International Journal of Enhanced Research in Educational Development (IJERED), ISSN: 2320-8708
Vol. 2, Issue 3, May-June, 2014, pp: (39-49), Impact Factor: 1.125, Available online at: www.erpublications.com

On an Irreducibility Theorem of A. Cohn

Archana Malik
M.D.U Rohtak, Haryana India

A classical result of Cohn [See eg. Polya and Szego (1964)] states as follows:-

Theorem 1.1: If a prime p is expressed in the decimal system as

$$p = \sum_{k=0}^{n} a_k 10^k, 0 \le a_k \le 9$$

then the polynomial $\sum_{k=0}^{n} a_k x^k$ is irreolucible in Z[x]. In this paper, we shall give a generalization of Cohn's Theorem. We shall also prove a criterion to test the irreducibility of a wide class of polynomials.

Theorem 1.2 : Let $f(x) \in Z[x]$ be a polynomial of degree n with z with zeroes $\alpha_i, \alpha_2, \ldots, \alpha_n$. If there is an integer b for which f(b) is a prime, $f(b-1) \neq 0$ and $Re(\alpha_i) < b - \frac{1}{2}$ for $1 \le i \le n$, then f(x) is irreducible in Z[x].

Proof : Let f(x) be reducible in Z[x] that is $f(x) = g(x)h(x)$ where $g(x), h(x) \in Z[x]$ with degree (g(x)), degree (h(x)) ≥ 1. If α_j are the zeroes of g(x) then $Re(\alpha_j) < b - \frac{1}{2}$. Let degree g(x) be m. Let g(x) be factored over the complex field that is $g(x) = a_m(x - \beta_1)(x - \beta_2) \ldots (x - \beta_r)(x - \beta_{r+1}) \ldots (x - \beta_m)$, where $\beta_1, \beta_2 \ldots \beta_r$, are reals and $\beta_{r+1}, \beta_{r+2} \ldots \beta_m$ are complex number with non-zero imaginary part. Further, complex roots occur in pairs so that

$$g(x) = a_m(x-\beta_1)(x-\beta_2)\ldots(x-\beta_r)(x-\gamma_{r+1}-i\gamma_{r+2})\ldots(x-\gamma_{r+1}+i\gamma_{r+2}) \ldots(x-\gamma_{m-1}-i\gamma_m)(x-\gamma_{m-1}+i\gamma_m)$$

Then
$$g\left(x+b-\frac{1}{2}\right) = a_m\left(x+b-\frac{1}{2}-\beta_1\right)\left(x+b-\frac{1}{2}-\beta_2\right)\ldots\left(x+b-\frac{1}{2}-\beta_r\right) \left(x+b-\frac{1}{2}-\gamma_{r+1}-i\gamma_{r+2}\right)\left(x+b-\frac{1}{2}-\gamma_{r+1}+i\gamma_{r+2}\right)\ldots\left(x+b-\frac{1}{2}-\gamma_{m-1}-i\gamma_m\right) \left(x+b-\frac{1}{2}-\gamma_{m-1}+i\gamma_m\right)$$

$$= a_m(x+\delta_1)(x+\delta_2)\ldots(x+\delta_r)(x+\delta_{r+1}-i\gamma_{r+2})(x+\delta_{r+1}+i\gamma_{r+2}) \ldots(x+\delta_{m-1}-i\gamma_m)(x+\delta_{m-1}+i\gamma_m) \quad \ldots(1.2.1)$$

where $\delta_j = b - \frac{1}{2} - \beta_j$ for j = 1, 2, ..., r

and $\delta_j = b - \frac{1}{2} - \gamma_j$ for j = r + 1, r + 3, ..., m – 1

$$= a_m(x+\delta_1)(x+\delta_2)\ldots(x+\delta_r)\left((x+\delta_{r+1})^2+\gamma_{r+2}^2\right)\ldots\left((x+\delta_{m-1})^2+\gamma_m^2\right) \quad \ldots(1.2.1)$$

