International Journal of Enhanced Research in Science Technology & Engineering, ISSN: 2319-7463
Vol. 3 Issue 3, March-2014, pp: (86-89), Impact Factor: 1.252, Available online at: www.erpublications.com

On Infinitesimal Sets and Hyper Real Numbers

Tahir H. Ismail
Department of Mathematics, College of Computer Sciences and Mathematics, University of Mosul, Iraq

Abstract: In this paper, we introduce and define a new type of numbers called hyper real numbers and infinitesimal sets as a new type of sets in the set of real numbers ℝ .We use these concepts to define and characterize a relation between infinitesimal sets and hype real numbers.

Keywords: Non Standard Analysis, Infinitesimal, Hyper Real Numbers.

Introduction

Through this paper we need the following definitions and notations:

Definition 1.1 [6],[4],[2]

Areal number ω is called unlimited if its absolute value is larger than any standard integer. So a nonstandard integer ω is also unlimited real number, ω+1/2 is an example of unlimited real number that is not integer.

Definition 1.2 [5],[8],[9]

A real number x is called infinitesimal if its absolute value is smaller than $\frac{1}{n}$ for any standard number n, of course 0 is infinitesimal.

Definition 1.3 [2]

A real number x is called limited if x is not unlimited.

Definition 1.4[3],[7]

Areal number x is called appreciable if x is neither unlimited nor infinitesimal.

Definition 1.5 [1],[3]

Two real numbers x and y are infinitely near, denoted by $x \simeq y$ if x-y is infinitesimal.

The axioms of IST is the axioms of ZFC together with three additional axioms which are called the transfer axiom, the idealization axiom and the standardization axiom .They are as follow:

Transfer Axiom [6]: for each standard formula $F(x,t_1,t_2,\ldots,t_n)$ with only free variables $x,,t_1,t_2,\ldots, t_n$, the following statement is an axiom

$$\forall^{st} t_{1,} t_2 \ , \ldots \ldots t_n \ (\forall^{st} F(x, t_1, t_2, \ldots., t_n) \Rightarrow \forall_x F(x, t_1, t_2 \ , \ldots \ldots., t_n))$$

Idealization axiom [6]:

For each standard formula B(x,y), with free variables x,y the following is an axiom

$$\forall^{stfin} A \exists_x \forall_y \epsilon A \wedge B(x,y) \Leftrightarrow \exists_x \forall^{st} B(x,y)$$

Standardization Axiom [6]

For every formula A(z) internal or external, with free variable z, the following is an axiom

$$\forall_x^{st} A \ \exists_y^{st} \forall_z^{st} \ (z\epsilon \, Y \Leftrightarrow z\epsilon X \wedge A(Z))$$

Definition 1.6 [3],[9]

If x is limited real number, then it is infinitely near to unique standard real number called the standard part of x, denoted by ${}^0x$.

Definition 1.7

An infinitesimal set, denoted by ⊙ is an external convex additive subgroup of ℝ.

Definition 1.8

A hyper real number is the algebraic sum of a real number and an infinitesimal set.

The following remark gives us some properties and example of infinitesimal set and hyper real numbers.

Remark 1.9

1. If $⊙_1$ is an infinitesimal set, a is a real number and α is a hyper real number, then $\alpha \equiv a+⊙_1 \equiv \{a+x : x \in ⊙_1\}$, $⊙_1$ is the infinitesimal part of the hyper real number α.
2. The unique internal infinitesimal sets in ℝ are {0}, but there are many external infinitesimal sets, the monad of zero.
3. Every infinitesimal set is a hyper real number. On the other hand, there are hyper real numbers which are neither real number nor infinitesimal set as in the following example.

