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Starlikeness and Convexity for Analytic Functions in the Unit Disc

Mohammed Ahamed Osman Tom

[1]- University of Tabuk, College of Science, Department of Mathematics
[2]-University of Bahri & College of applied Science and Industrial, Department of Mathematics

ABSTRACT

We investigate some results for sufficient conditions of functions $f(z)$ which are analytic in the open uint disc $\mathbb{U}$ to be starlike and convex in $\mathbb{U}$ .The objective of this paper is to derive some interesting sufficient conditions for $f(z)$ to be starlike of order α and convex of order α in $\mathbb{U}$ concerned with Jack's lemma.We consider the generalization of the starlikeness of complex order and the generalization of convexity of complex order for the analytic functions in the unit disc
$\mathbb{U} = \{z: |z| < 1\}$.

Keywords : Analatic , univalent ,starlike of order α, convex of order α, subordination .

1-INTRODUCTION

In this paper we discuss two classes of function $f(z)$ which are analytic in the open unit disc $\mathbb{U}$ under same conditions .Let $\mathcal{A}$ denote the class of functions that

are analytic in the open unit disc $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$, so that $f(0) = f'(0) - 1 = 0$.We denote by S the subclass of $\mathcal{A}$ consisting of univalent functions $f(z)$ in $\mathbb{U}$. Let $S^*(\alpha)$ be the subclass of $\mathcal{A}$ consisting of all functions $f(z)$ which satisfy

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha \quad (z \in \mathbb{U})$$

for some $0 \leqq \alpha < 1$ .A function $f(z) \in S^*(\alpha)$ is sais to be starlike of order α in $\mathbb{U}$.We denote by $S^* = S^*(0)$.
Also , let $\mathcal{K}(\alpha)$ denote the subclass of $\mathcal{A}$ consisting of all functions $f(z)$ which satisfy

$$\mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \quad (z \in \mathbb{U})$$

for some $0 \leqq \alpha < 1$. A function $f(z)$ in $\mathcal{K}(\alpha)$ is said to be convex of order α in $\mathbb{U}$. We say that $\mathcal{K} = \mathcal{K}(0)$. From the definitions for $S^*(\alpha)$ and $\mathcal{K}(\alpha)$,we know that $f(z) \in \mathcal{K}(\alpha)$ if and only if $zf'(z) \in S^*(\alpha)$.

Let $f(z)$ and $g(z)$ be analytic in . Then $f(z)$ is said to be subordinate to $g(z)$ if there exists an analytic function $\omega(z)$ in $\mathbb{U}$ satisfying

$\omega(0) = 0, |\omega(z)| < 1 \ (z \in \mathbb{U})$ and $f(z) = g(\omega(z))$ .We denote this subordination by

$$f(z) \prec g(z) \quad (z \in \mathbb{U}).$$

On the other hand let Ω be the family of functions $\omega(z)$ regular in the unit disc $\mathbb{U}$ and satisfying the condition $\omega(0) = 0$, $|\omega(z)| < 1$ for $z \in \mathbb{U}$. For arbitrary fixed numbers $A, B, -1 \leq B < A \leq 1$, denote by $P(A, B)$ the family of functions $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$ regular in $\mathbb{U}$, such that $p(z) \in P(A, B)$ if and only if $p(z) = \frac{1+A\omega(z)}{1+B\omega(z)}$ for some functions $\omega(z) \in \Omega$ and for evey $z \in \mathbb{U}$.This class was introduced by Janowski [8].

Further let $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$ and $g(z) = z + b_2 z^2 + b_3 z^3 + \cdots$ be analytic functions in the disc $\mathbb{U}$ .Then we say that the function $f(z)$ is subordinate to (z) , written $f \prec g$ or $f(z) \prec g(z)$, such that $f(z) = g(\omega(z))$, $\omega(z) \in \Omega$,for all $z \in \mathbb{U}$. In particular , if $g(z)$ is univalent in $\mathbb{U}$, then $f \prec g$ if and only if $f(0) = f(0)$ and $f(\mathbb{U}) \subseteq g(\mathbb{U})$.

