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A new improvement of Shishkin fitted mesh technique on deferred correction method with applications

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ABSTRACT

We consider a class of singular perturbation ODE-BVP associated with, the common Dirichlet boundary condition, as one of the most comprehensive and difficult boundary conditions which with boundary layers at end points or interior layer. A new improvement of general form of Shishkin piecewise uniform fitted mesh technique applied, of adjustable width. Taking into consideration the locating process to find, the true locations of the fine subintervals in which corresponding to singular boundary layers (viscous parts) or interior layers, occur in their solutions using some well known easy applied techniques, then the fine subinterval also divided to two other a little bit different distance subintervals because of the different rate of convection and diffusion phenomenon in the solutions, as alternative of the well known uniform or equidistant mesh for Deferred correction method, to introduce what a known by a fitted mesh. So 4 standard problems solved and then compared with versus numerical solution of uniform mesh and Shishkin mesh inside deferred correction method for different choices of ε and N. In addition to presenting illustrative Matlab plots of most of the mesh constructions, the solutions and the epsilon convergence for each case separately in order to show the verification of progress and efficiency of the new method relative to both methods.

Keywords: Deferred Correction method, Differential Equation, Mesh generation and refinement, Numerical Analysis, Singularly perturbed problems.

1. INTRODUCTION

A singularly perturbed differential equation (SPDE) problem is a differential equation problem with a small parameter ε multiplying some or all of the terms involving the highest order derivatives. The physical properties associated with a solution containing a boundary layer function are reflected by the mathematical properties of the solution of (SPDE). The solution and its derivatives approach a discontinuous limit as ε approches zero. These problems are characterized by the property that the solution has different asymptotic expansions in distinguished sub domains of the entire given domain. They present layers where the solution changes abruptly. If any discretization technique is applied, need to analyze carefully the dependence on the parameter ε of those constants that arise in consistency, stability and error estimates. Truncation error may depend on ε. Usually the pointwise error of such solutions increases as the mesh is refined, to a stage where **the mesh parameter (h) is of the same order of magnitude** as **the singular perturbation parameter ε.** One obvious requirement for a numerical method being applied to these kinds of problems is that the pointwise errors of its solutions be bounded independently of ε and that they decrease as the mesh is redefined, at the rate which should also be independent of ε.

2. THE NON-STIFF PROBLEMS REVIEW

The singularly perturbed differential equation problems can also called in general stiff problems, in order to becomes non-stiff problems we must take $\varepsilon = 1$ in all its appearance place inside the problem.

A. THE PROBLEM

Our problem is the singularly perturbed two point boundary value problem nonlinear ODE has a form:

$$\varepsilon y'' - f(x,y) = 0. \quad (1a)$$
$$y(a) = \alpha, \quad y(b) = \beta. \quad (1b)$$

It is also possible and it is usual in numerical analysis, without losing generality, reducing the restrictions in the equation (1b) after a few mathematical processors to homogeneous boundary conditions to reach the final formula:

$$y(0) = y(1) = 0, \quad (1c)$$

under additional conditions:

$$f(x,y) \in C^{\infty}[a,b] \to (-\infty,\infty) \quad (2a)$$
$$f_y(x,y) > -\frac{\pi^2}{(b-a)^2} \quad (2b)$$

Then equation has unique solution $y^* \in C^{\infty}[a,b]$, which can be approximated by a three point finite difference method. Let let $\varepsilon = 1$ without loss of generality and $h = \frac{b-a}{N}$ for $N > 1$, and let

$$x_i = a + ih, i = 0, 1, \dots, N, \quad (2b)$$

define a uniform mesh on [a, b]. The discrete problem is obtained by replacing y'' in equation (1) by a second order symmetric difference at every interior mesh point:

$$\frac{y_{i-1} - 2y_i + y_{i+1}}{h^2} - f(x_i, y_i) = 0, i = 1, \dots, N-1 \quad (3a)$$
$$Y_0 = \alpha, Y_N = \beta \quad (3b)$$

For short, we can denote equation (1) by

$$F(Y) = 0, \quad (4)$$

And in the same spirit of equation (4), the equation (3) will be denoted by

$$F_h(Y) = 0, \quad (5)$$

System of equations in the Euclidean space E^{N-1}, the unknown being the vector $Y^T = (Y_1, \dots, Y_{N-1})$. Also h will go to zero $(N \to \infty)$ and we expect that, in some sense $Y_j(h)$ will converge to the respective function values of the exact solution. i.e. for each function $Z(x)$ defined in $[a,b]$ and satisfy the equation (1b) we define $\varphi_h[Z(x)] = [Z(x_1), \dots, Z(x_{N-1})]^T$. The operator φ_h is some time referred to as a space discretization. So $Y(h)$ converge discretely to the exact solution $y^*(x)$ if:

$$\lim_{h\to 0}\left\|Y(h) - \varphi_h y^*\right\|_{(h)} = 0, \quad (6)$$

Where $\|.\|_{(h)}$ is the maximum norm on $E^{\frac{b-a}{h}}$. Where this convergence depends on Consistency and Stability of operator F_h.

Definition (1): The operator F_h is consistent of order $p > 0$, if for the solution $y(x)$ of equation (1) and $h < h_0$ it holds that:

$$\left\|F_h(\varphi_h y)\right\| = O(h^p) \quad (7)$$

Definition (2): The operator F_h is stable if for any pair of discrete functions U, V, and $h \le h_0$, $\exists\, c > 0, c \in R$, independent of h, such that:

$$\|U - V\| \le c\|F_h(U) - F_h(V)\|. \quad (8)$$

Lemma (1): If F_h is a stable then it is locally invertible around $\varphi_h y^*$, and the inverse mapping F_h^{-1} is uniformly Lipschitz continuous for all $h < h_0$.

Theorem (1): Let us assume that the continuous problem $F(y) = 0$ has a unique solution y^*. Let F_h be a stable discretization on the sphere $B_h = B(\varphi_h y^*, p)$, and the consistent of order p with F. Then there is an $\bar{h}_0 > 0$ such that:
For any $\bar{h}_0 > 0$ there exists a unque solution $Y(h)$ for the discrete problem $F_h(y) = 0$.
The discrete solution $Y(h)$ satisfy

$$\left\|Y(h) - \varphi_h y^*\right\| = O(h^p) \quad (9)$$

(i.e., they are convergent of order p) [i]

B. CONSISTENCY, STABILITY, AND CONVERGENCE

The local truncation error shows how much our discrete operator fails to represent the continuous operator (for which we have $F(y^*) = 0$):

$$\tau_h(x_i) = [F_h(\varphi_h y^*)]_i = \frac{y^*(x_{i-1}) - 2y^*(x_i) + y^*(x_{i+1})}{h^2} - f\big(x_i, y^*(x_i)\big) \quad (10)$$

And the Taylor's formula expanding around x for $\tau_h(x)$ by using the fact that $f(x, y^*(x)) = y^{*''}(x)$ is

$$\tau_h(x) = -\sum_{k=1}^{K} \frac{2}{(2k+2)!} y^*(x) h^{2k} + O\big(h^{2k+2}\big) \qquad (11)$$

