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Modelling Group Structures

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ABSTRACT

Axiomatic structures have been studied by various mathematicians, such as [1] and [2]. The formation of axiomatization of a commonly known kind of algebraic structure such as groups have steadily established. The consequences of basic terminologies are identified in the field of group theory.

Keywords: Axiomatization, Definable sets, Embedding, Homomorphism, Interpretation, Reducts, Sentence, Signature, Structure, Ultraproducts.

INTRODUCTION

In this paper we consider group in the signature of addition and multiplication structures, i.e. (0, 1; +, .) and (0, 1; -, .), where 0 and 1 are the constant symbols,"– " is the unary function symbol and " + "and " . " are binary functions.
To achieve our objectives, we started with some basic principles enabling us to progress through the whole of our process in this research.

1. BASICS

These basic definitions and properties are given in [1], [2], [3] and [4].

Definition 1.1: A signature $\sigma$is a system (3-tuples) , (C,F,R, σ), which consists of a set C of constant symbols, a set F of function symbols a set R of relation symbols and a signature symbol $\sigma : F \cup R \rightarrow \mathbb{N}\backslash\{0\}$.

Example 1.1 $< R\,;\, +,.\ ,\_,\ \geq;\ 0\,,1\ >$

$R \equiv$ Function symbol
$+,.,- \equiv$ Arity 2
$\geq \equiv$ Relation symbol
$0,1 \equiv$ Constant symbol

Definition 1.2 : A sign structure $\mathcal{M}$ is a quadruple (M , C , F, R) which consists of, $(\mathbb{C} : c \in \mathbb{C})$, $F = (f \in F)$ and $R : R \in \mathbb{R})$, where $f \in F$ and R is a $\sigma(R) - arity$ relation is M.

Example 1.2: Consider $\sigma = (\,0,1\ ;\ +\,;\ .\ ;\ <)$, where $\sigma\,(+) = \sigma(.) = \sigma(<) = 2$. Then $(R\,;0{,}1\ ;\ +,.\ ;<)$ is an ordered ring structure.

Take $\sigma = < .\ ,1\ ,\ (-1)^{-1} >$ is a $\sigma -$ structure for a group.

Definition 1.3: A homomorphism of $\sigma -$ structure from$\mathcal{A}$to $\mathcal{B}$ is a Relationθ: $\mathcal{A} \rightarrow \mathcal{B}$ such that

1. for all relation symbols R of σ , if $(\,a_1, \dots, a_n) \in R^{\mathcal{A}}\ then\, \theta(a_1), \dots, \theta(a_n) \in R^{\mathcal{B}}$.
2. For all function relation symbols of σ.
   $\theta\,(f^{\mathcal{A}}(a_1\,, \dots, a_n) = f^{\mathcal{B}}\,(\,\theta(f^{\mathcal{A}}(a_1)\,, \dots, \theta(a_n))$
3. For each constant symbol C of σ , $(c^{\mathcal{A}}) = c^{\mathcal{B}}$

An **embedding** is a homomorphism which is injective and for all relation symbols R, $(a_1, \ldots, a_n) \in R^{\mathcal{A}}$ *if and only if* $\theta(a_1), \ldots, \theta(a_n) \in R^{\mathcal{B}}$. An **isomorphism** of $\sigma -$ structure is a homomorphism $\theta : \mathcal{A} \to \mathcal{B}$ if there is $\varphi: \mathcal{A} \to \mathcal{B}$ with $\theta.\varphi$ the identity on $\mathcal{B}$. ie, $\varphi.\theta$ is the identity.

Lemma 1.1: Every isomorphism of $\sigma -$ structure is an embedding.

Proof : Suppose $\theta : \mathcal{A} \to \mathcal{B}$ is an isomorphism and R is a relation symbol of σ.
Suppose $\bar{a} = (a_1, \ldots, a_n) \in A$, and $(\mathcal{A}) \in R^{\mathcal{B}}$. Then writing θ^{-1} for inverse of θ, we have $\theta^{-1}(\theta(\mathcal{A})) \in R^{\mathcal{A}}$. So $\bar{a} \in R^{\mathcal{A}}$. Therefore θ is an embedding.

Example 1.3: There is an inclusion map $< Q, +, ., -, 0, 1, \leq > \to < \mathbb{R}, +, ., -, 0, \leq >$.

2. PRODUCTS AND EXPANSIONS

The structures $< Z; + >$ and $< Z; +, . >$ are different as they have different signatures.
We say that $< Z; + >$ is a reduct of $< Z; +, . >$. Also $< Z, +, . >$ is an expansion.