Next we consider the following class of functions defined in $\mathbb{U}$. Let $CS^*(A,B,b,q)$ denote the family of functions $f(z) = z + a_2z^2 + a_3z^3 + \cdots$ regular in $\mathbb{U}$ such that $f(z) \in CS^*(A,B,b,q)$ if and only if

$$1 + \frac{1}{b}\left(z\frac{f^{(q+1)}(z)}{f^{(q)}(z)} + q - 1\right) = \frac{1 + A\omega(z)}{1 + B\omega(z)},$$

where $b \neq 0$, b is a complex number , $f^{(q)}(z)$ denotes the derivative of $f(z)$ with respect to z of order $q \in \{0,1\}$ with $f^{(0)}(z) = f(z)$ and $\omega(z) \in \Omega$. The definition of the class $CS^*(A,B,b,q)$ is equivalent to $f(z) \in CS^*(A,B,b,q)$ if and only if

$1 + \frac{1}{b}\left(z\frac{f^{(q+1)}(z)}{f^{(q)}(z)} + q - 1\right) \prec \frac{1+Az}{1+Bz}$ for all $z \in \mathbb{U}$, $B \neq 0$ (1)
$1 + \frac{1}{b}\left(z\frac{f^{(q+1)}(z)}{f^{(q)}(z)}\right) \prec 1 + Az$ for all $z \in \mathbb{U}, B = 0$
The geometric meaning of (1) is that the image of $\mathbb{U}$ by
$1+\frac{1}{b}\left(z\frac{f^{(q+1)}(z)}{f^{q}(z)} + q - 1\right)$

is inside the open disc centered on the real axis with diameter end points

$\frac{1-A}{1-B}$ and $\frac{1+A}{1+B}$, $B \neq 0$
$1 - A$ and $1 + A$, $B = 0$

Some examples of functions in the classes $CS^*(A,B,b,0)$, $CS^*(A,B,b,1)$, $CS^*(1,-1,b,1)$ respectively , are the following

for $q = 0$, $f(z) = \begin{cases} z(1 + Bz)^{b(A-B)/B} & B \neq 0 \\ ze^{Abz} & B = 0 \end{cases}$,

for $q = 1$, $f(z) = \begin{cases} \int_0^z (1 + B\zeta)^{b(A-B)/B} d\zeta & B \neq 0 \\ \int_0^z e^{bA\zeta} d\zeta & B = 0 \end{cases}$
for A =1 , B = -1 , $q = 0$, $f(z) = \frac{z}{(1-z)^{2b}}$,
for A = 1 , B = -1 , $q = 1$, $f(z) = \int_0^z (1 - \zeta)^{-2b}\, d\zeta$,

Clearly we have the following classes:

(i) For $q = 0$, $A = 1$, $B = -1$, $C\,S^*(1,-1,b,0)$ is the class of starlike functions of complex order . This class was introduced by Aouf [3] .
(ii) For $q = 1$, $A = -1$, $CS^*(1,-1,b,1)$ is the class of convex functions of complex order . This class was introduced by Nasr and Aouf [4].
(iii) For $q = 0$, $B = -1$, $b = 1$, $CS^*(0,1,-1,1) = S^*$ is the class starlike functions [6] , [1].
(iv) For $q = 1$, $A = 1$, $B = -1$, $b = 1$, $CS^*(1,-1,1,1) = C$ is the class convex function . The class is well known [6] ,[1].

We note that by giving special values to be b (which are $b = 1 - \alpha$, $0 \leq \alpha < 1$; $b = 1 - (1 - \alpha)(\cos\lambda)e^{-i\alpha}$, $0 \leq \alpha < 1$, $|\lambda| < \pi/2$; $b = (1 - (\cos\lambda)e^{-i\lambda})$ we very important subclasses of starlike functions and convex functions,[6],[1].

Lemma 1. [2, 5]
Let $\omega(z)$ be analytic in $\mathbb{U}$ with $\omega(0) = 0$ .Then if $|\omega(z)|$ attains its maximum value on the circle $|z| = r$ at a point $z_0 \in \mathbb{U}$,then we have $z_0\omega'(z_0) = k\omega(z_0)$, where $k \geqq 1$ is real number.