This expression then shows that the discrete method is consistent of order $p = 2.$ to prove that the discrete method in equation (2) is stable for h sufficiently small, which through theorem (1) will give us the existence of unique discrete solutions of the nonlinear system of equations (2), and there discrete convergence of order h^2 to $y^*(x)$. The proof of the L_∞ stability is basically due to Lees 1964 [ii]. For every h We define the inner product of mesh functions by

$$(V, U) = h \sum_{i=1}^{N-1} V_i U_i. \qquad (12)$$

This inner product induces a norm over the mesh functions that we denote by

$$\|V\|_0 = (V, V)^{\frac{1}{2}} \qquad (13)$$

By the usual relationships between the standard L_∞ and L_2 norms $\big(\|x\| \le \|x\|_2 \le \sqrt{n}\|x\|\big)$ we have that

$$h^{\frac{1}{2}}\|V\| \le \|V\|_0 \le (b-a)^{\frac{1}{2}}\|V\|, \qquad (14)$$

Since

$$\|V\|_0 = h^{\frac{1}{2}}\|V\|_2 = (\frac{(b-a)}{n})^{\frac{1}{2}}\|V\|_2$$

Let us consider the difference operators Δ_+, Δ_-:

$$\Delta_+ u(x) = \frac{u(x+h) - u(x)}{h}$$
$$\Delta_- u(x) = \frac{u(x) - u(x-h)}{h} \qquad (15)$$

It is clear that $\partial^2 u(x) = \frac{u(x-h) - 2u(x) + u(x-h)}{h^2}$ satisfies:

$$\partial^2 u(x) = -\Delta_+ \Delta_- u(x) \qquad (16)$$

We need still another norm in our space, that will involve the difference operator Δ_-:

$$\|V\|_\Delta = (\Delta_- V, \Delta_- V)^{\frac{1}{2}} = \left(h \sum_{i=1}^{N} |\Delta_- V_i|^2\right)^{\frac{1}{2}}. \qquad (17)$$

Theorem (2): Let $\eta = \inf f_y$. The discretization in equations (3) is stable for satisfying:

$$\frac{-2\pi^2}{(b-a)^2}\left[1 - \frac{\pi^2 h_0^2}{24(b-a)^2}\right]^2 < \eta$$

Proof: the proof is in (Pereyra 1973) [1]

Theorem (1), (2), and equation (10) prove that the discretization in equations (3) is convergent of order 2, i.e.

$$\big\|Y(h) - \varphi_h y^*\big\| = O(h^2) \qquad (18)$$

An asymptotic expansion for the global discretization error

The variational equation associated with (1)

$$-e'' + f_y(x, y)e = g(x) \qquad (19a)$$
$$e(a) = e(b) = 0 \qquad (19b)$$

Has an unique solution $e(x) \in C^\infty[a, b]$ for each given C^∞ functions $y(x), g(x)$. If we use, for equation (19), the same discretization in equation (2) as we used for equation (1), then an expression similar to equation (10) holds. Therefore we have, at the solution of equation (1)

$$\tau_h'(x) = F_h'(\varphi_h y^*)\varphi_h e^*(x) = \varphi_h\{g - \sum_{k=1}^{K} a_k e^{*(2k+2)}(x)h^{2k}\} + O(h^{2K+2}), (20)2.21$$

Where $e^*(x)$ is the corresponding solution of equation (19), and $a_k = \frac{2}{(2k+2)!}$. and higher order derivatives of the mapping φ_h, F_h coincide, having the form

$$\varphi_h F^{(j)} e^j = F_h^{(j)} \varphi_h e^j = \varphi_h \frac{\partial^j f}{\partial y^j} e^j \qquad (21)$$

Theorem (3) 2.6 Let F, F_h be as above. Then for $h \le h_0$ the global discretization error has an asymptotic expansion in even powers of h:

$$Y(h) - \varphi_h y^* = e(h) = \varphi_h \sum_{k=1}^{K} e_k(x) h^{2k} + O(h^{2k+2}). \qquad (21)$$

The function $e_k(x)$ are independent of h and satisfy the linear two point boundary value problems:

$$F'(y^*) = -e_k'' + f_y(x, y^*(x))e_k = b_k, e_k(a) = e_k(b) = 0. \quad (22)$$

The functions b_k constructed as:

$$b_1 = -\frac{1}{12} y^{*(4)}(x),$$

$$b_2(x) = -\left[\frac{1}{360} y^{*(6)}(x) + \frac{1}{12} e_1^{(4)}(x) + \frac{1}{2} f_{yy}(x, y^*) e_1^2(x)\right],$$

In general, the determination of b_k involves derivatives of the solution y^*, and earlier error functions $e_v, v = 1, \dots, k-1$. Therefore the b_k can be determined recursively.

C. DEFERRED CORRECTION METHOD

In this method the approximating difference equations are solved as usual. Their solution is then used to calculate a correction term, at each mesh point of the solution domain, which is added to the approximating difference equation at each mesh point. The corrected equations are then resolved and the process repeated if necessary. The correction terms are numbers obtained by differencing the numerical solution in the x-direction [iii]. As early as 1947, Leslie Fox [iv] advocated a technique called Difference Correction. Through the years he ends his collaborators have applied this technique to a variety of problems in differential and integral equations. In Fox 1962 [v], a wealth of information on the state of the art in the English school can be found. It is there where we the term Deferred Corrections used interchangeably with that of difference corrections. The reasons for this switch in nomenclature are not apparent, except perhaps for the feeling that technique was in some way connected with the deferred approach to the limit that we were discussing in the earlier sections, and also because the name reflected the fact that a posteriori corrections were performed. We have preferred to adopt the latest name in our work on this technique since our approach is not tied up (at least in appearance) to expansions in terms of differences, as it was in the earlier development. We base our formulation of the method on the asymptotic expansion for the local truncation error:

$$\tau_h(x_i) = -\sum_{k=1}^{K} a_k\, y^{2k+2}(x_i) h^{2k} + o(h^{2k+2}) \qquad (23)$$

Which, only needs smoothness of the exact y(x) and the applications of Taylor's formula for its derivation. For any smooth function y(x) we can approximate linear combinations of its derivatives with any order of accuracy in h at any grid point by using sufficient ordinates in a neighborhood this is again conscience of a wise applications of the Taylor's expansions and numerical differentiations techniques. Thus, there exist weights w_s such that

$$T_l(x_i) = -\sum_{k=1}^{l} a_k\, y^{2k+2}(x_i) h^{2k+2} = \sum_{s=1}^{2l+2+q} w_s\, y(x_i + \alpha_s h) + o(h^q) \equiv s_l(y(x_i)) + o(h^q),\ \alpha_s\ Integers.\ (24)$$

We shall show later how to obtain w_s in an efficient and sufficiently accurate way. Observe that we have multiplied $\tau_h(s)$ by h^2. In this fashion s_l becomes a bounded operator (for $h \to 0$) and most of the danger of numerical differentiation formulas are avoided. In fact, Fox's difference correction procedure was mostly advocator desk calculator computation, where a table of difference manipulated by an able person was a real asset. Then main contributions of these notes, starting with a Stanford Report (Pereyra 1965)[vi], have been to put on a sound theoretical bases the asymptotic behavior of Avery general procedure modeled on Fox's difference corrections, and is even more relevant, he has produced tools and complete implementations of this technique in a variety applications. However, so many years and development later (with some minor changes) the words in Fox's (1963) [vii]. This idea (difference correction) does not seem to have penetrated deeply into the literature of automatic computation. Certainly we have to do some differencing involving extra programming, extra space, and some difficulties in automatic inspection of differences, but machines are getting larger and programming easier, and if we are concerned with accuracy, as we certainly should be, something like this was essential.

