On the Derivation of the Coordinates of Coupler Points in Spherical Mechanisms

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ABSTRACT

In great number of applications the output from a simple mechanism is the path traced by one of the points on the coupler link. These paths are generally called coupler curves. Coordinates of the coupler point must be obtained analytically for drawing of coupler curve. In this study, it is proposed relatively simple method for determining the coordinates of the coupler point in the spherical mechanisms. A coupler point of a link which belongs to a spherical mechanism forms a spherical triangle on the sphere surface together with two points whose coordinates are known and which belong to the link. From the position analysis of the mechanism, the coordinates of two points can be calculated. A set of linear equation with three unknowns was derived by means of spherical and spatial geometry for rectangular coordinates of the coupler point and they were found by solving the set of linear equations. Coordinates along whole motion of mechanism were calculated and thus the coupler curve can be drawn.

Keywords: Coupler curve, Spherical mechanism.

1. INTRODUCTION

One of the main issues on which the mechanism designers focus their attention, is the mechanisms that can trace trajectories providing the desired motion characteristics. Analysis and synthesis techniques to solve such issues have been developed for planar, spatial and spherical mechanisms by many researchers. There are several studies in the literature on this subject. Planar coupler curves are the curves of which the point on any link of mechanism traces on the fixed motion space during the motion of mechanism. As the links in a spherical mechanism move on the spherical surface, the coupler curve becomes a spherical curve on the spherical surface. Such spherical coupler curves have been discussed in several analytical studies in the literature. Chiang [1] gave the closed form equations of the spherical coupler curve, which is a curve of eighth degree in spherical four bar linkages, in Cartesian and spherical polar coordinate systems. Deng, et al [2] has studied an approach for the synthesis of spherical four bar mechanisms with two or three prescribed coupler curve cusps. Bagci [3] carried out study on a method for geometric structure and function synthesis of compatible projection of spherical mechanism.

He explained spherical slider crank mechanism, crank rocker mechanism and very simple geometric synthesis techniques with numerical examples and also presented computer-aided analytical solutions for each geometrical configuration. Chiang [4] performed a detailed comparison by investigating the equivalents, similarities and dissimilarities of the concepts between planar and spherical kinematics. Lu [5] described a triangular diagram for spherical symmetric coupler curve traced by a spherical crank rocker mechanism with special dimensions and developed a computer program capable of computing the coordinates of the points of the coupler curve, for a mechanism with given dimensions. Chu, et al [6] researched the mathematical formulas that can represent the coupler curves based on the relationship between coupler curves and basic dimensional types in spherical four bar mechanism. They also performed a mathematical presentation by giving examples about the formulas they developed. In the study of Sun, et al [7], based on the analysis of the function relationship between input and output angles of the spherical four bar mechanisms, the relation with input-output displacement equation and basic dimensional types were given using the Fourier analysis method.

In this study, a set of linear equations with three unknowns was derived by means of spherical and three dimensional geometry for coordinates of a coupler point belonging to a spherical mechanism. The coordinates of the point were determined by solving these equations and the study was intended to draw the coupler curve by calculating the coordinates through the whole motion of mechanism.

2. SPHERICAL GEOMETRY AND SPHERICAL MECHANISM

In the planar mechanisms, fixed link is expressed by lines passing through two points of the body. Similarly, fixed link in spherical geometry can be defined by a piece of great circle passing through two points of the body that is on the surface of a unit sphere with radius of one unit. Also the other links are expressed with the great circular arcs in different directions on the surface of the same unit sphere. In spherical geometry, the distance between two points on the surface of the unit sphere is measured by central angle subtended by the arc of the great circle passing through these two points.

The great circle in sphere is the intersection of the spherical surface and a plane that passes through the center of the sphere and the two points on the spherical surface. Accordingly, there is only one great circle passing through any two points on the spherical surface or this result may also be expressed as follows: as the shortest distance between two points in the plane is a line segment, the shortest distance between two points on the spherical surface is the arc of the great circle passing through these points [4].