2- MAIN RESULTS

Applying Lemma 1, we drive the following result.
Theorem 1
If $f(z) \in \mathcal{A}$ satisfies

$$\mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) < \frac{\alpha + 1}{2(\alpha - 1)} \qquad (z \in \mathbb{U})$$

for some $\alpha (2 \leqq \alpha < 3)$, or

$$\mathrm{Re}\left(1+\frac{zf''(z)}{f'(z)}\right)<\frac{5\alpha-1}{2(\alpha+1)} \qquad (z\in\mathbb{U})$$

for some $\alpha(1<\alpha\leqq 2)$, then

$$\frac{zf'(z)}{f(z)}\prec\frac{\alpha(1-z)}{\alpha-z} \qquad (z\in\mathbb{U})$$

and

$$\left|\frac{zf'(z)}{f'(z)}-\frac{\alpha}{\alpha+1}\right|<\frac{\alpha}{\alpha+1} \qquad (z\in\mathbb{U}).$$

This implies that $f(z)\in S^*$ and $\int_0^z\frac{f(t)}{t}dt\in\mathcal{K}$.

Proof

Let us define the function $\omega(z)$ by

$$\frac{zf'(z)}{f(z)}=\frac{\alpha\left(1-\omega(z)\right)}{\alpha-\omega(z)} \qquad (\omega(z)\neq\alpha).$$

Clearly , $\omega(z)$ is analytic in $\mathbb{U}$ and $\omega(0)=0$. We want to prove that $|\omega(z)|<1$ in $\mathbb{U}$. Since

$$1+\frac{zf''(z)}{f'(z)}=\frac{\alpha\left(1-\omega(z)\right)}{\alpha-\omega(z)}-\frac{z\omega'(z)}{1-\omega(z)}+\frac{z\omega'(z)}{\alpha-\omega(z)},$$

we see that

$$Re\left(1+\frac{zf''(z)}{f'(z)}\right)=Re\left(\frac{\alpha\left(1-\omega(z)\right)}{\alpha-\omega(z)}-\frac{z\omega'(z)}{1-\omega(z)}+\frac{z\omega'(z)}{\alpha-\omega(z)}\right)$$
$$<\frac{\alpha+1}{2(\alpha-1)} \qquad (z\in\mathbb{U})$$

for $2\leqq\alpha<3$, and

$$Re\left(1+\frac{zf''(z)}{f'(z)}\right)=Re\left(\frac{\alpha\left(1-\omega(z)\right)}{\alpha-\omega(z)}-\frac{z\omega'(z)}{1-\omega(z)}+\frac{z\omega'(z)}{\alpha-\omega(z)}\right)$$
$$<\frac{5\alpha-1}{2(\alpha-1)} \qquad (z\in\mathbb{U})$$

for $1<\alpha\leqq 2$ .If there exists a point $z_0\in\mathbb{U}$ such that

$$\max_{|z|\leqq|z_0|}|\omega(z)|=|\omega(z_0)|=1,$$

then Lemma 1 gives us that $\omega(z_0)=e^{i\theta}$ and $z_0\omega'(z_0)=k\omega(z_0)$, $k\geqq 1$.
Thus we have

$$1+\frac{z_0f''(z_0)}{f'(z_0)}=\frac{\alpha\left(1-\omega(z_0)\right)}{\alpha-\omega(z_0)}-\frac{z_0\omega'(z_0)}{1-\omega(z_0)}+\frac{z_0\omega'(z_0)}{\alpha-\omega(z_0)}$$
$$=\alpha+\alpha(1-\alpha+k)\frac{1}{\alpha-e^{i\theta}}-\frac{k}{1-e^{i\theta}}.$$

If follows that

$$Re\left(\frac{1}{\alpha-\omega(z_0)}\right)=Re\left(\frac{1}{\alpha-e^{i\theta}}\right)$$
$$=\frac{1}{2\alpha}+\frac{\alpha^2-1}{2\alpha(1+\alpha^2-2cos\theta)}$$

and

$$Re\left(\frac{1}{1-\omega(z_0)}\right)=Re\left(\frac{1}{1-e^{i\theta}}\right)$$
$$=\frac{1}{2}$$

Therefor , we have

$$Re\left(1+\frac{z_0f''(z_0)}{f'(z_0)}\right)=\frac{1+\alpha}{2}+\frac{(\alpha^2-1)(1-\alpha+k)}{2(1+\alpha^2-2\alpha cos\theta)}.$$