Since β's are one of α's so

$$Re(\beta_j) < b - \frac{1}{2} \text{ for } j = 1, 2, \ldots, m.$$

$\therefore \delta_1 > 0$ for all i occuring in question. So each term occurring in (1.2.1) is positive except possibly for a_m. So if

$$g'(x) = (x+\delta_1)\ldots(x+\delta_r)\left(x^2+2x\delta_{r+1}+\left(\delta_{r+1}^2+\gamma_{r+2}^2\right)\right)\left(x^2+2x\delta_{m-1}+\left(\delta_{m-1}^2+\gamma_m^2\right)\right).$$

Then each coefficient in $g'(x)$ is positive and no term is missing in $g'(x)$. So all the terms in $g\left(x+b-\frac{1}{2}\right)$ have the same sign and no term is missing.

Now, the coefficients of $g\left(x+b-\frac{1}{2}\right)$ are strictly alternating.

Thus

$$\left|g\left(b-\frac{1}{2}-t\right)\right| < \left|g\left(b-\frac{1}{2}+t\right)\right| \qquad \text{for any } t > 0.$$

If we take $t = \frac{1}{2}$, then

$$|g(b-1)| < |g(b)|$$

from the given conditions for f(x), these conditions also hold for g(x) that is

$$g(b-1) \in Z[x] \text{ and } g(b-1) \neq 0,$$

it follows that $|g(b-1)| \geq 1$ and $|g(b)| \geq 2$.

Similarly we get $|h(b)| \geq 2$ which gives us a contradiction to the assumption that f(b) = h(b) g(b) is a prime. So our supposition that f(x) is reducible, is wrong. Hence f(x) is irreducible in Z[x].

Theorem 1.3 : Let $f(x) = \sum_{k=0}^{n} a_k x^k, \in Z[x]$ be a polynomial with $a_n > 0$, $a_{n-1} \geq 0$ and $a_{n-2} \geq 0$.

Let m = maxi. $\left\{\frac{|a_k|}{a_n}\right\}$ for $0 \leq k \leq n-2$.

$$r_1 = \frac{1+\sqrt{4m+1}}{2} \quad r_2 = \left[\frac{s+\sqrt{s^2-4}}{54}\right]^{\frac{1}{3}} + \left[\frac{s-\sqrt{s^2-4}}{54}\right]^{\frac{1}{3}} + \frac{1}{3}$$

where s = 27m+2. If there is an integer

$$b > \text{maxi}\left\{\frac{r_1}{\sqrt{2}}, r_2\right\} + \frac{1}{2}$$

for which f(b) is prime and $f(b-1) \neq 0$ then f(x) is irreducible in Z[x].

Proof : Firstly let us consider the set

$$A = \left[Z \in \mathbb{C} : \text{Re}(z) \leq \text{maxi}\left\{\frac{r_1}{\sqrt{2}}, r_2\right\}\right].$$

In this theorem, we first prove that all zeroes of f lie in A by proving that $|f(z)| > 0$ for $z \in A^c$, the complement of A and then we will apply theorem 1.2 and get the required result.

Now since $r_1 = \frac{1+\sqrt{4m+1}}{2} \geq 1$ since $m \geq 0$.

$\Rightarrow \quad r_1 \geq 1$

and $\quad r_1 = \frac{1}{2} + \frac{\sqrt{4m+1}}{2}$

$\Rightarrow \quad r_1 - \frac{1}{2} = \frac{\sqrt{4m+1}}{2}$.

Squaring both sides, we get

$$\left(r_1 - \frac{1}{2}\right)^2 = \frac{4m+1}{4}$$

$\Rightarrow \quad r_1^2 + \frac{1}{4} - r_1 = \frac{4m+1}{4}$

$\Rightarrow \quad r_1^2 - r_1 + \frac{1}{4} = m + \frac{1}{4}$

$\Rightarrow \quad r_1^2 - r_1 = m$

$\Rightarrow \quad r_1^2 - r_1 - m = 0$.