Probably one of the main reasons for this neglect in recent times has been the widespread interest in other high order methods (splines, finite elements). Unfortunately, the theoretical developments in these areas have very much surpassed (and overshadowed) the practical, efficient implementation of the methods. Thus, we find ourselves in the sad situation of having a highly promising, very general, theoretical well supported technique, that is begging for an at least equal treatment in its practical aspects, while on the other hand, for some applications at least, it is fairly clear that the results obtained with our more traditional finite difference technique will be hard to beat. It wasn't surprise if it finally turns out that a successful implementation of high order splines methods comes about via a deferred correction type of approach, bypassing in some way the very expensive steps of high order quadrature formulae and complicated systems arising from the present approaches. See [viii] for a first timid step in that direction.

D. ALGORITHMS

There are many ways of producing differed corrections. Fox's way consisted essentially of representing y'' as series of difference. In the first steps, common to all producers, one would compute using only the first term of the expansion, in this case the basic method as equation (3) and then use these o(h^2) values in the difference expansion, and recomputed in order to obtain a more accurate solution. The process was thought as iterative, providing in infinitely many steps the exact solutions. This was never done in practice; in fact it is hard to find any published numerical example in which more than two corrections were performed, carrying perhaps three or four terms in the difference expansion. Naturally, the reason for this was that on a desk calculator any prolonged computation was a big undertaking.

Let $y^{(0)}$ be the o(h^2) solution to equation (3), and let s_1 be, as in equation (2), an o(h^6) approximation to $T_1 \equiv -a_1 y^{*(4)} h^4$, the first term in the local truncation error (multiplied by h^2). Observe that since there is already a factor h^4 in T_1, we only are requiring an $o(h^2)$ approximation to $y^{*(4)}$ at the grid points. If we have $y^*(x)$ a available then, there is no problem in obtaining the weight w_s for s_1. But all we have is $Y^{(0)}$. In principle it cannot be expected that from an $o(h^2)$ discrete approximation to a function one can obtain an $o(h^2)$ approximation to derivative. It is here where we make use of the expansion in equation (21) for the global discretization error. In fact we have that because of linearity and equation (2):

$$S_1\left(Y^{(0)}\right) - S_1(\varphi_h y^*) = S_1(\varphi_h e_1)h^2 + S_1(\varphi_h e_2)h^4 + O(h^6).$$

Such that $S_1 = O(1)$. But $S_1(\varphi_h y^*) = T_1 + O(h^6)$ and $S_1(\varphi_h e_k) = -a_1 e_k^{(4)} h^4 + O(h^6), k = 1, 2.$ Therefore, $S_1\left(Y^{(0)}\right) = T_1 + O(h^6)$, and we can use $Y^{(0)}$ instead of $\varphi_h y^*$ and still obtain the same asymptotic behavior. With $S_1(Y^{(0)})$ computed at every grid point we solve for a corrected value $Y^{(1)}$

$$F_h(Y) = h^{-2} S_1\left(Y^{(0)}\right). \qquad (25)$$

The local truncation error for this new discretization is $o(h^4)$ and therefore, since we are still talking about the same basic operator F_h, the stability condition proves that there exists a unique solution $Y^{(1)}$ to this problem and that

$$\left\|Y^{(1)} - \varphi_h y^*\right\| = o(h^4) \qquad (26)$$

Provided we can obtain an asymptotic expansion for $Y^{(1)} - \varphi_h y^*$

The procedure can be repeated, and each time two more orders in h will be gained. In general, the iterated deffered correction procedure can be described in the following way:

Let $Y^{(k)}$ be an $o(h^{2k+2})$ discrete solution.

Compute $h^{-2} S_{k+1}\left(Y^{(k)}\right)$, an h^{2k+2} approximation, to the first $(k+1)$ terms in the local truncation error expansion.

Solve $F_h(Y) = h^{-2} S_{k+1}\left(Y^{(k)}\right)$, for $Y^{(k+1)}$. (See Pereyra 1973)[i]

3. AN $o(h^8)$ METHOD FOR THE PRICE OF AN $o(h^2)$ METHOD

We consider problem in equation (1) again but we shall use the more accurate $o(h^4)$ discretization

$$\frac{Y_{i-1} - 2Y_i + Y_{i+1}}{h^2} - \frac{1}{12}[f_{i-1} + 10f_i + f_{i+1}] = 0, i = 1, \ldots, N-1 \quad (27)$$

Where $f_i = f(x_i, Y_i)$. we symbolize equation (27) by $G_h(Y)$. By recalling that $f\left(x, y^*(x)\right) = y^{*''}(x)$ it is easy then to derive via Tylor expansions that the local truncation error is in this case:

$$G_h(\varphi_h y^*) = \varphi_h \sum_{k=2}^{K} a_k f^{(2k)}\left(x, y^*(x)\right) \frac{h^{2k}}{(2k)!} + o(h^{2k+2}),$$

Where $a_k = \frac{1}{(k+1)(2k+1)} - \frac{1}{6}$.

The linearized equations that obtain at each Newton step v are the following:

$$\left[\frac{h^2}{12}f_y(x_{i-1},Y_{i-1}^v)-1\right]E_{i-1}+\left[\frac{5h^2}{6}f_y(x_i,Y_i^v)+2\right]E_i+\left[\frac{h^2}{12}f_y(x_{i+1},Y_{i+1}^v)-1\right]E_{i+1}=r_i^v \qquad (28)$$

Where

$$r_i^v=-\left[(-Y_{i-1}^v+2Y_i^v-Y_{i+1}^v)+\frac{h^2}{12}(f_{i-1}^v+10f_i^v+f_{i+1}^v)\right]. \qquad (29)$$

for short, we can call the left hand side of equation (28): $h^2G_h'(Y^v)E$. Once $\{E_i\}$ is obtained, then the new iterate results:

$$Y_i^{v+1}=Y_i^v+E_i \qquad (30)$$

because of the stability, it is enough to reduce the residuals r_i to a level compatible with the global discretization error in the final corected solution. In fact,
$\bar{r}^v=h^2G_h(Y^v), G_h\big(Y(h)\big)=0$, and therefor we have that
$\|Y^v-Y(h)\|\le c\|G_h(Y^v)\|=ch^{-2}\|\bar{r}^v\|$.
Thus,

$$\|Y^v-\varphi_h y^*\|\le\|Y^v-Y(h)\|+\|Y(h)-\varphi_h y^*\|\le ch^{-2}\|\bar{r}^v\|+Ch^4,$$

and a reasonable stopping criteria for Newton's method is then:

$$\|\bar{r}^v\|\le c_1h^{10}, \qquad (31)$$

Where c_1 is usually chosen to be a small constant unless some more precise information a bout c and C is a vailable. Let $Y^{(0)}$ be the computed $o(h^4)$ solution. If we now define

$$T(x_i)=-\frac{1}{10}\frac{d^4}{dx^4}f\big(x_i,y^*(x_i)\big)\frac{h^4}{4!}-\frac{11}{84}\frac{d^6}{dx^6}f\big(x_i,y^*(x_i)\big)\frac{h^6}{6!}, \qquad (32)$$