This implies that , for $2\leqq\alpha<3$,

$$Re\left(1+\frac{z_0f''(z_0)}{f'(z_0)}\right)\geqq\frac{1+\alpha}{2}+\frac{(\alpha-1)(1-\alpha+k)}{2(\alpha+1)}$$

$$\geqq \frac{1+\alpha}{2} + \frac{(\alpha+1)(2-\alpha)}{2(\alpha-1)}$$
$$= \frac{\alpha+1}{2(\alpha-1)}$$

and ,for $1 < \alpha \leqq 2$,

$$Re\left(1 + \frac{z_0 f''(z_0)}{f'(z_0)}\right) \geqq \frac{1+\alpha}{2} + \frac{(\alpha-1)(1-\alpha+k)}{2(\alpha+1)}$$
$$\geqq \frac{1+\alpha}{2} + \frac{(\alpha-1)(2-\alpha)}{2(\alpha+1)}$$
$$= \frac{5\alpha-1}{2(\alpha+1)}.$$

This contradicts the condition in the theorem 1. Therefoer , there is no $z_0 \in \mathbb{U}$ such that $|\omega(z_0)| = 1$ for all $z \in \mathbb{U}$, that is , that

$$\frac{zf'(z)}{f(z)} \prec \frac{\alpha(1-z)}{\alpha - z} \qquad (z \in \mathbb{U}).$$

Furthermore , since

$$\omega(z) = \frac{\alpha\left(\frac{zf'(z)}{f(z)} - 1\right)}{\frac{zf'(z)}{f(z)} - \alpha} \qquad (z \in \mathbb{U})$$

and $|\omega(z)| < 1$ $(z \in \mathbb{U})$, we conclude that

$$\left|\frac{zf'(z)}{f(z)} - \frac{\alpha}{\alpha+1}\right| < \frac{\alpha}{\alpha+1} \qquad (z \in \mathbb{U}) ,$$

Which implies that $f(z) \in S^*$.Futhermore, we if and only if $\int_0^z \frac{f(t)}{t} dt \in \mathcal{K}$.
Thaking $\alpha = 2$ in the theorem 1 , we have following corollary due to R.Singh and S.Singh [7] .

Corollarly 2 If $f(z) \in \mathcal{A}$ satisfies

$$Re\left(1 + \frac{zf''(z)}{f'(z)}\right) < \frac{3}{2} \qquad (z \in \mathbb{U}) ,$$

then

$$\frac{zf'(z)}{f(z)} \prec \frac{2(1-z)}{2-z} \qquad (z \in \mathbb{U})$$

and

$$\left|\frac{zf'(z)}{f(z)} - \frac{3}{2}\right| < \frac{3}{2} \qquad (z \in \mathbb{U}).$$

With theorem 1 ,we give the following example .

Example 3 For $2 \leqq \alpha < 3$, we consider the function $f(z)$ given by

$$f(z) = \frac{\alpha-1}{2}\left(1 - (1-z)^{\frac{2}{\alpha-1}}\right) \qquad (z \in \mathbb{U}) .$$

If follows that

$$\frac{zf'(z)}{f(z)} = \frac{2z(1-z)^{\frac{3-\alpha}{\alpha-1}}}{(\alpha-1)\left(1-(1-z)^{\frac{2}{\alpha-1}}\right)} \qquad (z \in \mathbb{U})$$

and

$$Re\left(1 + \frac{zf''(z)}{f'(z)}\right) = Re\left(\frac{\alpha-1-2z}{(\alpha-1)(1-z)}\right)$$
$$= Re\left(\frac{2}{\alpha-1} - \frac{3-\alpha}{(\alpha-1)(1-z)}\right)$$
$$< \frac{\alpha+1}{2(\alpha-1)} \qquad (z \in \mathbb{U}).$$

Therefore , the function $f(z)$ satisfies the condition in Theorem 1 .If we define the function $\omega(z)$ by

$$\frac{zf'(z)}{f(z)} = \frac{\alpha\big(1-\omega(z)\big)}{\alpha-\omega(z)} \qquad (\omega(z) \neq \alpha)\,,$$

then we see that $\omega(z)$ is analytic in $\mathbb{U}$, $\omega(0) = 0$ and $|\omega(z)| < 1 \ (z \in \mathbb{U})$

with Mathematica 5 .2. This implies that

$$\frac{zf'(z)}{f(z)} \prec \frac{\alpha(1-z)}{\alpha-z} \quad (z \in \mathbb{U}).$$