Thus r_1 is a positive zero of $x^2 - x - m$ and $x^2 - x - m > 0$ for $x > r_1$

$$r_2 = \left[\frac{s+\sqrt{s^2-4}}{54}\right]^{\frac{1}{3}} + \left[\frac{s-\sqrt{s^2-4}}{54}\right]^{\frac{1}{3}} + \frac{1}{3}$$

where $s = 27m + 2$ and $m = \max. \left\{\frac{|a_k|}{a_n}\right\}$ for $0 \le k \le n - 2$.

Since $s = 27m + 2$, $m \ge 0$

$\Rightarrow \quad s \ge 2$

Set $x = \left[\frac{s+\sqrt{s^2-4}}{54}\right]^{\frac{1}{3}}$. Then $x \ge \frac{1}{3}$

Now
$$\left[\frac{s-\sqrt{s^2-4}}{54}\right]^{\frac{1}{3}} = \left[\frac{s-\sqrt{s^2-4}}{54} \times \frac{s+\sqrt{s^2-4}}{s+\sqrt{s^2-4}}\right]^{\frac{1}{3}}$$

$$= \left[\frac{s^2-(s^2-4)}{54\left(s+\sqrt{s^2-4}\right)}\right]^{\frac{1}{3}}$$

$$= \left[\frac{4}{54\left(s+\sqrt{s^2-4}\right)}\right]^{\frac{1}{3}}$$

$$= \left[\frac{2}{27\left(s+\sqrt{s^2-4}\right)}\right]^{\frac{1}{3}}$$

$$= \left[\frac{54}{729\left(s+\sqrt{s^2-4}\right)}\right]^{\frac{1}{3}}$$

$$= \frac{1}{9}\left[\frac{54}{s+\sqrt{s^2-4}}\right]^{\frac{1}{3}}$$

$$= \frac{1}{9x}$$

$$r_2 = x + \frac{1}{9x} + \frac{1}{3}$$

Let $f(y) = y + \frac{1}{9y} + \frac{1}{3}$ where $y \geq \frac{1}{3}$.

Then $f'(y) = 1 - \frac{1}{9y^2} \geq 0$ for $y \geq \frac{1}{3}$.

$\therefore$ f(y) is an increasing function of y.

for $y = \frac{1}{3}$, $f(y) = \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 1$

$\therefore$ $r_2 \geq 1$ for $x = \frac{1}{3}$

i.e. $r_2 \geq 1$ for $s = 27m + 2$.

Let A^c be partitioned into two sets $A^c \cap B$ and $A^c \cap B^c$ where

$$B = \{Z \in \mathbb{C} : Re(z) < 0 \text{ or } |z| \leq r_1\}$$

Let $z \in A^c \cap B^c$. Then $(|z|^2 - |z| - m) > 0$ since $|z| > r_1$.

Now

$$Re\left(\frac{1}{z}\right) = Re\left(\frac{\bar{z}}{z\,\bar{z}}\right)$$

$$= Re\left(\frac{\bar{z}}{|z|^2}\right)$$

$$= \frac{1}{|z|^2} Re(\bar{z})$$

$$\Rightarrow Re\left(\frac{1}{z}\right) > 0$$

and

$$\left|\frac{f(z)}{z^n}\right| = \left|\frac{a_0}{z^n} + \frac{a_1 z}{z^n} + \ldots.. + \frac{a_{n-1} z^{n-1}}{z^n} + \frac{a_n z^n}{z^n}\right|$$

$$= \left|\frac{a_0}{z^n} + \frac{a_1}{z^{1-n}} + \ldots.. + \frac{a_{n-1}}{z} + a_n\right|$$

$$= \left|a_n + \frac{a_{n-1}}{z} + \sum_{k=2}^{n} \frac{|a_{n-k}|}{|z|^k}\right|$$

$$\leq \left|a_n + \frac{a_{n-1}}{z}\right| + \left|\sum_{k=2}^{n} \frac{a_{n-k}}{|z|^k}\right|$$