Example 1.10

If ω is unlimited positive real number, then the following sets are all hyper real numbers, which are neither real numbers nor infinitesimal sets

1. $1+⊙ \equiv \{ 1+\varepsilon : \varepsilon \epsilon ⊙ \}$
2. $\omega + L \equiv \{ \omega + t : t \in L \}$

Definition 1.11[3],[5]

An infinitesimal set $⊙_2$ is sub infinitesimal set of the infinitesimal set $⊙_1$, if $⊙_1 \sqsubset ⊙_2$ and denoted by $Max(⊙_1, ⊙_2) = ⊙_1$.

Remark 1.12

1) The sum of two infinitesimal sets $⊙_1$ and $⊙_2$ is defined by $⊙_1 + ⊙_2 = max(⊙_1, ⊙_2)$.

2) The product of two infinitesimal sets $⊙_1$and $⊙_2$ is defined by $⊙_1⊙_2 \equiv \{x\,y : x \in ⊙_1 \text{ and } y \in ⊙_2\}$.

3) The product of an infinitesimal set $⊙_1$ with any appreciable real number leaves $⊙_1$ invariant.

4) If an infinitesimal set does not contain 1, then by external induction it does not contain $(\frac{1}{n})$ for any standard integer n and is thus included in ⊙.

Definition 1.13

If a real number P does belong to an infinitesimal set $⊙_1$,then $\frac{⊙_1}{p}$ is an infinitesimal set included in ⊙ called the relative infinitesimal set of a hyper real number $\alpha \equiv P + ⊙_1$. The hyper real number $\alpha \equiv a + ⊙_1$ can be written in the following form $\alpha \equiv (1 + \frac{⊙_1}{a})$,and the hyper real number $(1 + \frac{⊙_1}{a})$ is called module of a .which is exactly the collection of real numbers that leave α invariant undermultiplication.

Proposition 1.14

If $\alpha \equiv a + ⊙_1$, $\beta \equiv b + ⊙_2$ are two hyper real numbers $b \notin ⊙_2$ and $a \notin ⊙_1$. Then

1) $\alpha + \beta \equiv a + b + ⊙_1⊙_2$

$\equiv a + b + max(⊙_1, ⊙_2)$

2) $\alpha - \beta \equiv a - b + max(⊙_1, ⊙_2)$

3) $\alpha . \beta \equiv a\,b + max(a⊙_2, b⊙_1, ⊙_1⊙_2)$

4) $\frac{1}{\beta} \equiv \frac{1}{b}(1 + \frac{⊙_2}{b})$

5) $\frac{\alpha}{\beta} \equiv \frac{a}{b} + \frac{1}{b^2}$ $(max(a⊙_2, b⊙_1, ⊙_1⊙_2)$

Proof: (1), (2) and (3) are obvious and direct

Proof. (4):

We have

$$\frac{1}{\beta} \equiv \frac{1}{b+⊙_2} \equiv \frac{1}{b\,(1+\frac{⊙_2}{b})} \equiv \frac{1}{b}\left(\frac{1}{1+\frac{⊙_2}{b}}\right) \quad (1)$$

Since $\frac{\circledcirc_2}{b}$ is the relative infinitesimal set contained in $\circledcirc_2$, we have $\frac{\circledcirc_2}{b} \subset \circledcirc$, this means that $\frac{\circledcirc_2}{b}$ is infinitesimal real number, therefore,

$$\frac{1}{(1+\frac{\circledcirc_2}{b})} \equiv \frac{\left(1+\frac{\circledcirc_2}{b}\right)(1+\frac{\circledcirc_2}{b})}{(1+\frac{\circledcirc_2}{b})}$$

$$\equiv \left(1+\frac{\circledcirc_2}{b}\right)$$

From (1) we get

$$\frac{1}{\beta} \equiv \frac{1}{b}(1+\frac{\circledcirc_2}{b})$$

Proof (5):

