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On Non Standard Description of Solutions of Functional Equations

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Abstract: In this paper, we use some concepts of nonstandard analysis to establish whether all solutions of the functional equation

$$\boldsymbol{f(x+y) = f(x) + f(y)}$$

can be described in terms of a functional equation of the form $f(x) = mx$, where m is unlimited or limited.

Keywords: Non Standard analysis, infinitely near, functional equation.

1. Introduction

It is well known that all the continuous solutions of the functional equation

$$f(x+y) = f(x) + f(y) \qquad (1)$$

Where f is a real valued function given by $f(x) = mx$.

Many papers on the subject have dealt mainly with the problem of finding the conditions on an additive function that ensure its continuity, one such condition is that the function be bounded on some interval. (see [1], [10], [7], [2]).

The dichotomy between the continuous and discontinuous solution of (1) is striking particularly in view of the simplicity of the equation. The problem of finding solutions to (1) is nearly related to the problem of finding all the characters of an additive subgroup S of IR, that is , all the complex valued functions X on S such that $|X(x)| = 1$ for all $x \in S$, and

$$X(x+y) = X(x) + X(y) \qquad (2)$$

The solutions of this character problem are of two kinds continuous and discontinuous, all of the forms having the form $X(x) = e^{imx}$ $(m \in \mathbb{R})$. The following theorem gives important information about the discontinuous solution of (2).

<u>Theorem 1.1</u>[8]

If S is a subgroup of $\mathbb{R}$ and X is character of S. Then for every $\varepsilon > 0$ and every finite subset $\{t_1 , t_2 , \dots\dots, t_n\}$ of S, there exists a continuous character $X_0(x) = e^{imx}$ such that

$$|X(x_j) - X_0(x_j)| < \varepsilon , \ j = 1,2, \dots\dots, m$$

It follows from the approximation theorem that for each character$X$ of a subgroup S of $\mathbb{R}$ relation$T_1(x, \varepsilon, m)$ defined by the inequality $|X(x) - e^{imx}| < \varepsilon$ is concurrent in sense of Robinson A. [9]. Hence there exists an element m of an enlargement $^*\mathbb{R}$ of $\mathbb{R}$ such that $X(x) = e^{imx}$ for all $\in S$, thus every character of S, continuous or not, is of the form $X(x) \simeq e^{*imx}$ for some $m \in {}^*\mathbb{R}$.

To deal with additive functions, we introduce another form of the approximation theorem.

Through this paper we need the following definitions and notations.

<u>Definition 1.2</u> (see [6], [3], [4])

A real number ω is called unlimited if its absolute value is larger than any standard in tagger, so a nonstandard ω is also unlimited real number.

<u>Definition 1.3</u> (see [5], [8], [11])

A real number is called infinitesimal if its absolute is smaller than $\frac{1}{n}$ for any standard number, of course 0 is infinitesimal.

<u>Definition 1.4</u> [6]

A real number is called appreciable if it is neither unlimited nor infinitesimal.

Definition 1.5 [4]

Two real number x and y are called infinitely near, denoted by $x \simeq y$, if $x - y$ is infinitesimal.

2. The Main Results

Theorem 2.1

If f is a real valued function of a subgroup $S$of $\mathbb{R}$, such that

$$f(x+y) = f(x) + f(y)(mod\ N) \qquad (3)$$

for some positive number N. Then for each $\varepsilon > 0$ and every finite subset $\{t_1,\ t_2, \dots \dots, t_n\}$ of S, there exists a number m such that

$$|f(x_j) - mx_j| < \varepsilon \quad (mod\ N),\ j = 1,2, \dots \dots, m$$

In other words such that $|f(x_j) - mx_j|$ is always within, of some integral multiple of N.

Proof:

Each character in $\mathbb{R}$ is of the form

$$X(x)\text{=exp}(e\pi j f(x)/N)$$

Where f satisfies (3). By the character approximation theorem, there exists for each $\delta > 0$ a number m such that

$$\left|X(x_j) - \exp\left(\frac{2\pi jm\, x_j}{N}\right)\right| < \delta \qquad j = 1,2, \dots \dots \dots, n$$

Since the Logarithmic function is continuous, it follows that to each $\varepsilon > 0$ there corresponds a number m such that

$$\left|\frac{2\pi f(x_j)}{N} - 2\pi m x_j/N\right| < \frac{2\pi\varepsilon}{N}(mod\ 2\pi),\ \ j = 1,2, \dots \dots, n$$

In other words, such that

$$|f(x_j) - mx_j| < \varepsilon\ (mod\ N),\ \ j = 1,2 \dots \dots, n$$

For unlimited values of m the linear function $(m\,x)$ is clearly unlimited for all nonzero standard$x$. Thus to get a standard additive function, we must reduce the values of the function mx to finite values.

