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# Commutative Rule involving the Laplace Operator

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**Abstract**: Commutation rules between the Laplace operator and other basic operators (include the divergence, the curl, and the gradient operator) are established. These rules were seldom noticed in the past. However, it is shown that they lead to an alternative derivation of the electromagnetic fields due to an arbitrary current distribution in a homogeneous environment without referring to the vector potential concept. The derived mathematical result can still be linked to show the physical insight of the original problem.

**Keywords:** vector analysis, operator, potential, commutative, Laplace.

## INTRODUCTION

In the course of electromagnetic waves [1-4], both scalar and vector distributions are frequently encountered. The gradient, the divergence, and the curl are three basic vector operators. The gradient operates on the scalar function and the other two operate on the vector functions. The Laplace operator operates on both scalar and vector functions. In vector analysis, many vector analysis formulas are derived. However, the commutative properties involving the Laplace operator are seldom noticed.

In a scalar system, we know that the impulse response of a system is the derivative of the unit-step response. For a unit-step response of f(x), we write L (f(x)) =u(x), where u(x) is the unit-step function and L is the operator. Taking the derivative operation on both sides of the equation, we get $\mathrm{L}[\mathrm{df}(\mathrm{x})/\mathrm{dx}] = \delta(\mathrm{x})$, if the L-operator and the derivative operator are commutative to each other. In a vector system, the curl, the divergence, and the gradient involve a lot of derivative operations. We expect that commutative rules should also be existed in a vector system. The vector system could be a static E-field, a static H-field or an electromagnetic wave system.

## DERIVATION

In electromagnetic waves, vector operator L exists such that $\mathrm{L}(\vec{\mathrm{E}})$ is related to $\vec{\mathrm{J}}$ and $\nabla\cdot\vec{\mathrm{J}}$. The operator L involves the Laplace operator. In this paper, vector commutative properties involving the Laplace operator will be investigated. We found that the commutative property is useful to get an analytic solution for $\vec{\mathrm{E}}$ or $\vec{\mathrm{H}}$ without referring to the conventional vector potential approach. Many commutative properties can be derived from the following vector identity formulas

$$\nabla\times\nabla\times\vec{\mathrm{A}} = \nabla(\nabla\cdot\vec{\mathrm{A}}) - \nabla^2\vec{\mathrm{A}}$$

(A) Let $\vec{\mathrm{A}} = \nabla\psi$

$$\nabla\times\nabla\times(\nabla\psi) = \nabla(\nabla\cdot\nabla\psi) - \nabla^2(\nabla\psi)$$

$$0 = \nabla(\nabla^2\psi) - \nabla^2(\nabla\psi)$$

$$\nabla(\nabla^2\psi) = \nabla^2(\nabla\psi)$$

Therefore, the Laplace and the gradient operators are commutative to each other.

(B) Taking divergence operation to both sides of the vector identity formula, we get

$$0 = \nabla\cdot\left[\nabla(\nabla\cdot\vec{\mathrm{A}})\right] - \nabla\cdot(\nabla^2\vec{\mathrm{A}})$$

$$\nabla^2\left(\nabla\cdot\vec{\mathrm{A}}\right) = \nabla\cdot\left(\nabla^2\vec{\mathrm{A}}\right)$$

Therefore, the Laplace and the divergence operators are commutative to each other.

(C) Let $\vec{\mathrm{A}} = \nabla\times\vec{\mathrm{F}}$

The left-hand side of the vector identity becomes:

$$\nabla\times\nabla\times\left(\nabla\times\vec{\mathrm{F}}\right)=\nabla\times\left[\nabla\left(\nabla\cdot\vec{\mathrm{F}}\right)-\nabla^2\vec{\mathrm{F}}\right]=-\nabla\times\left(\nabla^2\vec{\mathrm{F}}\right)$$

The right-hand side of the vector identity becomes:

$$\nabla\left(\nabla\cdot\nabla\times\vec{\mathrm{F}}\right)-\nabla^2\left(\nabla\times\vec{\mathrm{F}}\right)=-\nabla^2\left(\nabla\times\vec{\mathrm{F}}\right)$$

$$\nabla\times\left(\nabla^2\vec{\mathrm{F}}\right)=\nabla^2\left(\nabla\times\vec{\mathrm{F}}\right)$$

Therefore, the Laplace and the curl operators are commutative to each other.