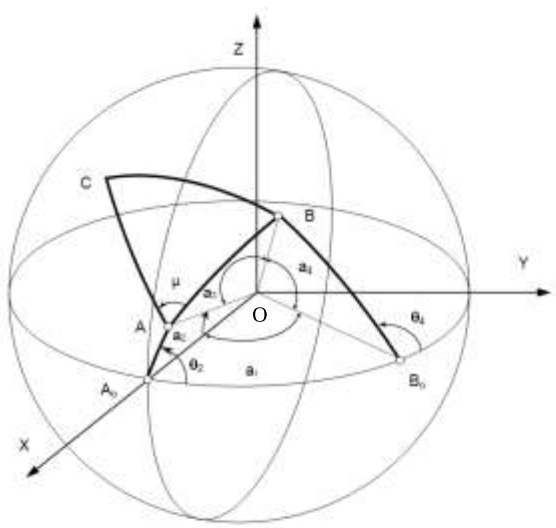
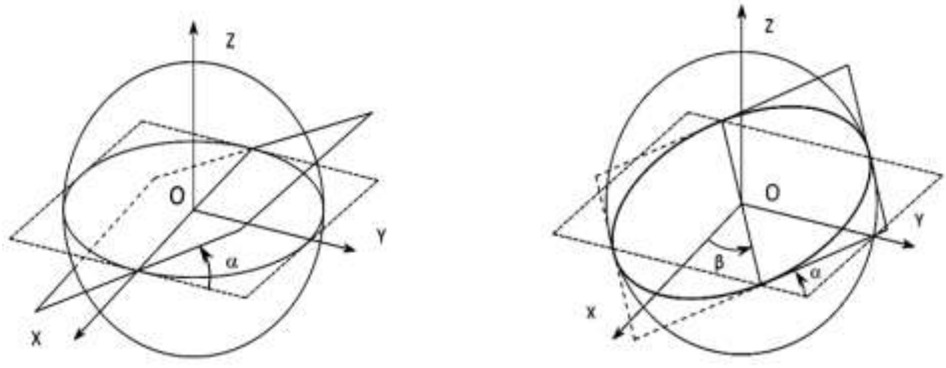


Figure 1. Spherical four bar mechanism

As an example, a spherical four bar mechanism can be seen in Fig. 1. Here, lengths of a_1, a_2, a_3, a_4 which represent the links, are the central angles subtended by the arcs of the great circle representing the links. For simplicity, the plane on which the arc of the great circle belonging to the fixed link located is taken as OXY plane and the axis X is passed through a fixed joint A_o. In spherical mechanisms, all axes of rotation of the revolute pairs pass through the center of the sphere. The point C on the link 3 is a coupler point of this link. The points A, B, C form a spherical triangle on the sphere. The arcs AC and BC, which represent the location of the point C on the spherical surface, are also the arcs of great circle and their lengths are expressed as the central angles subtended by these arcs. The angle μ, which is between the great circle arcs AC and AB, is a spherical angle measured from the spherical surface. The lengths of the links and the coupler points in spherical mechanisms that have more links are also expressed in a similar manner.



