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Game Coloring Number of Planar Graphs

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Abstract: In this paper we discuss the game coloring of planar graphs. This parameter provides an upper bound for the game chromatic number of graph. We describe the problem and its solution given by Xuding Zhu [1] and point out an error in it.

Keywords: Graph theory, Approximation algorithm, Planar graphs, Chromatic number.

1. Introduction

Let G= (V, E) be a graph and let X be a set of colors. The game chromatic number of G is defined through a two person game called coloring game. Alice and Bob take alternate turns with Alice having the first move. Each play of either player consists of coloring an uncolored vertex of G with a color from X. Adjacent vertices must be colored by distinct colors. If after n = |V| moves, the graph G is colored, Alice is the winner. Otherwise at any stage, if there is an uncolored vertex v such that each color of X is assigned to at least one of its neighbours, then Bobs is the winner. The game chromatic number of G, denoted by $x_a(G)$, is the least cardinality of a color set X for which Alice has a winning strategy. The coloring game on planar graphs was invented by Steven J. Brams, and was published by Gardner [2]. The game chromatic number of planar graphs was first studied by Keirstead and Trotter [3]. Recently Xuding Zhu made a significance contribution in this field[1,4]. Since it seems very difficult to determine the game chromatic number of even small graphs, Xuding Zhu in [1] discusses a variation of the game chromatic number, the game coloring number.

2. Game Coloring

Suppose (V,E) is a graph and X is an infinite set of colors. The game coloring number of G is defined through a two-person game: the coloring game. Alice and Bob, with Alice playing first, take turns in playing the game. Each play by either player consists of coloring an uncolored vertex of G. A player during his/her turn must first select an uncolored vertex u. If one of the already used colors, is not assigned to any of neighbours of u, then the player must assign one of the already used colors to u. Otherwise a new color must be used. The game ends when all vertices are colored. For a vertex v of G, let b(v) be the number of neighbours of v that are colored before v is colored. The score of the game is s = 1 + maxb (v) where v ϵ V. Alices goal is to minimize the score, while Bobs goal is to maximize it . The game coloring number col_a (G) of Gis the least s such that Alice has a strategy that results in a score at most s. It is easy to see that for any graph G, $x_a(G) \leq col_a$ (G).The next two Lemmas are trivial and we are quoting then without proof.

Lemma 1. Suppose H is a spanning subgraph of G. Then $col_a(H) \leq col_a(G)$
Lemma 2. Suppose G=(V,E) and $E=E_1UE_2$. Let $G_1=(V_1,E_1)$ and $G_2=(V_2,E_2)$. Then $col_a(G) \leq col_a(G_1)+\Delta(G_2)$, where $\Delta(H)$ denotes maximum degree of graph H.

2.1 Xuding Zhu's Strategy of Game Coloring For Planar Graphs

In [1] Xuding Zhu first decomposes the planar graph G into two graphs by partitioning its edges and occasionally adding some new edges such that these graphs satisfy some properties. He also orients the edges of these graphs. His strategy for Alice is to focus on only one of these graphs. The game coloring number is deduced by using Lemma 2.

2.1.1 Decomposition of Planar Graphs

In the following "i-vertex" will refer to a vertex of degree i and "I,j-edge" will refer to an edge between an i-vertex and a j-vertex. We call an edge 'e' a light edge if it is either a 3, j-edge for some $j \leq 10$, or a 4, j-edge for some $j \leq 8$or a 5, j-degree foe some $j \leq 6$.Borodin in [5] has proved that, every planar graph with minimum degree ≥ 3 contains a light edge. Xuding Zhu uses this fact for the decomposition of planar graphs.
Let G=(V,E) be a directed graph. If e=uvϵ E, then we say that edge e is directed from u to v. u is called an in-neighbour of v and v is called an out-neighbour of u. The in-degree (resp. out-degree) of v is the number of in-neighbours (resp. outneighbours) of v. The degree of v is the sum of its in-degree and out-degree.

Lemma 3. [1] Suppose G=(V,E) is a connected planar graph without 2,2-edges and 1-vertices. Then there are two directed graphs $G_1 = (V,E_1)$ and $G_2 = (V,E_2)$ that satisfy the following conditions:

1. EC E_1 U E_2 and E∩E=ø, where E_1 and E_2 are undirected edges associated with E_1 and E_2 respectively.
2. G_1 has maximum degree at most 8, and has maximum out-degree at most 3.
3. G_2 is acyclic, and each vertex has out-degree 2, except two vertices, say r', r, which are joined by a directed edge r'r, and have out-degree 1 and 0 respectively.
4. Suppose u, v are the two out-neighbours of a vertex x in G_2, then either uvϵ E_1 U E_2 or vuϵ E_1 U E_2.

In the following we give a sufficient description of an algorithm to compute G_1 and G_2. The edges of G_1 will referred as red and those of G_2 as blue.

The graph G_1 and G_2 are more or less obtained from G by coloring its edges by two colors red and blue, and assigning an orientation at the same time. In the process of coloring the edges of G, we keep a track of a planar graph G_a, which is a subgraph of G and a few additional edges. The algorithm for constructing graphs G_1 and G_2 is given in Algorithm 1.

