $T_2(x) = \frac{\partial T_1}{\partial x} f(x)\,;\, T_3(x) = \frac{\partial T_2}{\partial x} f(x)$ (3.3)

Then, with $T_1(\varkappa) = 0$,we can find $T_1 = T_1(x)$,which satisfies the conditions (2.41),(2.42) and

$$\frac{\partial T_3}{\partial x} g(x) \neq 0 \qquad (3.4)$$

thus,
$$\frac{\partial T_1}{\partial x} g = \frac{\partial T_1}{\partial x_1} = 0 \qquad (3.5)$$

and therefore,

$$T_2(x) = \frac{\partial T_1}{\partial x_2}[-bx_2 + \rho - cx_1x_3] + \frac{\partial T_1}{\partial x_3}\theta x_1 x_2 \qquad (3.6)$$

Using (2.42) in (2.46),we get $cx_3 \frac{\partial T_1}{\partial x_2} = \theta x_2 \frac{\partial T_1}{\partial x_3}$, (3.7)

Equation (3.7) will be satisfied if T_1 is of the form $T_1 = c_1[\theta x_2^2 + cx_3^2] + c_2$ (3.8)

Let $c_1 = 1$ and $c_2 = -\theta(\varkappa)_2^2 - c(\varkappa)_3^2 = -\theta(\rho/b)^2 - c\omega_0^2$ (3.9)

using (3.9), we get $T_2(x) = 2\theta x_2(\rho - bx_2)$ (3.10)

and $T_3(x) = 2\theta(\rho - 2bx_2)(bx_2 + \rho - cx_1x_3)$ (3.11)

hence,

$$\frac{\partial T_3}{\partial x} g = \frac{\partial T_3}{\partial x_1} = 2c\theta(\rho - 2bx_2)(x_3) \qquad (3.12)$$

and the (3.4) is satisfied, whenever $x_2 \neq \rho/b\,, x_3 \neq 0$.

Let D_x contains the point $\varkappa$ and $\varkappa_3 > 0$,then

$$D_x = \left\{x \in R^3\,;\, x_2 > \frac{\rho}{2b}\,, x_3 > 0\right\} \qquad (3.13)$$

Thus, the map (2.12) is diffeomorphism on $D_x$and the state equation in the $z-$ coordinates are defined in

$$D_z = T(D_x) = \{z \in R^3\} \qquad (3.14)$$

such that $\{z_1 > \theta\phi^2(z)_2 - \theta(\rho/b)^2 - c\omega_0^2\}, \left\{z_2 < \frac{\theta\rho}{2b}\right\}$ and $\phi(\cdot)$ is the inverse of the map $2\theta x_2(\rho - bx_2)$. Thus, domain D_z contains the origin $z = 0$.

Now, we can verify the same using differential geometry approach as explained in the previous section. As we know

$$ad_f g = [f, g] = \begin{bmatrix} a \\ cx_3 \\ -\theta x_2 \end{bmatrix}; \; ad_f^2 g = [f, ad_f g] = \begin{bmatrix} a^2 \\ (a+b)cx_3 \\ (b-a)\theta x_2 - \theta\rho \end{bmatrix};$$

Considering $G = [g, ad_f g, ad_f^2 g] = \begin{bmatrix} 1 & a & a^2 \\ 0 & cx_3 & (a+b)cx_3 \\ 0 & -\theta\, x_2 & (b-a)\theta\, x_2 - \theta\, \rho \end{bmatrix}$;

thus, the determinant of G, $\det G = c\theta(-\rho + 2bx_2)x_3$. Hence, G has a rank 3 for $x_2 \neq \rho/2\,b, x_3 \neq 0.$ Now,

$$[g, ad_f g] = \frac{\partial(ad_f g)}{\partial x} g = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & c \\ 0 & -\theta & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Hence the distribution $\Delta = span\{g, ad_f g\}$ is involutive as $[g, ad_f g] \in D$ and $D_x = \left\{x \in R^3 ;\ x_2 > \frac{\rho}{2b}, x_3 > 0\right\}$, which is same as (3.14) and this completes the verification and the development of exact feedback linearized model of the DC motor.

CONCLUSION

The results of geometric approach are compared and found similar, if other tools are used e.g.. input-state linearization and the input-output linearization, to obtain the exact feedback linearization mathematical model of non linear model of a DC motor.

REFERENCES

[1]. Krause, Oleg, Sudhoff," Analysis of electric machinery and Drives systems," IEEE Press, 2002, ch. 2.

[2]. Ned Mohan "First Course on Power Electronics and Drives", MNPERE Minneapolis, 2003, ch. 11.

[3]. Serge Lang,"Differential and Riemannian Manifolds", Springer-Verlag, 1995, ch.1.

[4]. Tom M. Apostol," Mathematical Analysis",Addison-Wesley Publishing Company, World Student Series Edition, 1981, ch. 4, ch. 5.

[5]. Sharipov R. A. ,"Course of Differential Geometry: the textbook", Russian Federal Committee for Higher Education, 2004, ch. 2.

[6]. Jeffrey M. Lee," Manifolds and Differential Geometry", American Mathematical Society, 2009, ch. 1, ch. 8.

[7]. Michael Müger," An Introduction to Differential Topology, de Rham Theory and Morse Theory", 2005.

[8]. M. Vidyasagar,"Non linear system analysis", Prentice-Hall, 199, ch. 6 , ch.7.

[9]. Horacio J. Marquez ,"Nonlinear Control Systems Analysis and Design", John Wiley & Sons Inc., 2003, ch. 6, ch. 10.

[10]. Lui Lam,"Nonlinear Physics for Beginners Fractals, Chaos, Solitons, Pattern Formation, Cellular Automata and Complex", World Scientific Publishing Co. Pte. Ltd., 199, ch. 1.

[11]. Gildas Besancon (Ed.), "Nonlinear Observers and Applications", Springer, 2007, ch.1 ,ch.3, ch. 5.

[12]. Wassim M. Haddad, Vijay Sekhar Chellaboina, "Nonlinear Dynamical Systems and Control: A Lyapunov-Based Approach", Princeton University Press, 2008, ch. 2, ch. 4,ch.7,ch.8 ,ch.9.

[13]. Walter Rudin,"Real And Complex Analysis", McGrawHill, 1970, ch. 17.

[14]. V.S. Varadarajan,"Lie Groups, Lie Algebras, and Their Representations", Springer-Verlag, New York, 1984, ch.1, ch.2.

[15]. Francesco Iachello,"Lie Algebras and Applications", Springer-Verlag Berlin Heidelberg, 2006, ch.1 ch.3., ch. 4.