$$\geq \left|a_n + \frac{a_{n-1}}{z}\right| - \sum_{k=2}^{n} \frac{|a_{n-k}|}{|z|^k}$$

$$\geq Re\left(a_n + \frac{a_{n-1}}{z}\right) - \sum_{k=2}^{n} \frac{ma_n}{|z|^k} \quad \text{(Since } |a_k| < ma_n \text{ for } k = 1, 2, \ldots, n-2)$$

$$> Re\left(a_n + \frac{a_{n-1}}{z}\right) - \sum_{k=2}^{\infty} \frac{ma_n}{|z|^k}$$

$$= Re\left(a_n + \frac{a_{n-1}}{z}\right) - (ma_n) \sum_{k=2}^{\infty} \frac{1}{|z|^k}$$

$$= Re\left(a_n + \frac{a_{n-1}}{z}\right) - ma_n \left(\frac{1}{|z|^2} + \frac{1}{|z|^3} + \ldots.. + \infty\right)$$

$$= \mathrm{Re}(a_n + a_{n-1}) - ma_n \frac{\frac{1}{|z|^2}}{1 - \frac{1}{|z|}}$$

$$= \mathrm{Re}\left(a_n + \frac{a_{n-1}}{z}\right) - \frac{ma_n}{|z|^2 - |z|}$$

$$= a_n + a_{n-1}\,\mathrm{Re}\left(\frac{1}{z}\right) - \frac{ma_n}{|z|^2 - |z|}$$

$$\geq a_n - \frac{ma_n}{|z|^2 - |z|}\left(\because \mathrm{Re}\left(\frac{1}{z}\right) > 0\right)$$

$$= \frac{a_n\left(|z|^2 - |z| - m\right)}{|z|^2 - |z|} > 0 \text{ since } z \in A^c \cap B^c$$

$\Rightarrow$ $|f(z)| \neq 0$ for $z \in A^c \cap B^c$

$\Rightarrow$ $f(z) \neq 0$ for $z \in A^c \cap B^c$

In second case, let $z \in A^c \cap B$ then if $z \in A^c$.

$\Rightarrow$ $\mathrm{Re}(z) >$ maxi. $\left\{\frac{r_1}{\sqrt{2}}, r_2\right\} > 0$ and if $z \in B \Rightarrow \mathrm{Re}(z) < 0$ or $|z| \leq r_1$. But $\mathrm{Re}(z) < 0$ not possible so $|z| \leq r_1$. Since any complex number z can be written as

$$z = r(\cos\theta + i\sin\theta) \text{ where } -\pi < \cos\theta \leq \pi$$

Now $z \in B \Rightarrow \mathrm{Re}(z) \leq r_1$, i.e., $r \leq r_1$

$z \in A^c \Rightarrow \mathrm{Re}(z) > \frac{r_1}{\sqrt{2}}$ i.e., $r > \frac{r_1}{2}$

$\therefore$ $r_1\cos\theta > r\cos\theta > \frac{r_1}{\sqrt{2}}$

$\Rightarrow$ $\cos\theta > \frac{1}{\sqrt{2}}$

$\Rightarrow$ $|\theta| < \frac{\pi}{4}$.

$\Rightarrow$ $|\arg. z| < \frac{\pi}{4}$

Since $\mathrm{Re}\left(\frac{1}{z}\right) = \frac{1}{|z|^2}$ $\mathrm{Re}(z)$

Replacing z by z^2, we get

$$\mathrm{Re}\left(\frac{1}{z^2}\right) = \frac{1}{|z|^4}\mathrm{Re}(z^2) = \frac{r^2\cos 2\theta}{r^4} = \frac{\cos 2\theta}{r^2} > 0$$

since $|\theta| < \frac{\pi}{4}$.