And

$$S\left(f\big(x,Y^{(0)}\big)\right)=T_1(x)+o(h^8). \qquad (33)$$

Then by solving

$$G_h'\big(Y^{(0)}\big)E=S\big(f^{(0)}\big) \qquad (34)$$

and putting

$$Y^{(1)}=Y^{(0)}-E, \qquad (35)$$

We shall have an $O(h^8)$ approximation. (See -Pereyra 1973) [i]

4. DESCRIPTION OF THE MESH REFINEMENT METHOD:

A. SHISHKIN MESH

It is also possible, without losing generality, reducing the restrictions in the equation (1) after a few mathematical processors to homogeneous. We now describe the Shishkin mesh for convection-diffusion problem Let $q\in(0,1)$ and $\sigma>0$ be two mesh parameters. We define a mesh transition point λ by

$$\lambda=min\{q,\sigma\varepsilon ln\,N\} \qquad (36)$$

Then the intervals $[0,\lambda]$ and $[\lambda,1]$ are divided into qN and $(1-q)N$ equidistant subintervals (assuming that qN is an integer). This mesh may be regarded as generated by the mesh generating function

$$\varphi(\xi)=\begin{cases}\dfrac{\sigma\varepsilon}{\beta}\tilde{\varphi}(\xi)\ with\ \tilde{\varphi}(\xi)=ln\,N\dfrac{\xi}{q}for\ \xi\in[0,q],\\ 1-\left(1-\dfrac{\sigma\varepsilon}{\beta}ln\,N\right)\dfrac{1-\xi}{1-q}\quad for\quad \xi\in[q,1]\end{cases} \qquad (37)$$

if $q\ge\lambda,\beta=1$; Again the parameter q is the amount of mesh points used to resolve the layer. The mesh transition point λ has been chosen such that the layer term $exp(-\beta x/\varepsilon)$ is smaller than N^{-1} on $[\lambda,1]$. Typically σ will be chosen equal to the formal order of the method or sufficiently large to accommodate the error analysis [ix INT]. The course part of this Shishkin mesh has spacing h=(1-q)(1- λ)/N, so N-1≤h≤ qN-1. The fine part has spacing $h=q\lambda/N=q(\frac{\sigma}{\beta})\varepsilon N^{-1}lnN$, so $h\ll\varepsilon$. Thus there is a very abrupt change in mesh size as one passes from the coarse part to the fine part. The mesh is not locally quasi-equidistant, uniformly in ε. On the mesh $x_i=ih\ for\ i=0,\dots,N/2$ and $x_i=1-(N-i)h$ for $i=\frac{N}{2}+1,\dots,N$.

A key property, nonequidistant of the Shishkin mesh, for convection diffusion-problems are some time described as "layer resolving" meshes. One might infer from this terminology that wherever the derivatives of u are large, the mesh is chosen so fine that the truncation error of the difference scheme is controlled. But the Shishkin mesh does not fully resolve the layer: for

$$|u'(x)|\approx C\varepsilon^{-1}e^{-b(1)(1-x)/\varepsilon}$$

so

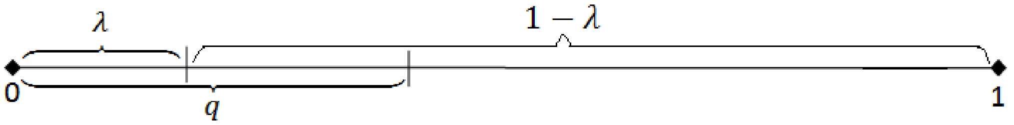
$$|u'(1-\lambda)|\approx C\varepsilon^{-1}e^{-2lnN}=C\varepsilon^{-1}N^{-2}$$

Which in general is large since typically $\varepsilon\ll N^{-1}$ that is $|u'(x)|$ is still large
on part of the first coarse-mesh interval $\big[x_{N/2-1},x_{N/2}\big]$. [x]

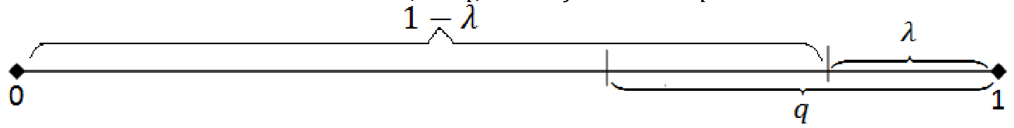
B. CONSTRUCTION THE "PIECEWISE UNIFORM" SHISHKIN FITTED MESH: [xi], [xii], [xiii], [xiv],[xv]

We will use four construction of Shishkin piecewise uniform fitted mesh, as decoding of the Shishkin function in equation (37) , each of, which is a mesh vary depending on the location of the singularity, as follows: 38a, 38b, 38c and 38d , below represents mapping to fix the location of boundary and interior layers, puts fine part of the mesh, of thickness not exceeding the value of transition point indicator λ as in equation(36), at the left, the right, the center, and both extreme points (left and right) respectively.

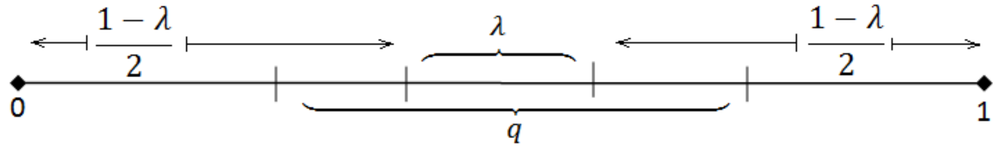
$$x_i = \frac{\lambda}{qN} i, x_{j+qN} = 1 - \lambda + \frac{(1-\lambda)}{(1-q)N} j, i = 0,1,\dots,qN, j = 0,1,\dots,(1-q)N \quad (38a)$$



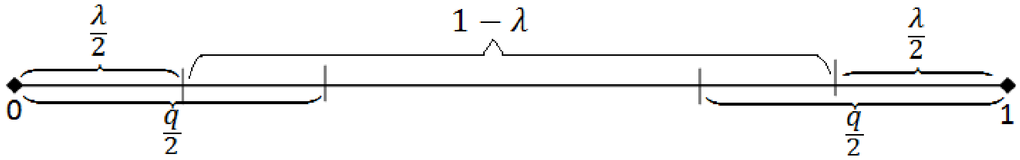
$$x_i = \frac{(1-\lambda)}{(1-q)N} i, \qquad x_{j+(1-q)N} = 1 - \lambda + \frac{\lambda}{qN} j \qquad (38b)$$
$$i = 0,1,\dots,(1-q)N, \qquad j = 0,1,\dots,qN$$



$$x_i = \frac{(1-\lambda)}{(1-q)N} i, x_{j+\frac{(1-q)}{2}N} = \frac{1-\lambda}{2} + \frac{\lambda}{qN} j, x_{i+\frac{1+q}{2}N} = \frac{1+\lambda}{2} + \frac{\lambda}{qN} i \qquad (38c)$$
$$i = 0,1,\dots,\frac{(1-q)}{2}N, \qquad j = 0,1,\dots,qN$$



$$x_i = \frac{\lambda}{qN} i, x_{j+\frac{q}{2}N} = \frac{\lambda}{2} + \frac{1-\lambda}{(1-q)N} j, x_{i+(1-\frac{q}{2})N} = 1 - \frac{\lambda}{2} + \frac{\lambda}{qN} i \qquad (38d)$$
$$i = 0,1,\dots,\frac{q}{2}N, \qquad j = 0,1,\dots,(1-q)N$$



With regard to norm which is used in these types of problems like equation (1), the reference [xvi], resolved by the favor of the use of any Norm, do's not involve averaging namely maximum norm, which is defined by

$$\|f\|_\infty = \max_{x\in[0,1]} |f(x)|. \qquad (39a)$$

So the maximum error e between the numerical solution U_ε and the exact solution u^E is

$$e = \|U_\varepsilon - u^E\|_\infty = \max_{x\in[0,1]} |U_\varepsilon(x) - u^E(x)|. \qquad (39b)$$

In this norm we see that differences between distinct functions are detected, irrespective of how small ε is. Which mean that, the maximum norm is an appropriate norm for the study of boundary layer phenomena.