Let K be the additive subgroup of ${}^*\mathbb{R}$ generated by the set of unlimited power of 2, that is let

$$K = \{z2^N : z\ and\ N\ are\ integers, N > 0\ is\ unlimited\}$$

If K_n is the additive subgroup of IR generated by $\frac{n}{2}$ (n is standard positive integer), then K can also be written $K = \underset{n \in IN}{\cap} K_n$.

Definition 2.2

$a \simeq b\ (mod\ K)$ means that the distance between $a - b$ and some element of K is infinitesimal.

Theorem 2.3

For$\in {}^*\mathbb{R}$, let S denote the subgroup of $\mathbb{R}$ consisting of all x such that $(m\,x)$ is congruent to some limited number $Z(x)$ modulo K, define a function f on S by $f(x) = z(x)$. Then f is additive on S.

Conversely if f is additive on a subgroup $S$of $\mathbb{R}$. Then there exists a number $m \in {}^*\mathbb{R}$ such that for each $x \in \mathbb{R}$, mx is limited modulo, and $f(x) \simeq mx(mod\ K)$.

Proof:

Suppose$m \in {}^*\mathbb{R}$, and if $x, y \in \mathbb{R}$ have the property that mx and my are limited modulo K, then certainly

$m(x+y) = mx + my$ has the some property.

Now by definition of

$$f(x) + f(y) \simeq m(x) + m(y) = m(x+y) \simeq f(x+y)(module\ K)$$

Since the values of f are standard, and no two standard numbers can be congruent modulo Kwithout being equal

$$f(x+y) = f(x) + f(y).$$

Now suppose that f is an additive function on a subgroup $S$of $\mathbb{R}$, then certainly

$$f(x+y) = f(x) + f(y)(modulo\ 2^n) for\ every\ positive\ integer\ n$$

By the second approximation theorem, for any $\varepsilon > 0$, for positive integers $n_1, n_2, \dots \dots, n_p$, and for elements $x_1, x_2, \dots \dots, x_k \in S$, there exists some $m \in \mathbb{R}$ such that

$$|f(x_j) - mx_j| < \varepsilon\ (mod\ 2^{ij}), j = 1,2, \dots \dots, k\ , i = r_1, r_2, \dots \dots, r_k$$

Choose$N = \max(2^i)\ , i = n_1, n_2, \dots \dots, n_p$.

The relation $T_2(\varepsilon, x, n, m)$ defined by the inequality

$$|f(x_j) - mx_j| < \varepsilon\ (mod\ 2^N)$$

Is therefore can, hence there exists $m \in {}^*\mathbb{R}$ such that, for every $x \in S$ and every limited positive integer I

$$f(x) \simeq mx(mod\ 2^I)$$

This implies that $f(x) \simeq mx(mod\ K)$.

Remark 2.4:

Since the elements of K are numerous, in the sense that each interval of unlimited length in ${}^*\mathbb{R}$ contains unlimited many elements of K. It is natural to ask whether, for each unlimited $m \in {}^*\mathbb{R}$, mx is necessarily finite modulo K for every $x \in \mathbb{R}$?

Theorem 2.5:

There exists an element $y \in {}^*\mathbb{R}$ such that the distance from y to each element of K is unlimited.

Proof:

Consider the sequence of numbers defined inductively by the condition$y_1 = 1, y_{n+1} = 2^n - y_n$.

If$\|\boldsymbol{x}\|_{\boldsymbol{0}}$ denotes the distance form $x$to the nearest integral multiple of 2^n, then the induction argument shows that

$$\|y_n\|_j = y_j \quad , j = 1, 2, \dots \dots, n$$

For each n, the relation $T_n(n, y)$ defined by statement $\|y_n\| = y_m$ is concurrent therefore there exists$y \in {}^*\mathbb{R}$ such that $\|y_n\| = y_n$for every standard positive integer n.

Now suppose that y within a limited distance, say 2^k , of some element of K . This element is evidently a multiple of 2^{k+j} for each standard j , since the sequence $\{y_j\}$ is increasing and not bounded which implies contradiction to inequalities

$$|y_n| < 2^k < y_{k+j}$$

Hence y is of unlimited distance from each element of K.

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