Now $r_2 = \left[\frac{s + \sqrt{s^2 - 4}}{54}\right]^{\frac{1}{3}} + \left[\frac{s - \sqrt{s^2 - 4}}{54}\right]^{\frac{1}{3}} + \frac{1}{3}$

$\Rightarrow \qquad r_2 - \frac{1}{3} = a + b$ where $a = \left[\frac{s+\sqrt{s^2-4}}{54}\right]^{\frac{1}{3}}$ and $b = \left[\frac{s-\sqrt{s^2-4}}{54}\right]^{\frac{1}{3}}$.

Cubing both sides, we get

$$\left[r_2 - \frac{1}{3}\right]^3 = (a+b)^3$$

$$\Rightarrow \qquad r_2^3 - \frac{1}{27} - 3r_2.\frac{1}{3}\left(r_2 - \frac{1}{3}\right) = a^3 + b^3 + 3a^2b + 3ab^2$$

$$\Rightarrow \qquad r_2^3 - \frac{1}{27} - r_2^2 + \frac{r_2}{3} = a^3 + b^3 + 3ab(a+b)$$

$$\Rightarrow \qquad r_2^3 - \frac{1}{27} - r_2^2 + \frac{r_2}{3} = \frac{s+\sqrt{s^2-4}}{54} + \frac{s-\sqrt{s^2-4}}{54} + 3\left(\frac{s+\sqrt{s^2-4}}{54}\right)^{\frac{1}{3}}$$

$$\left(\frac{s-\sqrt{s^2-4}}{54}\right)^{\frac{1}{3}}\left(r_2 - \frac{1}{3}\right)$$

$$\Rightarrow \qquad r_2^3 - \frac{1}{27} - r_2^2 + \frac{r_2}{3} = \frac{s}{27} + 3\left(\frac{s^2-(s^2-4)}{54\times 54}\right)^{\frac{1}{3}}\left(r_2 - \frac{1}{3}\right)$$

$$= \frac{s}{27} + 3\left(\frac{4}{54\times 54}\right)^{\frac{1}{3}}\left(r_2 - \frac{1}{3}\right)$$

$$= \frac{s}{27} + \frac{3}{9}\left(r_2 - \frac{1}{3}\right)$$

$$= \frac{s}{27} + \frac{r_2}{3} - \frac{1}{9}$$

$$r_2^3 - \frac{1}{27} - r_2^2 + \frac{r_2}{3} = \frac{s}{27} + \frac{r_2}{3} - \frac{1}{9}$$

$$\Rightarrow \qquad r_2^3 - r_2^2 = \frac{s}{27} - \frac{1}{9} + \frac{1}{27}$$

$$= \frac{s-3+1}{27}$$

$$= \frac{s-2}{27} = m. \text{ (given)}$$

$$\Rightarrow \qquad r_2^3 - r_2^2 - m = 0.$$

Thus r_2 is a root of the equation

$x^3 - x^2 - m = 0$ that is r_2 is a zero of $x^3 - x^2 - m$.

$$\left|\frac{f(z)}{z^n}\right| = \left|\frac{a_0}{z^n} + \frac{a_1}{z^{1-n}} + \ldots + \frac{a_{n-2}}{z^2} + \frac{a_{n-1}}{z} + a_n\right|$$

$$= \left|\frac{a_{n-2}}{z^2} + \frac{a_{n-1}}{z} + a_n + \sum_{k=3}^{n}\frac{a_n - k}{(z)^k}\right|$$

$$\geq \left|a_n + \frac{a_{n-1}}{z} + \frac{a_{n-2}}{z^2}\right| - \sum_{k=3}^{n}\frac{|a_{n-k}|}{|z|^k}$$

$$\geq \left|a_n + \frac{a_{n-1}}{z} + \frac{a_{n-2}}{z^2}\right| - \sum_{k=3}^{n} \frac{ma_n}{|z|^k}$$

$$> \operatorname{Re}\left(a_n + \frac{a_{n-1}}{z} + \frac{a_{n-2}}{z^2}\right) - \sum_{k=3}^{\infty} \frac{ma_n}{|z|^k}$$