Xuding Zhu claims that G_a is always planar without I-vertices, parallel edges, loops, and 2,2-edges, and that the coloring process terminates in O(│E│) steps.

In section 2.2, we present a counter example which disproved the above claim. For completeness we are presenting Alice's strategy given by Xuding Zhu[1] in Appendix A.

Input: A connected planar graph G=(V,E) without 2,2-edges and I-vertices.

Output: Output two directed graphs $G_1=(V,E_1)$ and $G_2=(V,E_2)$.

Initialize, $G_a=G$

Repeat

If G_a is isomorphic to K_3, then color all edges of G_a blue and assign orientations to the edges so that it is acyclic. Otherwise, suppose │V(G_a)│≥4. If G_a has a vertex say v, of degree 2 with edges vu and vw, then we do the following:

1. Color the two edges incident on v blue, and orient these two blue edges from v to the respective neighbours.

2. Delete v (together with the two incident edges) from G_a.

3. If uv is not an edge of G_a U G_1 U G_2, then add the edge uv to G_a.

If G_a contains no vertex of degree 2, then G_a has a light edge, say 'e'. In this case we color 'e' red, orient it from an end vertex of degree ≤ 5 to the other end vertex and delete 'e' from G_u.

Until G_u≠φ

unt

Algorithm 1: Algorithm for decomposition of planar graph.

2.2 Lacuna In The Decomposition Of The Planar Graph

Consider the planar graph G=(V,E) given in figure 1.1

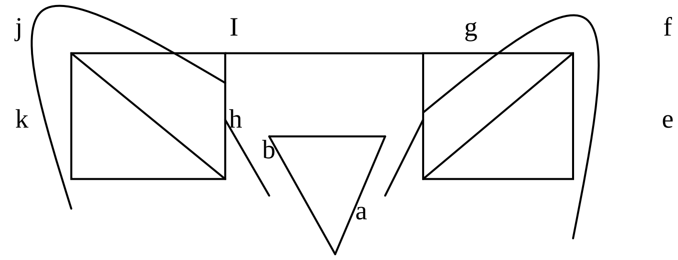

Figure 1.1: Planar graph G, without 2,2-edge and I-vertex: A counter example for the decomposition algorithm.

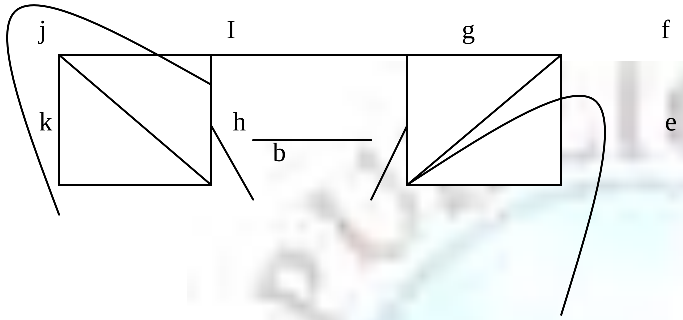

Figure 1.2: Resulting planar graph after applying steps 1, 2 and 3 of Algorithm 9 on vertex a of figure 6.1.

We apply the decomposition algorithm on G. Initialize G_a=G. Clearly G_a has a vertex 'a' of degree two, hence we color edges ab and ac blue and orient them away from a and then delete a from G_a. Since there is already an edge bc, we do not have to add any new edge.
After inserting the above step we have a 2,2-edge, that is bc, disproving the claim of xuding Zhu that G_a is always free from 2,2-edge. The resulting planar graph is shown in figure 6.2 . If we resume with the algorithm even in the presence of the said 2,2-edge the partition process comes to completion without any hurdle. But this is not the case in general.

Consider the graph G_a given below in figure 1.3.

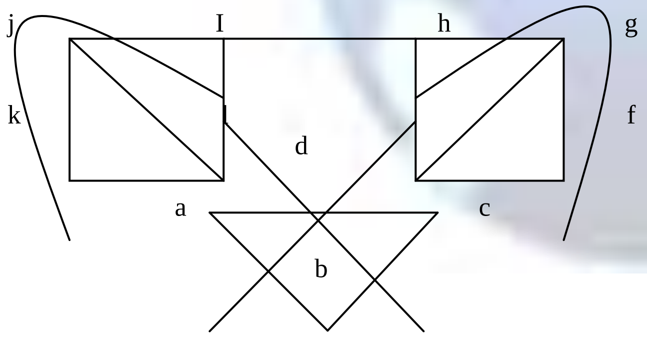

Figure 1.3: Planar graph G_a, without 2,2-edge and I-vertex: Another counter example for the decomposition algorithm.