5. THE OUTLINE

To find the approximation solution to the problem of equation (1) with the new improvement Shishkin fitted Mesh:
5.1. Decide on how many mesh points (sub intervals multiplicand number N).
5.2. Determine the Shishkin transition point indicator λ as in equation (36).

5.3. Allocate fine part of Shishkin mesh on the interval [0,1], corresponding to singularly boundary layer dropping on X-axis, which is easy process by deducing it through applying, uniform mesh with the deferred correction method numerical solution, once and observing the plot of the exit solution with the boundary conditions to discover lineament of location of the singular (stiff) layers.

5.4. The first improvement is by little tuning the value of σ in equation (36) until the solution became softer. The second improvement let $0 < v < 1$ be arbitrary real number and [f1,f2] be the fine subinterval with length $D = f_2 - f_1$:

a- For equations 38a, 38b and 38d we divide the fine subinterval to two another non equal subintervals as follows:

$$[f_1, f_2] = [f_1, f_1 + vD] \cup [f_1 + vD, f2]$$

Then the fine subinterval itself is divided into fine and coarse regions depending on the value of v in the arrangement and intensity without a change in the total length D since $D = vD + (1 - v)D$, in the case of if $v = 1$ then the mesh will return to normal construction of the Shishkin mesh.

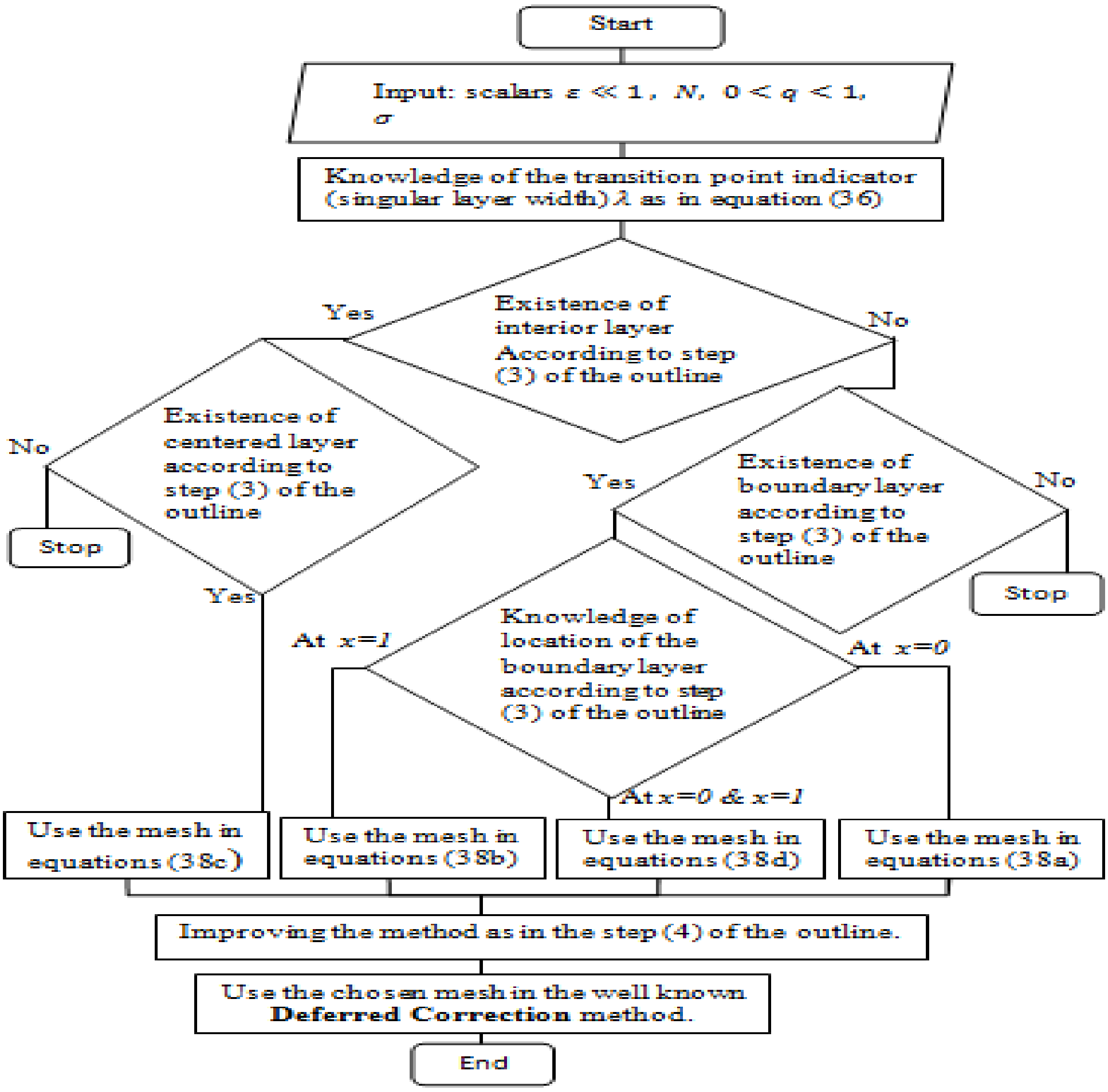
b- For equations 38c we divide the fine subinterval let denote it by [f1,f2] to three another non equal subintervals as follows:

$$[f_1, f_2] = \left[f_1, f_1 + \frac{v}{2}D\right] \cup [f_1 + \frac{v}{2}D, f_2 - \frac{v}{2}D] \cup \left[f_2 - \frac{v}{2}D, f_2\right]$$

Then the fine subinterval itself is divided into three (two of them same) fine and coarse regions depending on the value of v in the arrangement and intensity without a change in the total length D since $D = \frac{v}{2}D + (1 - v)D + \frac{v}{2}D$, in the case of if $v = 1$ then the mesh will return to normal construction of the Shishkin mesh.