$$= \operatorname{Re}(a_n) + a_{n-1} \operatorname{Re}\left(\frac{1}{z}\right) + a_{n-2} \operatorname{Re}\left(\frac{1}{z^2}\right) - ma_n \cdot \left(\frac{1}{|z|^3} + \frac{1}{|z|^4} + \ldots\right)$$

$$= a_n + a_{n-1} \operatorname{Re}\left(\frac{1}{z}\right) + a_{n-2} \operatorname{Re}\left(\frac{1}{z^2}\right) - \frac{ma_n}{|z|^3 - |z|^2}$$

$$\geq a_n - \frac{ma_n}{|z|^3 - |z|^2}$$

$$= \frac{a_n\left(|z|^3 - |z|^2 - m\right)}{|z|^3 - |z|^2} > 0 \text{ since } |z| > r_2$$

which is a positive zero of $x^3 - x^2$ 0 m and $r_2 \geq 1$ gives $\left|\frac{f(z)}{z^n}\right| > 0$

$\Rightarrow \quad f(z) \neq 0 \quad$ for $\quad z \in A^c \cap B$.

Thus we have shown that all zeros of f lie in A by proving that $|f(z)| > 0$ for $z \in A^c$. Hence all the hypothesis of theorem 1.2 are satisfied for integer b which gives f(x) is irreducible in Z[x].

Remark 1.3.1 : We can note that theorem 1.3 does not depend upon the actual value of a_{n-1}.

Remark 1.3.2 : In the part of the proof of theorem 1.3 where $z \in A^c \cap B^c$, if we take $z \in B^c$ only in place $A^c \cap B^c$, we get that the result is true for this z i.e., $|f(z)| > 0$ for any $z \in B^c$.

Corollary 1.3.1 : Let $b \geq 2$ be an integer and let B = 1 if b = 0 and

$$B = \left[\frac{(b-1)\left(2b-1-\sqrt{2}\right)}{2}\right] \text{ is } b \geq 3$$

where brackets are greatest integer function. Also, let a prime p be expressed as

$$p = \sum_{k=0}^{n} a_k b^k$$

where $a_n > 0$, $a_{n-1} \geq 0$, $a_{n-2} \geq 0$,

and $\frac{|a_k|}{a_n \leq B}$ for $0 \leq k \leq n-2$

and define $f(x) = \sum_{k=0}^{n} a_k x^k$.

If f(b – 1) = 0 then f(x) is irreducible in Z[x].

Proof : Because all the hypothesis of Theorem 1.3 except one are satisfied so to apply Theorem 1.3 we have only to show that

$$b > \text{maxi.} \left\{\frac{r_1}{\sqrt{2}}, r_2\right\} + \frac{1}{2}.$$

Let r_1^* and r_2^* be the positive zeros of $x^2 - x - B$ and $x^3 - x^2 - B$ respectively.

Let us denote $g(x) = x^2 - x - B$. Now since $m \leq B$, we have

$$r_1^* = \frac{1+\sqrt{4B+1}}{2} \geq \frac{1+\sqrt{4m+1}}{2} = r_1$$

$\Rightarrow \quad r_1^* \geq r_1$.

We have already proved that

$$f(y) = y + \frac{1}{9y} + \frac{1}{3} \text{ is an increasing function of y.}$$

Also r_2^* is a zero of $x^3 - x^2 - B$.

and $$r_2^* = \left[\frac{s+\sqrt{s^2-4}}{54}\right]^{\frac{1}{3}} + \left[\frac{s-\sqrt{s^2-4}}{54}\right]^{\frac{1}{3}} + \frac{1}{3}$$

where s = 27B + 2.

Now since $m \le B$.

$s^* = 27B + 2 \ge 27m + 2 = s$

$\Rightarrow \quad s^* \ge s$

$$\therefore \quad \left[\frac{s+\sqrt{s^2-4}}{54}\right]^{\frac{1}{3}} \le \left[\frac{s^*+\sqrt{s^{*2}-4}}{54}\right]^{\frac{1}{3}}.$$

Thus we have to show that b > maxi. $\left\{\frac{r_1^*}{\sqrt{2}}, r_2^*\right\} + \frac{1}{2}$.