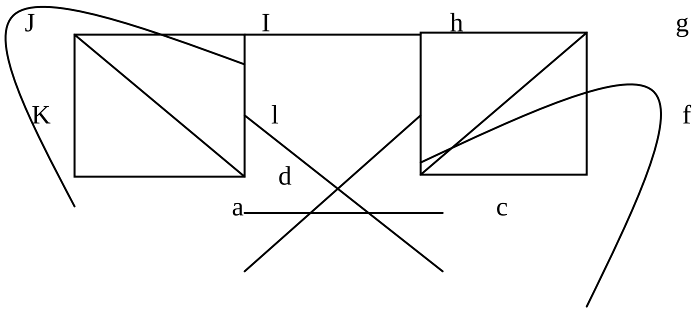

Figure 1.4: Resulting planar graph after applying steps 1,2 and 3 of Algorithm 9 on vertex b of figure 1.3.

Initially algorithm will color edges ba and bc blue and remove vertex b from G_a. The resulting graph will have a 2,2-edge ac as shown in figure 1.4. In this case the algorithm will fail to proceed any further since it will create aI-vertex. Hence we conclude that Xuding Zhu's claim of non-appearance of 2,2-edges is incorrect and in presence of these edges the decompositioning algorithm may fail.

Appendix A

In reference to the section 2.1, we are presenting Alice's strategy of coloring game proposed by Xuding Zhu [1].

Based on the decomposition of planar graph (section 2.1.1) into G_1 and G_2, Xuding Zhu in [1] has given a strategy for Alice, so that no matter how Bob plays the coloring game, the score of the game will be at most 19. In his strategy Alice will only take the graph G_2 into consideration. We need to define some terms before describing the strategy.

Suppose $x \in V-\{r,r'\}$ and u, v are the two out-neighbours of x in G_2, by Lemma 3, either uv or vu $\in E_1 \cup E_2$. Assume that vu$\in E_1 \cup E_2$. We call u, v the parents of x, call u the major parent of x, and v the minor parent of x. We call x a major son of u, and call it the minor son of v. We call the edge xu a major edge and xv a minor edge. Two vertices x, y are called brothers if x and y have the same parents. We call the edge rr' a major edge, and r' has a single major parent, no minor parent, and r has no parents.

Let T be a directed spanning tree of G_2 induced by the major edges of G_2. In the process of coloring game, Alice will keep track of a set, which includes r and induced graph on it is a sub-graph of T. We call this set the active set, and denote it by T_a. the vertices of T_a are called as active vertices. We define two operations on any directed graph in G_2, the extension and the switch as follows:

Suppose $P=\{y_1, y_2, \ldots, y_x\}$ is a directed path in G_2 not containing any T_a-vertex. Let P' be the unique directed path of T connecting y_x to T_a (i.e, the first vertex of P' is y_x, the last vertex of P' is a vertex of T_a, and all inner vertex of P'(if any) are not in T_a). The concatenation of P and P' is called the extension of P. If the last vertex of P is in T_a then, the extension of P is itself.

Suppose $P=\{y_1, y_2, \ldots, y_x\}$ is a directed path in G_2, and suppose that the last edge, $y_{x-1}y_x$, of P is a major edge. Let y' be the minor parent of y_{x-1}. Then the directed path $P'=\{y_1\, y_2, \ldots, y_{x-1}, y'\}$ is called the switch of P. Note that if the last edge of P is a major edge and not equal to r'r, then its switch is unique. Otherwise its switch is not defined.

Alice's strategy is as follows:

Initially, Alice's color r, and set $T_a=\{r\}$. Suppose at certain stage of the game,Bob has colored the vertex x. Then Alice select the next vertex to color by the following rule:

Let y be the major parent of x, and let P_1=xy. Let P_2 be the extension of P_1. Alice will repeat the following procedure until she select a vertex to color.

Suppose the presently found directed path is P_{2e} for some $e \geq 1$, and that the last edge of P_{2e} is vu.

1. If vu= r'r, then select any free (uncolored) vertex x such that all its predecessors in G_2 have been colored.
2. If vu is a major edge, and the no. of active brothers of v is even and that u is a free (uncolored) vertex, then select u.
3. If vu is a major edge, and that either v has an odd number of active brothers, or u is a colored vertex, then let P_{2e+1} be the switch of P_{2e} and let P_{2e+2} be the extension of P_{2e+1}, and go back to repeat the procedure (with P_{2e} replaced by P_{2e+2}).
4. If vu is a minor edge, and u is a free (uncolored) vertex, then select u.
5. If vu is a minor edge, and u is a colored vertex, then select any free (uncolored) vertex x such that all its predecessors in G_2 have been colored.

After Alice selected the next vertex to color, say v, add the vertices of the directed path P_{2e} and the vertex v to T_a, where P_{2e} is the last path found in the procedure above. Also color the vertex v with the first available color from the color set X.

For completeness, we quote the theorem of Xuding Zhu, which bounds the score of the coloring game to 19.

Theorem 1. [1] If Alice uses the strategy described above, then the score of the coloring game is at most 19

Conclusion

Initially algorithm will color edges ba and bc blue and remove vertex b from G_a. The resulting graph will have a 2,2-edge ac as shown in figure 1.4. In this case the algorithm will fail to proceed any further since it will create aI-vertex. Hence we conclude that Xuding Zhu's claim of non-appearance of 2,2-edges is incorrect and in presence of these edges the decompositioning algorithm may fail.

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