(a) **(b)**

Figure 2. Rotated planes

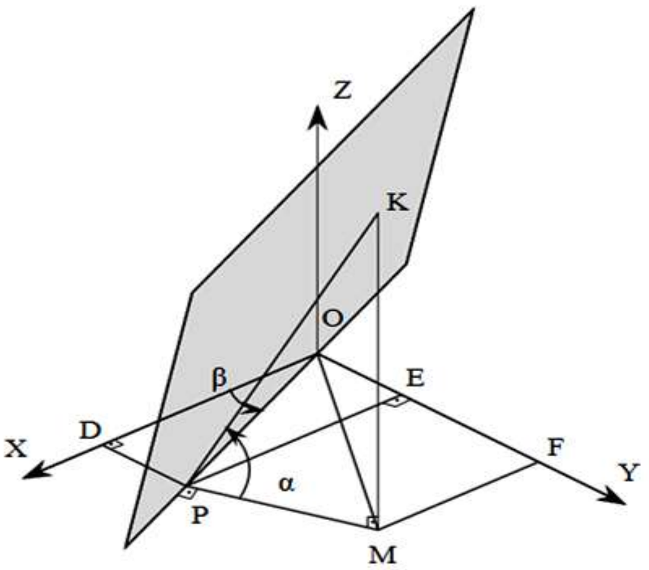


Figure 3. Axes frame and rotated plane

Let's take the plane where the great circle arc representing the fixed link situated, as OXY plane (equatorial plane) in OXYZ Cartesian coordinate system whose origin is on the center of the sphere as can be seen in Fig. 2a. Let's rotate this plane by an angle of α around axis X in the positive direction by right-hand rule (Fig. 2a). Then rotate by an angle of β around axis Z in the positive direction by the right-hand rule again (Fig. 2b). The great circle, which is the intersection of the sphere with the obtained plane, can be seen in Fig. 2b. All great circles on the surface of the sphere can be obtained in this way. This case may be expressed as follows: any great circle on the surface of the sphere is the intersection of the sphere with a plane rotated by an angle of α around axis X and rotated by an angle of β around axis Z. The aforementioned plane always passes through the center of the sphere as it is shown in Fig. 2 a, b. The whole great circles on the surface of the sphere are determined as the intersection of the sphere with the plane by changing the angles of α and β between -180° and +180°.

Suppose that point M is the projection of any point K in the OXY plane (Fig. 3) and point P is the end of the vertical line drawn from point M to the line that is the intersection of the plane with OXY, point D is the end of the vertical line drawn from point P to the axis X. It is obvious that the length KM is the z coordinate of the point K. As the KPM angle represents the α angle;

$$\tan\alpha = \frac{KM}{MP} = \frac{z}{MP}$$

$$z = MP \tan\alpha \tag{1}$$

The top view of the plane OXY is shown in Fig. 4. Here, the coordinates of the point K in this plane are OG = x and OF = y. FR=PN=ycosβ and FM=OG can be seen in Fig. 4.
Writing MP = PN – MN and considering

$$MN = FM\sin\beta = x\sin\beta;$$

$$MP = y\cos\beta - x\sin\beta \tag{2}$$

If the equation (1) is substituted into this expression and the expression is arranged;

$$z + \tan\alpha(x\sin\beta - y\cos\beta) = 0 \tag{3}$$

The equation (3) is the general equation of the plane passing through the origin and rotated by an angle of α around axis X and by an angle of β around axis Z. This equation is also valid for the points on the great circle that is the intersection of this plane with the sphere.

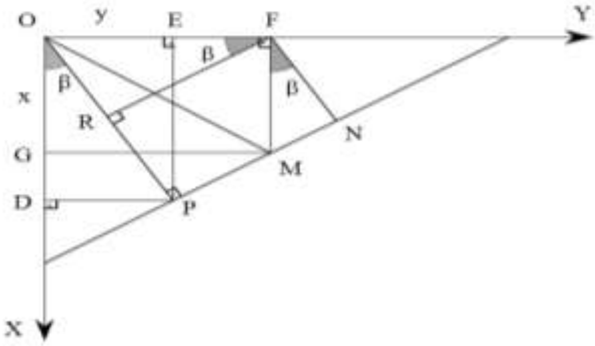


Figure 4. Projection on plane OXY

3. SPHERICAL COUPLER CURVE

For instance, let's try to determine the coordinates of the coupler point C of spherical four bar mechanism in Fig. 1. As θ_2 and θ_4 is the angle of rotation of the link 2 and link 4 respectively, based on the spherical geometry, the coordinates of the point A and B can be written as follows:

$$x_A = \cos a_2, \quad y_A = \sin a_2 \cos\theta_2, \quad z_A = \sin a_2 \sin\theta_2$$
$$x_B = \cos a_4 \cos a_1 - \sin a_4 \sin a_1 \cos\theta_4$$
$$y_B = \cos a_4 \sin a_1 + \sin a_4 \cos a_1 \cos\theta_4$$
$$z_B = \sin a_4 \sin\theta_4 \qquad (4)$$