$\because \quad r_1^*$ is a zero of $x^2 - x - B$.

$\therefore \quad \frac{r_1^*}{\sqrt{2}}$ is a zero of $\left(\sqrt{2}x\right)^2 - \left(\sqrt{2}x\right) - B = 2x^2 - \sqrt{2}x - B = h(x)$.

Now $h\left(b - \frac{1}{2}\right) = 2\left(b - \frac{1}{2}\right)^2 - \sqrt{2}\left(b - \frac{1}{2}\right) - B$.

$$= \frac{2}{4}(2b-1)^2 - \frac{\sqrt{2}}{2}(2b-1) - B$$

$$= \frac{2b-1}{2}\left(2b-1-\sqrt{2}\right) - B > 0$$ for $b \ge 3$ by definition of B. Also if b = 2, $h\left(b - \frac{1}{2}\right) > 0$, since B = 1.

Thus $h\left(b - \frac{1}{2}\right) > 0$ for $b \ge 2$.

$$g\left(b - \frac{1}{2}\right) = \left(b - \frac{1}{2}\right)^3 - \left(b - \frac{1}{2}\right)^2 - B$$

$$= \left(b - \frac{1}{2}\right)^2\left[\left(b - \frac{1}{2}\right) - 1\right] - B$$

$$= \left(b - \frac{1}{2}\right)^2\left(b - \frac{3}{2}\right) - B$$

For b = 2.

$$g\left(b - \frac{1}{2}\right) = \left(b - \frac{1}{2}\right)^2\left(b - \frac{3}{2}\right) - B$$

$$= \left(2 - \frac{1}{2}\right)^2\left(2 - \frac{3}{2}\right) - 1$$

$$= \frac{9}{4}\cdot\frac{1}{2} - 1$$

$$= \frac{9}{8} - 1 = \frac{1}{8} > 0.$$

For b = 3, B = $\left[\frac{5\left(5-\sqrt{2}\right)}{2}\right]$

$< [5 \times 1.8]$

$\Rightarrow$ B = 8

$$g\left(b-\frac{1}{2}\right)=\left(3-\frac{1}{2}\right)^2\left(3-\frac{3}{2}\right)-8$$

$$=\frac{75}{8}-8$$

$$=\frac{11}{8}>0$$

Now let b > 3, Then b ≥ 4 since b is an integer.

$$g\left(b-\frac{1}{2}\right)=\left(b-\frac{1}{2}\right)^2\left(b-\frac{3}{2}\right)-B$$

$$=\left(b-\frac{1}{2}\right)^2\left(b-\frac{3}{2}\right)-\frac{(2b-1)\left(2b-1-\sqrt{2}\right)}{2}$$

$$>\left(b-\frac{1}{2}\right)^2\left(b-\frac{3}{2}\right)-\left(b-\frac{1}{2}\right)\left(2b-1-\sqrt{2}\right)$$

$$=\left(b-\frac{1}{2}\right)\left[\left(b-\frac{1}{2}\right)-\left(b-\frac{3}{2}\right)\left(2b-1-\sqrt{2}\right)\right]$$

$$=\left(b-\frac{1}{2}\right)\left(b^2-2b+\frac{3}{4}-2b+1+\sqrt{2}\right)$$

$$=\left(b-\frac{1}{2}\right)\left(b^2-4b+\frac{7}{4}+\sqrt{2}\right)$$

$$=\left(b-\frac{1}{2}\right)\left(b(b-4)+\frac{7}{4}+\sqrt{2}\right)$$

> 0 for $b \geq 4$.

Thus $g\left(b-\frac{1}{2}\right) > 0$ for $b \geq 2$.