For $1 < \alpha \leqq 2$, we consider

$$f(z) = \frac{\alpha+1}{2(2\alpha-1)}\left(1-(1-z)^{\frac{2(2\alpha-1)}{\alpha+1}}\right) \quad (z \in \ \mathbb{U}).$$

Then we have that

$$\frac{zf'(z)}{f(z)} = \frac{2(2\alpha-1)z(1-z)^{\frac{3(\alpha-1)}{\alpha+1}}}{(\alpha+1)\left(1-(1-z)^{\frac{2(2\alpha-1)}{\alpha+1}}\right)}$$

and

$Re\left(1+\frac{zf''(z)}{f'(z)}\right) = Re\left(\frac{\alpha+1-2(2\alpha-1)z}{(\alpha+1)(1-z)}\right) < \frac{5\alpha-1}{2(\alpha+1)} \qquad (z \in \mathbb{U}).$

Thus , the function $f(z)$ satisfies the condition in Theorem 1 . Define the function $\omega(z)$ by

$$\frac{zf'(z)}{f(z)} = \frac{\alpha\big(1-\omega(z)\big)}{\alpha-\omega(z)} \quad (\omega(z) \neq \alpha).$$

That $\omega(z)$ is analytic in the in $\mathbb{U}$, $\omega(0) = 0$ and $|\omega(z)| < 1$ $(z \in \mathbb{U})$ with Mathematica 5.2. Therefour , we have that

$$\frac{zf'(z)}{f(z)} \prec \frac{\alpha(1-z)}{\alpha-z} \qquad (z \in \mathbb{U}).$$

In particular , if we take $\alpha = 2$ in this example , then $f(z)$ becomes

$$f(z) = z - \frac{1}{2}z^2 \in S^*,$$

where S^* denotes the classof starlike function in $\mathbb{U}$.

2-Some results for the class C $S^*(A, B, b, q)$

Lemma 4. [2]
Let $\omega(z)$ be a non-constant and analytic function in the unit disc $\mathbb{U}$ with $(0) = 0$. If $|\omega(z)|$ attains its maximum value on the circle $|z| = r$ at the point z_1 , then $z_1\omega'(z_1) = k\omega(z_1)$ and $k \geq 1$.

Lemma 5
Let $f(z) = z + a_2z^2 + a_3z^3 + \cdots$ be an analytic functions in yhe unit disc U. If $f(z)$ satisfies

$$\begin{cases} \frac{1}{b}\left(z\frac{f^{(q+1)}(z)}{f^{(q)}(z)} + q - 1\right) \prec \frac{(A-B)z}{1+Bz} = F_1(z)\,, & B \neq 0 \\ \frac{1}{b}\left(z\frac{f^{(q+1)}(z)}{f^{(q)}(z)} + q - 1\right) \prec Az = F_2(z)\,, & B = 0 \end{cases} \tag{2}$$

then $f(z) \in CS^*(A, B, b, q)$ and the result is sharp as the function

$$f_*^{(q)}(z) = \begin{cases} z^{1-q}(1+Bz)^{\frac{b(A-B)}{B}}, & B \neq 0 \\ z^{1-q}e^{Abz}, & B = 0. \end{cases}$$

Proof
Let $B \neq 0$.We define a function $\omega(z)$ by

$$\frac{f^{(q)}(z)}{z^{1-q}} = \big(1+B\omega(z)\big)^{\frac{b(A-B)}{B}} \,, \tag{3}$$

where $\big(1+B\omega(z)\big)^{\frac{b(A-B)}{B}}$ has value 1 at the origin . Then $\omega(z)$ is analytic in , $\omega(0) = 0$ and

$$\frac{1}{b}\left(z\frac{f^{(q+1)}(z)}{f^{(q)}(z)} + q - 1\right) = \frac{(A-B)z\omega'(z)}{1+B\omega(z)}. \tag{4}$$

Now it is easy to realize that the subordination (2) is equivalent to $|\omega(z)| < 1$,for all $z \in \mathbb{U}$. Indeed assume the contrary : Ther exist $z_1 \in \mathbb{U}$ such that $|\omega(z)| = 1$.Then by I.S. Jack's lemma $z_1\omega'(z_1) = k\omega(z_1)$, $k \geq 1$ and for such z_1 we have

$$\frac{1}{b}\left(z_1\frac{f^{(q+1)}(z_1)}{f^{(q)}(z_1)} + q - 1\right) = k\frac{(A-B)\omega(z_1)}{1+B\omega(z_1)} \notin F_1(\mathbb{U})$$

because $|\omega(z_1)| = 1$ and $k \geq 1$. But this is a contradiction to the condition (2) of this lemma and so assumption is wrong i.e.,$|\omega(z)| < 1$ for all $z \in \mathbb{U}$.