Products and Ultra products

Definition 2.1: Let I be a non-empty set and $(\mathcal{M}_i)_{i \in I}$ be a family of a structures. The **product**$\mathcal{M} = \pi_{i \in I} \mathcal{M}_i$ is $\sigma -$ structure defined by:

1. The domain of $\mathcal{M}$ is a set of functions $\alpha = I \to U_{i \in I} \mathcal{M}_i \; st \; \alpha(i) \in \mathcal{M}_i, i \in I$
$$(\alpha(i))_{i \in I} = \alpha$$
2. For each constant symbol $c \; of \; \sigma C^{\mathcal{M}} = (C^{\mathcal{M}_i})_{i \in I}$
3. For f, f^n symbol , anity n and $\bar{a} \in \mathcal{M}^n$, $f^{\mathcal{M}}(\bar{a}) = f^{\mathcal{M}i}(\bar{a}(i)),\; where \; i \in$ I
4. For R relation symbol anity n and $\bar{a} = (a_1, \ldots, a_n) \in \mathcal{M}^n$
$\bar{a} \in R^{\mathcal{M}} \Leftrightarrow \bar{a}(i) \in R^{\mathcal{M}i} \in R^{\mathcal{M}i}, \forall \; i \in I$.

Example 2.1: Consider a family $(G_i) \; i \in I$ of abelian groups.$\sigma = < +, -, 0 >$. Let $\mathcal{G} = \Pi_{i \in I} G_i$, the product of abelian groups. For $g \in \mathcal{G}$ the support is defined by:

Supp(g) =$\{i \in I = \mathcal{G}(i) \neq 0\}$.

Lemma 2.1: $H = \{ a \in \mathcal{G} = supp(a) \; is \; finte \}$.H is a subgroup of$\mathcal{G}$.

Proof: Supp $(O^{\mathcal{G}}) = supp\,(O^{\mathcal{G}}) i \in I = \emptyset.\, O^{\mathcal{G}} \in H \Rightarrow O \in H.\; Supp\,(-a) = supp(a)$,
$supp(a+b) \subseteq supp(a) \cup supp(b)$. If a,$b \in H$, then $-a \; and \; a + b \in H$
$Therefore \; H$ is a subgroup of $\mathcal{G}$.

H is called the direct sum of the $\mathcal{G}_i$, written $H = \bigoplus_{i \in I} \mathcal{G}_i$
As $\mathcal{G}$ is abelian, H is a normal subgroup of $\mathcal{G}$.So the quotient $\mathcal{G}/H$ is a group.
An element of $\mathcal{G}/H$ is an equivalence class of elements of $\mathcal{G}$ under the equivalence relation $a \sim b$ if and only if$-b \in H$.
That means a, b are the same class except possibly at finitely many indices.

Convention: If it does not confuse we use the same letter for a structure as its domain e.g $\mathcal{G}, G$ and $\mathcal{M} \; for \; \mathcal{M}$. In $\mathcal{G}/H \;\; g_1 + H = g_2 + H$ if and only if$g_1 - g_2$ is finite, if and only if$g_1(i) = g_2(i)$ except for finitely many indices.

3. FORMAL LANGUAGES AND THEIR INTERPRETATIUON

We want to see new to build a formal first order language $L(\sigma)$ and the signature σ from a σ_ structure. We can use $L(\sigma)$ to describe σ_ structures .

Terms: We have an infinite supply of variables. We use letters $x, y, z, x_1, x_2, \ldots etc$ to represent variables. The set of terms of $L(\sigma)$ is defined recursively:

i. Every variable is a term.
ii. Every constant symbol of σ is a term.
iii. If f is a function symbol of σ, of arity n , and $t_1, \ldots, t_n$ is a term, then f ($t_1 \ldots t_n$) is a Term.

iv. Only things build from i,…,iii is finitely many steps are terms .

Example 3.1: $\sigma = < s \quad , f , \quad c >$

Function system .1 $\quad f^{n}$ symb.2 $\quad$ Constant symbol

$c,\ x,\ s(c),\ f(x,c),\ f(f(s,c)\ ,\ s(c)x$ are terms.

Note: often we use a binary function symbols like +, . and write $t_1 + t_2$ instead of $+(t_1, t_2)$Similarly we write x^{-1} instead of $i(x)$.