(D) Replacing ψ in formula (A) by $\nabla\cdot\vec{\mathrm{F}}$, then use the result of (B),

$$\nabla\left[\nabla^2(\nabla\cdot\vec{\mathrm{F}})\right]=\nabla^2\left[\nabla(\nabla\cdot\vec{\mathrm{F}})\right]$$

$$\nabla\nabla\cdot\left(\nabla^2\vec{\mathrm{F}}\right)=\nabla^2\left(\nabla\nabla\cdot\vec{\mathrm{F}}\right)$$

(E) Let $\vec{\mathrm{F}}=\nabla\times\vec{\mathrm{G}}$ in formula (C),

$$\nabla\times\left[\nabla^2(\nabla\times\vec{\mathrm{G}})\right]=\nabla^2(\nabla\times\nabla\times\vec{\mathrm{G}})$$

$$\nabla\times\nabla\times(\nabla^2\vec{\mathrm{G}})=\nabla^2(\nabla\times\nabla\times\vec{\mathrm{G}})$$

The last two formulas state that the Laplace operator is also commutative with respect to the operator $\nabla(\nabla\cdot)$ and the operator $\nabla\times(\nabla\times)$.If we denote $\nabla\times(\nabla\times)\times\cdots\cdots\times(\nabla\times)=(\nabla\times)^{\mathrm{n}}$ for n-times of curl operation, it is easy to show that $\nabla^2(\nabla\times)^{\mathrm{n}}=(\nabla\times)^{\mathrm{n}}(\nabla^2)$. We can also show that $\nabla^2[\nabla(\nabla\cdot)]^{\mathrm{n}}=[\nabla(\nabla\cdot)]^{\mathrm{n}}(\nabla^2)$ for n-times operation of $\nabla(\nabla\cdot)$.

In a simple media with the constitutive parameters of $\varepsilon_0,\mu$, the impressed source $\vec{\mathrm{J}}^{\mathrm{i}}$ generates fields $\vec{\mathrm{E}}$ and $\vec{\mathrm{H}}$ according to the Maxwell's equation:

$$\nabla\times\vec{\mathrm{E}}=-\hat{\mathrm{z}}\vec{\mathrm{H}}$$

$$\nabla\times\vec{\mathrm{H}}=\vec{\mathrm{J}}^{\mathrm{i}}+\hat{\mathrm{y}}\vec{\mathrm{E}}$$

, where $\hat{\mathrm{z}}=\mathrm{j}\omega\mu$ , and $\hat{\mathrm{y}}=\mathrm{j}\omega\varepsilon_0$.The $\vec{E}$-field satisfies:

$$\nabla\times\nabla\times\vec{E}+\hat{z}\hat{y}\vec{E}=-\hat{z}\vec{J}^i$$

Since, $\nabla\cdot\vec{\mathrm{E}}=(\nabla\cdot\vec{\mathrm{D}})/\varepsilon_0=\rho/\varepsilon_0$, wher $\rho=-\nabla\cdot\vec{\mathrm{J}}^{\mathrm{i}}/(j\omega)$, and $\mathrm{k}^2=-\hat{\mathrm{z}}\hat{\mathrm{y}}$.

We get, $\nabla^2\vec{\mathrm{E}}+\mathrm{k}^2\vec{\mathrm{E}}=\hat{\mathrm{z}}\vec{\mathrm{J}}^{\mathrm{i}}-\frac{1}{\hat{y}}\left[\nabla\left(\nabla\cdot\vec{\mathrm{J}}^{\mathrm{i}}\right)\right]$

With the property of superposition, $\vec{\mathrm{E}}$ can be expressed as $\vec{\mathrm{E}}=\vec{\mathrm{E}}_1+\vec{\mathrm{E}}_2$ , where,

$$\nabla^2\vec{\mathrm{E}}_1+\mathrm{k}^2\vec{\mathrm{E}}_1=\hat{\mathrm{z}}\vec{\mathrm{J}}^{\mathrm{i}}\text{ , and}$$

$$\nabla^2\vec{\mathrm{E}}_2+\mathrm{k}^2\vec{\mathrm{E}}_2=-\frac{1}{\hat{y}}\left[\nabla\left(\nabla\cdot\vec{\mathrm{J}}^{\mathrm{i}}\right)\right]$$

The analytic form for $\vec{\mathrm{E}}_1$ is:

$$\vec{\mathrm{E}}_1=-\vec{a}\left[\frac{\hat{\mathrm{z}}}{4\pi}\iiint\frac{\mathrm{J}^{\mathrm{i}}(\vec{r'})e^{-jk\left|\vec{r}-\vec{r'}\right|}}{\left|\vec{r}-\vec{r'}\right|}d\tau'\right]$$

Where, $\vec{a}$ is a constant vector. The integral takes on the volume where the current exists. For arbitrarily located current distribution, it can be decomposed into three components in a rectangular coordinate system. The derivation of this equation will be discussed in the appedix.