On the other hand we have

$$\frac{1}{b}\left(z\frac{f^{(q+1)}(z)}{f^{(q)}(z)} + q - 1\right) \prec \frac{(A-B)z}{1+Bz} \Leftrightarrow \frac{1}{b}\left(z\frac{f^{(q+1)}(z)}{f^{q}(z)} + q - 1\right) = \frac{(A-B)\omega(z)}{1+B\omega(z)}$$
$$\Leftrightarrow 1 + \frac{1}{b}\left(\frac{f^{(q+1)}(z)}{f^{(q)}(z)} + q - 1\right) = \frac{1+A\omega(z)}{1+B\omega(z)} \qquad (5)$$

The equivalencies (5) show that $f(z) \in CS^*(A,B,b,q)$.

Let $B = 0$.Define a function by $\frac{f^{(q)}(z)}{z^{1-q}} = e^{Ab\omega(z)}$ .Then is analytic in $\mathbb{U}$ and $\omega(0) = 0$ and

$$\frac{1}{b}\left(z\frac{f^{(q+1)}(z)}{f^{q}(z)} + q - 1\right) = Az\omega'(z). \qquad (6)$$

Similarly by using I.S. Jack's lemma we obtain

$$1 + \frac{1}{b}\left(z\frac{f^{(q+1)}(z)}{f^{(q)}(z)} + q - 1\right) = 1 + A\omega(z). \qquad (7)$$

The equality (7`) shows that $f(z) \in CS^*(A,B,b,q)$.
The sharpness of the result follows from the fact that for

$$f_*^{(q)}(z) = \begin{cases} z^{1-q}(1+Bz)^{\frac{b(A-B)}{B}}, B \neq 0 \\ z^{1-q}e^{Abz}, \quad B = 0 \end{cases}$$

We receive

$$\left(z\frac{f_*^{(q+1)}(z)}{f_*^{(q)}(z)} + q - 1\right) = \begin{cases} \dfrac{(A-B)z}{1+B} = F_1(z), B \neq 0 \\ Az = F_2(z), \quad B = 0 \end{cases}$$

Lemma 6

If $(z) \in CS^*(A,B,b,q)$, then the set of the values of $\left(z\frac{f^{(q+1)}(z)}{f^{(q)}(z)}\right)$ is the disc with the center $C(r)$ and the radius $\rho(r)$, where

$C(r) = \frac{(1-q)+(q-1)B^2-b(AB-B^2)r^2}{1-B^2r^2}$, $\rho(r) = \frac{|b|(A-B)}{1-B^2r^2}$, $B \neq 0$
$C(r) = 1$ $\qquad \rho(r) = |Ab|r$, $B = 0$

Proof
If $(z) \in P(A,B)$, then

$$\left|p(z) - \frac{1-ABr^2}{1-B^2r^2}\right| \leq \frac{(A-B)}{(1-B^2r^2)} \qquad (8)$$

The inequality (8) was proved by Janowski [8].
By using the definition of the class C $S^*(A,B,b,q)$ and the inequality (8) we get

$$\left|1 + \frac{1}{b}\left(z\frac{f^{(q+1)}(z)}{f^{(q)}(z)} + q - 1\right) - \frac{1-ABr^2}{1-B^2r^2}\right| \leq \frac{(A-B)r}{1-B^2r^2}. \qquad (9)$$

After a berif calculation from (9) we obtain

$$\left|z\frac{f^{(q+1)}(z)}{f^{(q)}(z)}-\frac{(1-q)+[(1-q)B^2-b(AB-B^2)r^2]}{1-B^2r^2}\right|\le\frac{|b|(A-B)r}{1-B^2r^2}, \quad B\neq 0$$