From (4) we have,

$$\frac{\propto}{\beta} \equiv (a+\circledcirc_1)(\frac{1}{b}\left(1+\frac{\circledcirc_2}{b}\right))$$

$$\equiv \frac{1}{b}\left(a+\frac{a\circledcirc_2}{b}+\circledcirc_2+\frac{\circledcirc_1\circledcirc_2}{b}\right)$$

$$\equiv \frac{a}{b}+\frac{1}{b}\left(\frac{a\circledcirc_2+b\circledcirc_1+\circledcirc_1\circledcirc_2}{b}\right)$$

$$\equiv \frac{a}{b}+\frac{1}{b^2}(a\circledcirc_2+b\circledcirc_1+\circledcirc_1\circledcirc_2)$$

Therefore, $\frac{\propto}{\beta} \equiv \frac{a}{b}+\frac{1}{b^2}\max(a\circledcirc_2, b\circledcirc_1, \circledcirc_1\circledcirc_2)$

Remark 1.15

From the above proposition we have the following:

1. If $\propto$ is a hyper realnumber, then $(\propto - \propto)$ is not equal to zero but to $\circledcirc_1$, that is the infinitesimal part of $\propto$ which contain zero.
2. If $\propto \equiv a+\circledcirc_1$ is hyper real number, then $\frac{\propto}{\propto}$ is not equal to 1, but is either equal to module of $\propto$ or to $1+\frac{c}{\propto^2}\circledcirc_1$, where c is real number.
3. If $\propto \equiv \alpha+\circledcirc_1$ and $\beta \equiv b+\circledcirc_2$ are two hyperreal numbers and suppose that $\circledcirc_1$ contains $\circledcirc_2$, and that there exists a real x belonging to both $\propto$ and β .

Since $\propto$ is invariant, it can be translated by an element of $\circledcirc_1$, *and since* β isinvariant, it can be translated by an element of $\circledcirc_2$, it follows that $\propto \equiv x+\circledcirc_1$ *and* $\beta = x+\circledcirc_2$. This means that $\propto$ contains β. So we have the following proposition.

Proposition 1.16

If $\propto$ and β are two hyperreal numbers .Then either $\propto$ and β are disjoint or one contains the other.

Proof:

Suppose that $\propto \equiv a+\circledcirc_1$ and $\beta \equiv b+\circledcirc_2$ are two hyper realnumbers, we have to prove that either $\propto$ and β are disjoint or one contains the other. If $\propto$ and β are disjoint, then no one of them contains the elements of the other and we have nothing to prove.

Now if $\propto$ and $\beta$are not disjoint, this means that $\propto \equiv x+\circledcirc_1$and $\beta \equiv x+\circledcirc_2$

Then either $\Theta_1 \subset \circledcirc_2$ or $\circledcirc_2 \subset \Theta_1$. Hence $\propto \subset \beta$ or $\beta \subset \propto$.

References

[1]. I.v. Berg, "Extended use of IST," Annals of pure and Applied logic, vol. 58, pp.73-92, 1998.

[2]. M. Davis, Applied Nonstandard Analysis, New York, John Wiley & Sons, 1977 .

[3]. F. Diener and M. Diener, Nonstandard Analysis in Practice, Spring–verlag , Berlin, Heidelberg, 1995 .

[4]. F. Diener and G. Reeb, Nonstandard Analysis, Herman, 1989.

[5]. I.O. Hamad, "A new type of Infinitesimal in Nonstandard Analysis," Zanko Journal of pure and Applied Science, Salaheddin University ,Vol.16 ,No.1, 2004.

[6]. E. Nelson, "Internal Set theory: A new approach to Nonstandard Analysis," Bull. Amer. Math. Soc.,Vol.83, No.6, pp. 1165-1198, 1977.

[7]. A. Robinson , Nonstandard Analysis, 2^{ed}, North-Holland, Pup. Com., Amsterdam, 1970.

[8]. K.D. Stroyan and W.A. Luxembourg, Introduction to the Theory of Infinitesimal, New York , Academic Press, 1976.

[9]. N.Vokil, Real Analysis through modern infinitesimal, Cambridge University Press, 2011.