5.5. Apply the steps of deferred correction method to find the approximation solution.

6. FLOWCHART OF THE NEW METHOD



7. TEST PROBLEM : [xvii], [xviii]

1) $-\varepsilon u''(x) - u(x) = 0, u(0) = 1, u(1) = 0$

$$u^E(x) = e^{-x/\sqrt{\varepsilon}} - \frac{e^{((x-2)/\sqrt{\varepsilon})}}{1 - e^{-2/\sqrt{\varepsilon}}}.$$

2) $-\epsilon u''(x) + u(x) = (\epsilon\pi^2 + 1)\cos(\pi x), u(-1) = -1, u(1) = 0$

$$u^E(x) = \cos(\pi x) + \frac{e^{\frac{(x+1)}{\sqrt{\varepsilon}}} - e^{\frac{-(x+1)}{\sqrt{\varepsilon}}}}{e^{\frac{2}{\sqrt{\varepsilon}}} - e^{\frac{-2}{\sqrt{\varepsilon}}}}$$

3) $u''(x) = -\frac{3\varepsilon u}{(\varepsilon + x^2)^2}, u(0.1) = -u(-0.1) = 0.1/\sqrt{\varepsilon + 0.01}.$

$$u^E(x) = x/\sqrt{\varepsilon + x^2}.$$

4) $-\varepsilon u''(x) - u(x) = -(1 + \varepsilon\pi^2)\cos(\pi x), u(-1) = 0, u(1) = 0$

$$u^E(x) = \cos(\pi x) + e^{\frac{(x-1)}{\sqrt{\varepsilon}}} + e^{\frac{-(x+1)}{\sqrt{\varepsilon}}}$$

8. NUMERICAL RESULT

A. TOTAL NUMBER OF IMPLEMENTATION

We present the computational performance of a Matlab implementation on a set of (9120) singularly perturbation BV (test problem × ε × N × algorithm) as in table (1). The Matlab implementations based on the implementation of the eight order Deferred Correction method-uniform mesh provided by V. Pereyra 1973 [i] without slightest changes in composition of the Deferred Correction operator except the mesh was changed three time; uniform mesh represented in equation (2b); as (algorithm1), Shishkin mesh represented in equations (38); as (algorithm 2) and the new proposed improvement mesh represented in step(4) of the outline; as (algorithm 3). The comparisons of algorithms based on maximum error as in the equation (39).

Table (1): Total number of implementation

No. of Prob.	No. of Algorithms	No. of ε values	No. of N values	Total No. of Implementation
4	3	10	76	9120

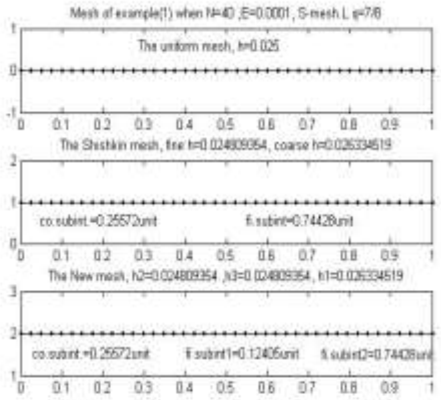
B. DETAILS

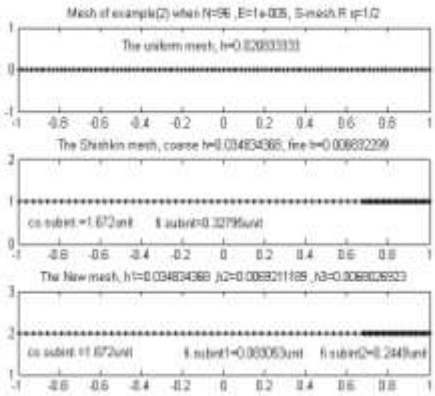
Details of table (1) are given in the following context arranged in table (2)

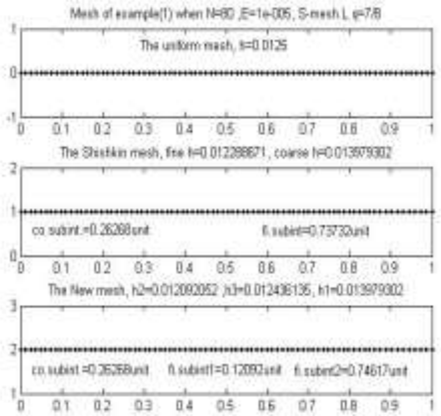
Table (2): Context of the choice of each of the fine layer location, perturbation parameter ε, Shishkin ratio q and sub intervals multiplicand number N in the numerical results and comparisons.

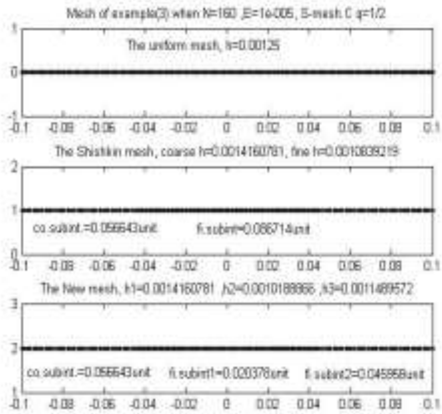
Prob. No.	Type of layer	ε	q	N						#N
1	Boundary at x=0	1.00E-04	7/8	40	48	56	…	80	88	7
		1.00E-05		16	24	32	…	72	80	9
2	Boundary at x=1	1.00E-03	1/2	32	40	48	…	72	80	7
		1.00E-04		24	32	40	…	64	72	7
		1.00E-05		40	48	56	...	88	96	8
3	Centered at x=0	1.00E-05	1/2	160	168	176	…	200	208	7
		1.00E-06		312	320	328	…	360	368	8
4	Boundary at both end points x=-1 & x=1	1.00E-04	1/2	56	64	72	…	104	112	8
		1.00E-05		208	216	224	…	256	264	8
		1.00E-06		1048	1056	1064	…	1088	1096	7
Number of taken values of N										76

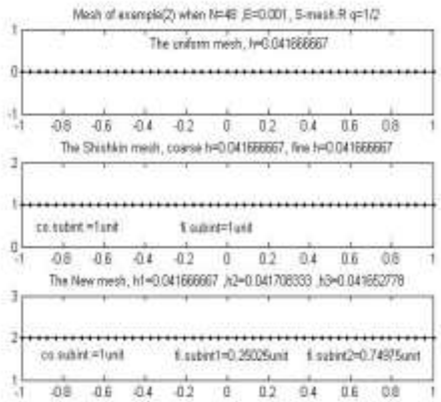
C. SOME MATLAB PLOTS OF MESHES CONSTRUCTION