$$\left|z\frac{f^{(q+1)}(z)}{f^{(q)}(z)}+q-1\right|\le|Ab|r, \qquad B=0.$$

Theorem 7

If $(z)\in CS^*(A,B,b,r)$, then

$$M_1(A,B,r)\le\left|f^{(q)}(z)\right|\le M_2(A,B,b,q), B\neq 0$$

$$N_1(A,r)\le\left|f^{(q)}(z)\right|\le N_2(A,r) < \quad B=0\,. \qquad (10)$$

where

$$M_1(A,B,r)=r^{1-q}(1-Br)\frac{(A-B)(|b|+Re\, b)}{2B}(1+Br)\frac{(A-B)(Re\, b-|b|)}{2B},$$

$$M_2(A,B,b,q)=r^{1-q}(1-Br)\frac{(A-B)(|b|-Re\, b)}{2B}(1+Br)\frac{(A-B)(|b|+Re\, b)}{2B},$$

$$N_1(A,r)=r^{1-q}e^{-|Ab|r}, N_2(A,r)=\ r^{1-q}e^{|Ab|r}$$

These bonuds are sharp because the extremal function is

$$f_*^{(q)}(z)=\begin{cases} z^{1-q}(1+Bz)\dfrac{b(A-B)}{B}, & B\neq 0\\ z^{(1-q)}e^{Abz}, & B=0\end{cases}$$

Proof

By using Lemma 6 and after a berif calculations we get

$$\frac{(1-q)-|b|(A-B)r+[(q-1)B^2-Re\, b(AB-B^2)]r^2}{1-B^2r^2}\le Re\, z\frac{f^{(q+1)}(z)}{f^{(q)}(z)}$$

$$\le\frac{(1-q)+|b|(A-B)r+[(q-1)B^2-Re\, b(AB-B^2)]r^2}{1-B^2r^2}, B\neq 0$$

$$(1-q)-|Ab|r\le Re\, z\frac{f^{(q+1)}(z)}{f^{(q)}(z)}\le(1-q)+|Ab|r, B=0$$

Since

$$Re\, z\frac{f^{(q+1)}(z)}{f^{(q)}(z)}=\frac{\partial}{\partial r}log\left|f^{(q)}\left(re^{i\theta}\right)\right|,\ |z|=r$$

and using preceding inequalities we obtain

$$\frac{(1-q)-|b|(A-B)r+[(q-1)B^2-Re\, b(AB-B^2)]r^2}{r(1-B^2r^2)}\le\frac{\partial}{\partial r}log\left|f^{(q)}\left(re^{i\theta}\right)\right|$$

$$\le\frac{(1-q)+|b|(A-B)r+[(q-1)B^2-Re\, b(AB-B^2)]r^2}{r(1-B^2r^2)}, B\neq 0$$

$$\frac{(1-q)}{r}-|Ab|\le\frac{\partial}{\partial r}log\left|f^{(q)}\left(re^{i\theta}\right)\right|\le\frac{(1-q)}{r}+|Ab|, \qquad B=0$$

Integrating both sides of these inequalities from 0 to r we obtain (10).

Corollary 8 For $q=0,\ A=1,\ B=-1,\ b=1$ we obtain

$$\frac{r}{(1+r)^2}\le|f(z)|\le\frac{r}{(1+r)^2}.$$

This is the distortion theorem of starlike functions .The result is well known [6],[1].

Corollary 9 For $q=1, A=1, B=-1, b=1$ we get

$$\frac{1}{(1+r)^2}\le\left|f'(z)\right|\le\frac{1}{(1-r)^2}.$$

This is the distortion theorem of the derivative of convex function this result is well known [6],[1].

Corollary 11 For $q=0$, $A=1$, $B=-1$ the following result is obtained

$$\frac{r}{(1+r)^{(\mathrm{Re}\,b+|b|)}(1-r)^{(\mathrm{Re}\,b-|b|)}} \le |f(z)| \le \frac{r}{(1-r)^{(\mathrm{Re}\,b+|b|)}(1+r)^{(|b|+\mathrm{Re}\,b)}}.$$

This is the distortion theorem for the starlike functions of complex order .

Corollary 12 For $q = 1$, $A = 1$, $B = -1$ the following result is obtained

$$\frac{1}{(1-r)^{(\mathrm{Re}\,b-|b|)}(1+r)^{(\mathrm{Re}\,b+|b|)}} \le |f'(z)| \le \frac{1}{(1-r)^{(|b|-\mathrm{Re}\,b)}(1+r)^{(|b|+\mathrm{Re}\,b)}}.$$

This is the distortion theorem for the derivative of convex functions of complex order.

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