Hence, we then have

$$b-\frac{1}{2} > \text{maxi.}\left\{\frac{r_1^*}{\sqrt{2}}.r_2^*\right\}$$

$$b-\frac{1}{2} > \text{maxi.}\left\{\frac{r_1}{\sqrt{2}}.r_2\right\} \text{ since } r_1^* \geq r_1 \text{ and } r_2^* \geq r_2.$$

Thus all the hypothesis of Theorem 1.3 are satisfied. Hence by applying that theorem we get f(x) is irreducible in Z[x].

Corollary 1.3.2 : If a prime p is expressed in the number system with base b ≥ 2 as $p=\sum_{k=0}^{n} a_k b^k$, $0 \leq a_k \leq b-1$ then the polynomial $\sum_{k=0}^{n} a_k x^k = f(x)$ is irreducible in Z[x].

Proof : In order to prove it, we shall prove that f(x) satisfies all the conditions of Corollary 1.3.1, that is

$f(b-1) \neq 0$.

and $\frac{|a_k|}{a_n} \leq B$ for $0 \leq k \leq n-2$.

where B =1 if b = 2

and B = $\left[\frac{(2b-1)\left(2b-1-\sqrt{2}\right)}{2}\right]$ if $b \geq 3$

where brackets are greatest integer function.

Now $f(x) = \sum_{k=0}^{n} a_k x^k$, $a_n \neq 0$, $0 \leq a_k \leq b-1$

$$f(b-1) = \sum_{k=0}^{n} a_k (b-1)^k$$

$\geq a_n(b-1)^n > 0 \ \forall \ b \geq 2$ since all a_n's are positive and $(b-1) \geq 1$.

For b = 2, B = 1

Since $0 \leq a_n \leq 1$ and $a_n \neq 0$

$\Rightarrow \quad a_n = 1.$

Also $0 \leq a_k \leq 1 = B$

$\Rightarrow \quad a_k \leq B$

$\Rightarrow \quad B \geq a_k$

So $\quad \frac{|a_k|}{a_n} \leq B$ for b = 2

Let b ≥ 3. Then

$$B = \left[\frac{(2b-1)\left(2b-1-\sqrt{2}\right)}{2}\right]$$

Now $\quad \frac{(2b-1)\left(2b-1-\sqrt{2}\right)}{2} \geq b-1$

$\Leftrightarrow \quad (2b-1)\left(2b-1-\sqrt{2}\right) \geq 2b-2$

$\Leftrightarrow \quad 2b-1-\sqrt{2} \geq \frac{2b-2}{2b-1}$

which holds if

$2b-1-\sqrt{2} \geq 1.$

or if $\quad 2b \geq 2+\sqrt{2}$

which holds if $2b \geq 4$

$\Rightarrow \quad b \geq 2$

Thus $B \geq b-1$ only if $b \geq 2$.

But in case b = 2, we have proved that $\frac{|a_k|}{a_n} < B$ so we take $b \geq 3$. Thus if $b \geq 3$ then

$$\frac{|a_k|}{a_n} \leq |a_k| \leq b-1 \leq B$$

$\Rightarrow \quad \frac{|a_k|}{a_n} \leq B$

so $\quad \frac{|a_k|}{a_n} \leq B$ for $b \geq 2$.

Hence, all the conditions of for 1.3.1. are satisfied so by applying for 1.3.1 we get f(x) is irreducible in Z[x].

References

[1]. Hardy, G.M., and W.Right, E.M. (1938(a)), An introduction to the theory of numbers.
[2]. John Brillhart, Michael Filaseta and Andram Odlyzko (1981), On an irreducibility theorem of A, Cohn, Can. J. Math. Vol. XXXIII, No. 5, pp. 1055-1059.
[3]. Michael Filaseta, (1982), A further Generalization of an irreducibility theorem of A. Cohn., Can. J. Math. Vol. XXXIV, No. 6, pp.1390-1395.
[4]. Michael Filaseta, (1988), Irreducibility criteria for polynomials with non-negative coefficients, Can. J. Math. Vol. XL, No. 2, pp.339.351.