Terms are just strings of symbols without meaning. Note that terms of $L(\sigma)$ are called σ_terms or L_terms or,$(\sigma)\_terms$.

Interpretation of Terms

σ_terms can be interpreted as functions in $\sigma\_structures$.

Example 3.2: $R =< R; +, . , - , 0 , 1 >$

The term [$((x.x) + x) + 1$] can be interpreted as the polynomial functions $\mathbb{R} \to \mathbb{R}$

$x \to x^2 + x + 1$

Let $\mathcal{A}$ be a $\sigma\_Structure$, to$\sigma_\ term\ t(\bar{x})$ suing variable from $\bar{x} = (x_1, \dots, x_n)$, we associate a function $t^{\mathcal{A}}: A^n \to A$ as follows

i. If $t = x_i$ then $t^{\mathcal{A}}(a) = a_i$.

ii. If $t = c$ then $t^{\mathcal{A}} = c^{\mathcal{A}}$.

iii. If $t = f(t_1, \dots, t_n)$ then $t^{\mathcal{A}}(\bar{x})$ is given by composition:

$$t^{\mathcal{A}}(\bar{x}) = f^{\mathcal{A}}(t^{\mathcal{A}}(\bar{x}), \dots, t_n^{\mathcal{A}} \ (\bar{x}))$$

Formulas

We now define $L(\sigma) - formulas$ ($L - formulas$), which are interpreted as statements about the structure:

i. If t_1 and t_2 are $\sigma\_\ terms$, then $t_1 = t_2$ is an $L(\sigma)$ formula .

ii. If $t_1 , \dots, t_n$ are $\sigma\_\ terms$ and R is a relation symbol of arity n, then $R(t_1, \dots, t_n)$ is an $L(\sigma) - formula.$

iii. If $\emptyset , \psi$ are $L(\sigma) - formulas$, then $(\emptyset \wedge \psi)$ and $\neg\ \emptyset$ are $L(\sigma)\_formula.$

iv. If $\emptyset$ is an $L(\sigma) - formula$ and x is a variable, then $\exists\ x\ \emptyset$ as an $L(\sigma) - formula$.

v. Only something built from (i) – (iv) in finitely many steps is an $(\sigma) - formula.$

Formulas from (i) and (ii) are called **atomic formulas.** Those built from (i) , (ii) and (iii) are quantifiers – free formulas. We define the language $L(\sigma)$ to be the set of all$(\sigma) - formulas.$

$L(\sigma)$ is called a first – order **language** because we only quantify over elements, not over a set of elements.

Theorem 3.1 Let$\sigma_{ring} =< +,\ .;\ -, 0, 1\ >$ be the structure of the ring of integers considered as a σ_{ring}structure. Then every polynomial function in Z [x_1, …., x_n] is the interpretation I Z of a σ_{ring}term.

Proof 0,1, -1 are polynomials. So is x_i . If p and q are polynomials then p + q is a polynomial. pq is also a polynomial.

Now 0, 1 and -1 are the interpretations of the terms written in the same way.

Similarly variables x_i are terms.

Suppose p and q are polynomial functions and assume that there are terms τ and σ such that τ = p and σ = q

Then the terms (τ+ σ) and $\tau\sigma$ have interpretation in Z which are p +q and p q respectively. By induction every polynomial is the interpretation in Z of a σ_{ring}term.

Theorem 3.2: Let M ⊨Ts where Ts is the theory of the successor. Suppose a, b ∈M \ N. Then there is an automorphism π of M such that π (a) = b.

Proof: Assume that M = M_1 for some I, as every symbol is isomorphic to M_i for some i.

Since a, b ∈ M \ N, we have a = (i, n) and b = (j, m) for some i, j ∈ I and some n, m ∈ Z.
x ~y if and only if x = s (y) for some r ∈ Z.
If x ~y define π by π (x) = x for every x such that: a ~ b and π (i, r) = (i, r + m –n), and π (j, r) = (i, r)
Both cases show that π (a) = b and that π is an automorphism.

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REFERENCES

[1]. P. Rothmale, "Introduction to Model Theory", Algebra, Logic, Applications Series, Volume 15, 2000.
[2]. J. J Rotman, "The Theory of Groups, An Introduction", second Edition, Allyn and Bacon, Inc, 1973.
[3]. M.Drosle and R. Gobel, "Advances in Algebra and Model Theory", C R C 1998.
[4]. W. Felscher, "Lectures in Mathematical Logic", Algebra, Logic, Applications Series, Volume 1, 2000.