$\vec{\mathrm{E}}_2$ can be expressed in terms of $\vec{\mathrm{E}}_1$, a simple proof is given below:

$$\text{Since, } \nabla^2\vec{E}_1 + k^2\vec{E}_1 = \hat{z}\vec{J}^i$$

$$\text{We get : } \nabla\nabla\cdot\left[\nabla^2\vec{E}_1 + k^2\vec{E}_1\right] = \nabla\nabla\cdot(\hat{z}\vec{J}^i)$$

By commutative property, the above equation can be written as :

$$\nabla^2\left[\nabla\nabla\cdot\vec{E}_1\right] + k^2[\nabla\nabla\cdot(\vec{E}_1)] = \nabla\nabla\cdot(\hat{z}\vec{J}^i)$$

$$\text{Therefore, } \vec{E}_2 = \frac{-\nabla\nabla\cdot(\vec{E}_1)}{\hat{z}\hat{y}}$$

Conventionally, the $\vec{E}$-field is derived with the potential approach. The potential approach gives its physical appeal. In our expression, one can easily show that $\vec{E}_1$ is the expression for the conventional vector potential multiplied by $-j\omega$, and $\vec{E}_2$ is the negative gradient of the conventional scalar potential. Therefore, the conventional vector and scalar potentials can easily be extracted out.

## Conclusion

In general, $\alpha\beta \neq \beta\alpha$ for two linear operators $\alpha$ and $\beta$ . But, if they are equal, $\alpha$ and $\beta$ are said to commute [5]. To the authors' best knowledge, the commutative properties between several useful operators are seldom noticed in the EM community. In this short note, we revisit those basic vector operators. Starting from the definition of the vector Laplace operator, we found that many commutative rules exist between the vector Laplace operator and the basic operators. The vector Laplace operator is also commutative with the $(\nabla\nabla\cdot)$ operator. It leads to derive the analytic solution for $\vec{E}$-field in free space without explicitly referring to the vector potential approach. The Lorentz-Gauge condition is also not required in our derivation. We can also show that the scalar Laplace operator is also found commutative with the gradient operator. Though no new expressions are derived, the mathematical skill paves a different and straight-forward way to solve the problem.

## Appendix:

Since the operator approach is highlighted in this paper, derivation of $\overrightarrow{E_1}$ is shown below using only the basic vector opertions.

$$\text{Given } \; r^2 = x^2+y^2+z^2 \text{ or } r=\sqrt{x^2+y^2+z^2},$$

, it is easy to show that $\nabla r^2 = 2r\overrightarrow{a_r}$, $\nabla r = \overrightarrow{a_r}$, $\nabla\frac{1}{r} = -\frac{1}{r^2}\overrightarrow{a_r}$, $\nabla\frac{1}{r^2} = -\frac{2}{r^3}\overrightarrow{a_r}$, $\nabla\cdot\vec{r} = 3, and\ \nabla e^{-jkr} = -jke^{-jkr}\overrightarrow{a_r}$ .Therefore, $\nabla^2 e^{-jkr} = \nabla\cdot(-jke^{-jkr}\nabla r) = \nabla(-jke^{-jkr})\cdot\nabla r + (-jke^{-jkr})\nabla\cdot\nabla r = -k^2e^{-jkr}\nabla r\cdot\nabla r + (-jke^{-jkr})\frac{2}{r} = -k^2e^{-jkr} + (-jke^{-jkr})\frac{2}{r}$, and

$$\nabla^2\frac{e^{-jkr}}{r} = \nabla\cdot\left(e^{-jkr}\nabla\frac{1}{r} + \frac{1}{r}\nabla e^{-jkr}\right) = e^{-jkr}\nabla^2\frac{1}{r} + 2(\nabla e^{-jkr})\cdot\nabla\frac{1}{r} + \frac{1}{r}\nabla^2 e^{-jkr} = -4\pi\delta(\vec{r}) - \frac{1}{r}k^2e^{-jkr} \qquad (1)$$

Eq.(1) can be rewritten as :$(\nabla^2+k^2)\frac{e^{-jkr}}{4\pi r} = -\delta(\vec{r})$, or $(\nabla^2+k^2)\psi\left(\left|\vec{r}-\overrightarrow{r'}\right|\right) = -\delta\left(\vec{r}-\vec{r}'\right)$

Mutiplying both sides of the above equation by a constant vector $\vec{a}$, it yields :

$$(\nabla^2+k^2)\vec{a}\psi\left(\left|\vec{r}-\overrightarrow{r'}\right|\right) = -\vec{a}\delta\left(\vec{r}-\vec{r}'\right)$$

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