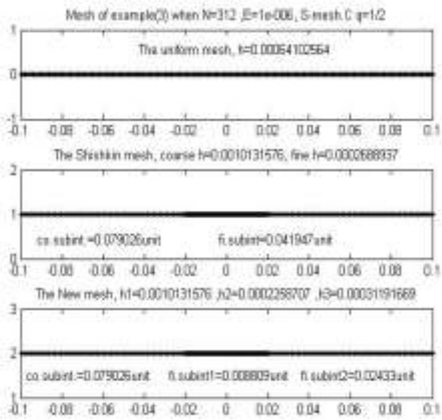


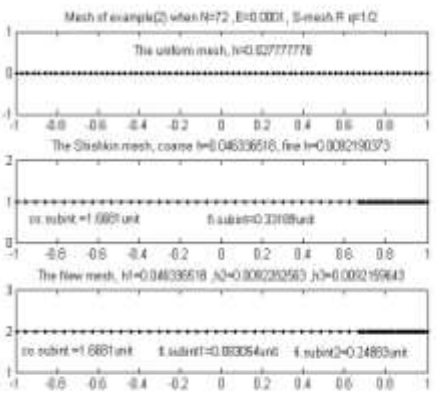


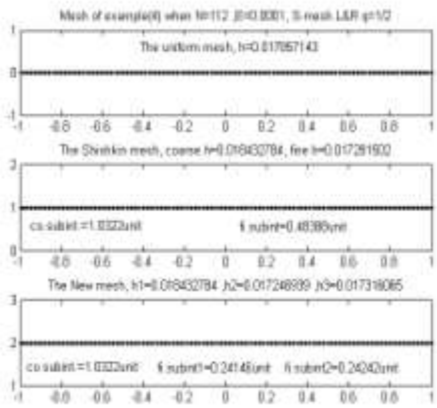












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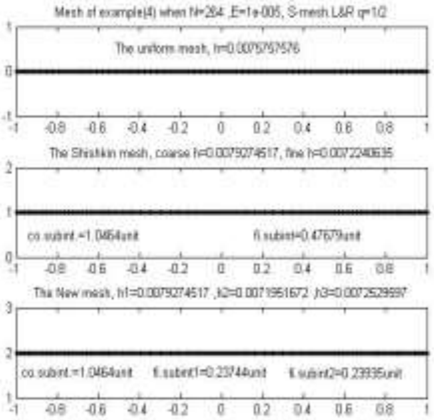
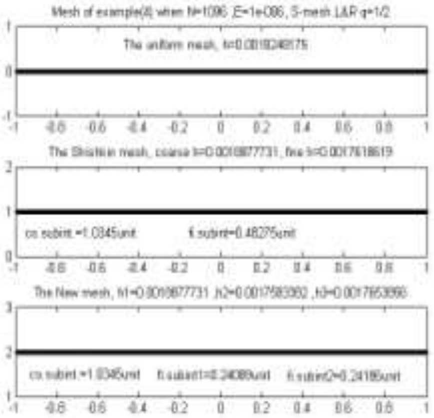
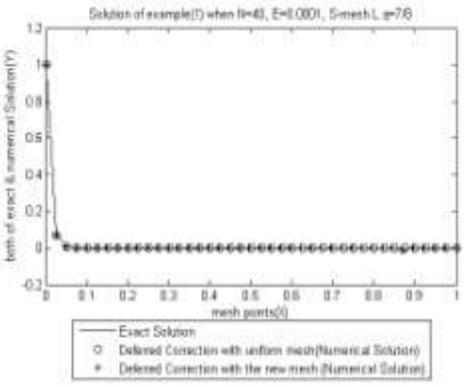
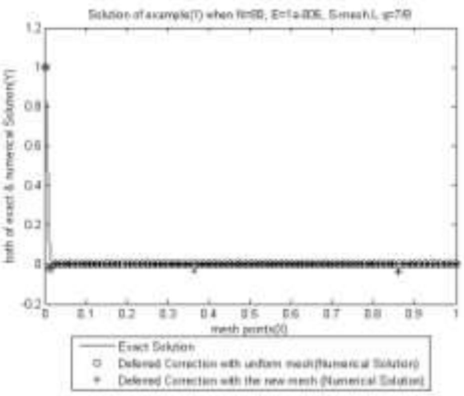


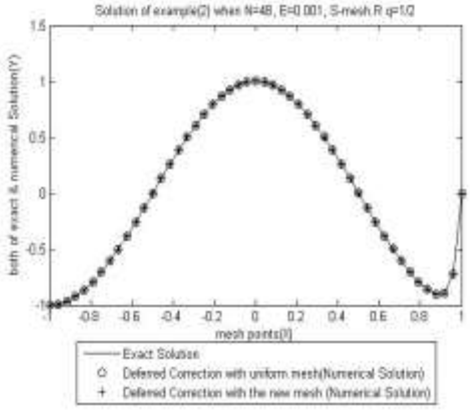


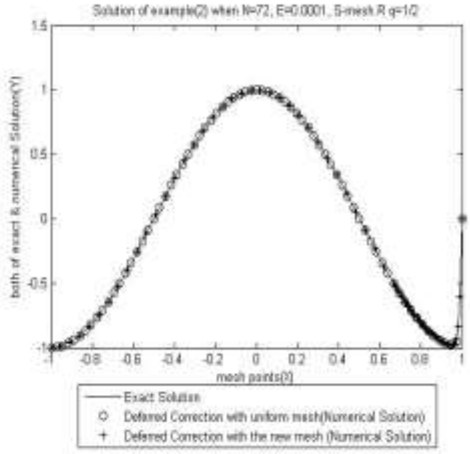
D. Matlab plots illustrate solutions of the new algorithm together with the exact solutions

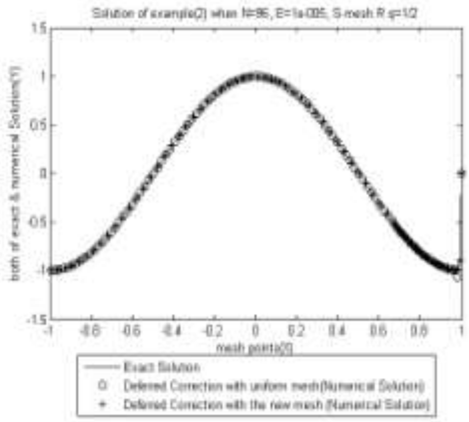


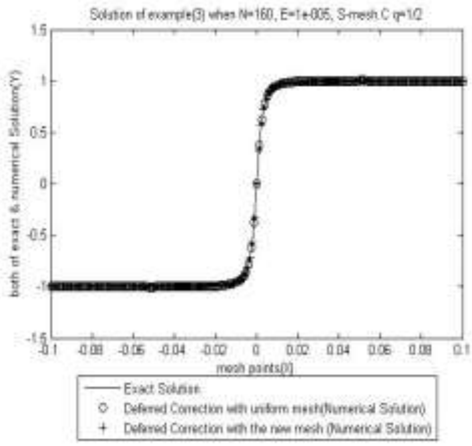
E.

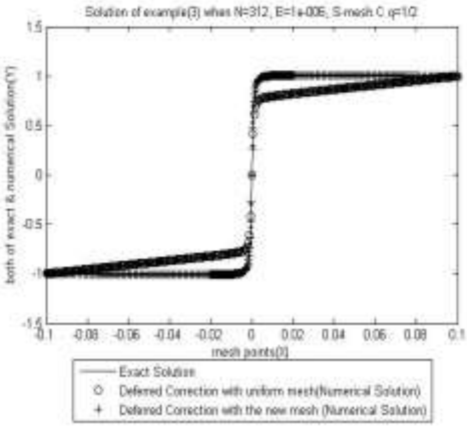


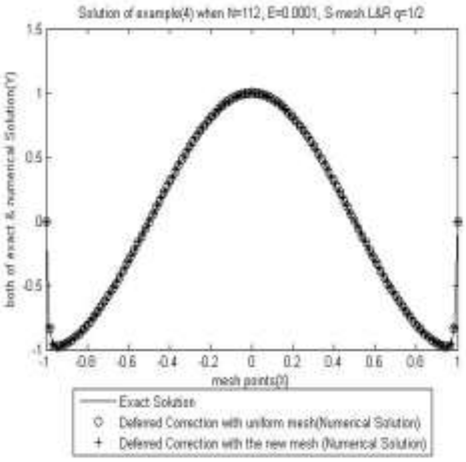


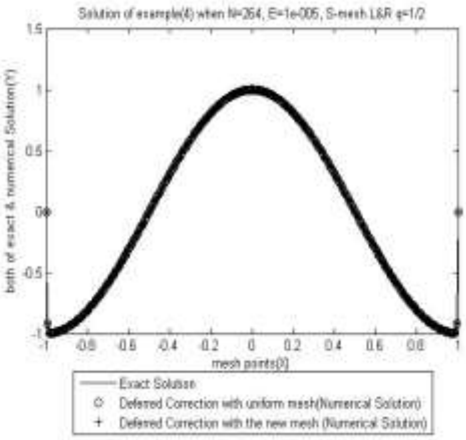


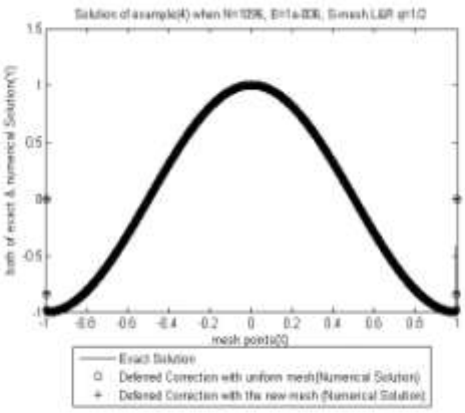




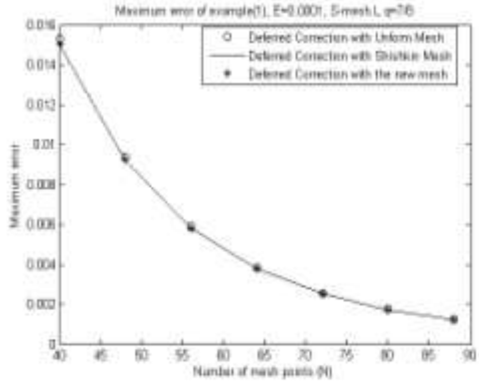


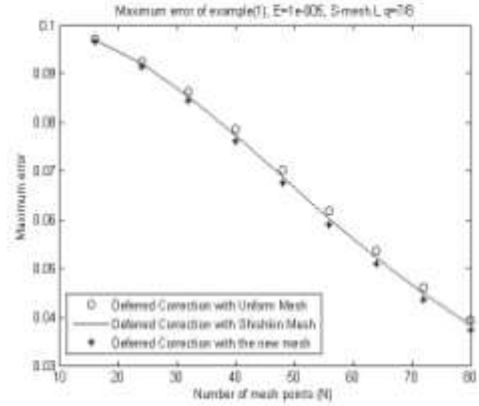


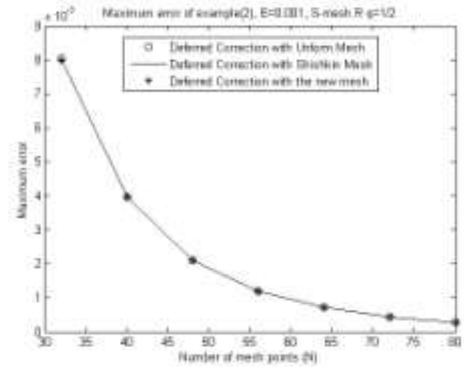


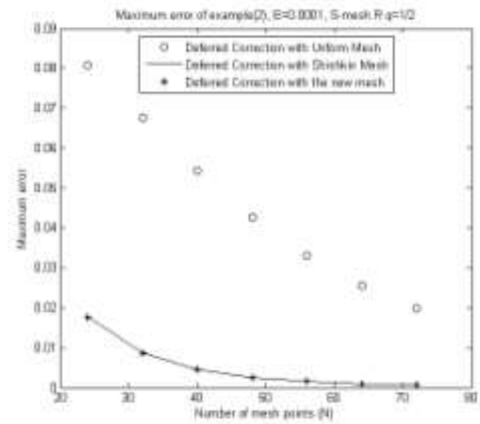


F. Matlab plots compare the convergence of the three algorithm:









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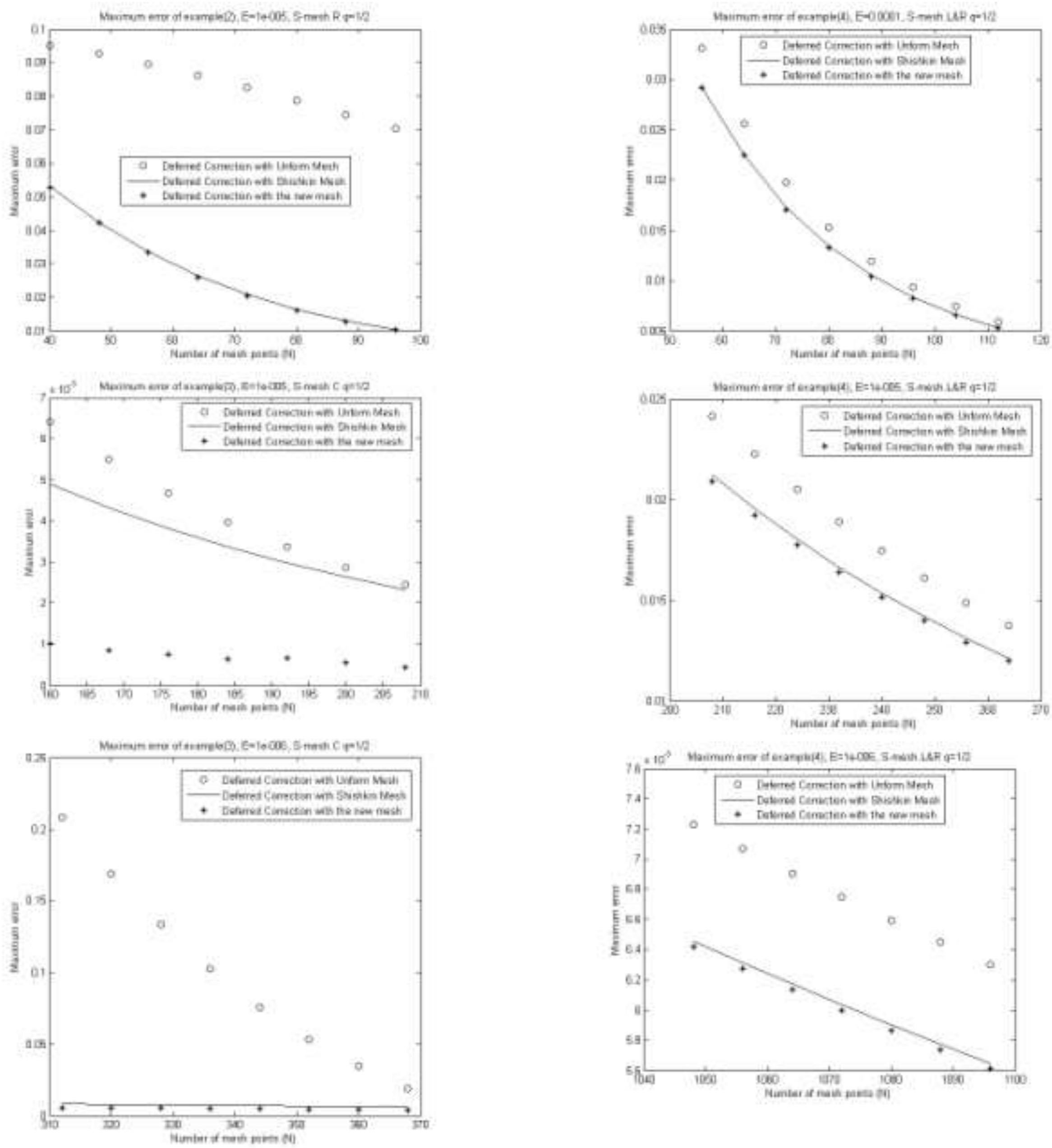

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