In spherical four bar mechanism, the equation to calculate the θ_4 angle with a given θ_2 angle is as follows [9]:

$$\theta_4 = 2\arctan\left(\frac{p - \sqrt{p^2 - u^2 + q^2}}{q + u}\right) \qquad (5)$$

where,

$p = \sin a_2 \sin a_4 \sin\theta_2$
$q = \sin a_2 \sin a_4 \cos a_1 \cos\theta_2 - \cos a_2 \sin a_4 \sin a_1$
$u = \cos a_3 - \cos a_1 \cos a_2 \cos a_4 - \sin a_2 \cos a_4 \sin a_1 \cos\theta_2$

When the plane on which the arc of the great circle AB located, is rotated by an angle of μ around the axis OA, the point B moves to the point B' and the plane where the great circle arc AB located coincides with the plane where the great circle arc AC located. Thus, the point A, C and B' locates on the same plane (Fig. 5).

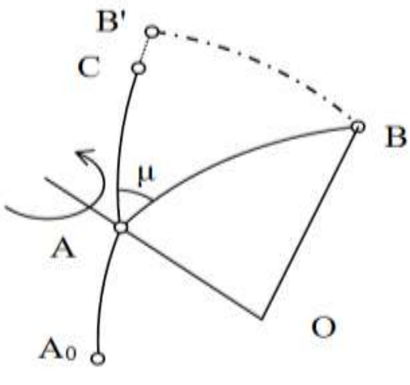


Figure 5. Rotation of OAB plane

The coordinates of point B' are determined by the following equation [8]:

$$X_{B'} = T X_B \qquad (6)$$

where, **T** is the rotation matrix around axis OA:

$$\mathbf{T}=\begin{bmatrix} x_A{}^2K+\cos\mu & x_Ay_AK-z_A\sin\mu & x_Az_AK+y_A\sin\mu \\ x_Ay_AK+z_A\sin\mu & y_A{}^2K+\cos\mu & y_Az_AK-x_A\sin\mu \\ x_Az_AK-y_A\sin\mu & y_Az_AK+x_A\sin\mu & z_A{}^2K+\cos\mu \end{bmatrix}$$

$$K=1-\cos\mu$$

$\mathbf{X}_B = [\, x_B \;\; y_B \;\; z_B \,]^T$ and $\mathbf{X}_{B'} = [\, x_{B'} \;\; y_{B'} \;\; z_{B'} \,]^T$

The equation of the plane where the points A and B' situated, is the same as the equation (3). Since the coordinates of these two points are known, the values of the angles α and β in the equation are simply determined as follows:

$$z_A + \tan\alpha(x_A\sin\beta - y_A\cos\beta) = 0$$
$$z_{B'} + \tan\alpha(x_{B'}\sin\beta - y_{B'}\cos\beta) = 0 \tag{7}$$

The angle of β is determined with function atan2 by solving the two equations above;

$$\beta=\text{arctan2}\left(\frac{z_A y_{B'} - z_{B'} y_A}{z_A x_{B'} - z_{B'} x_A}\right) \tag{8}$$

Then, the angle of α is calculated by one of the equations above. Since the angles of α and β are determined, the equation of this plane can be written in terms of the coordinates of point C:

$$z_C + \tan\alpha(x_C\sin\beta - y_C\cos\beta) = 0 \tag{9}$$

Spherical distance between the points A and C, angle AOC or BOC is determined by scalar multiplication of vectors $\overrightarrow{OA}$ and $\overrightarrow{OC}$ or $\overrightarrow{OB}$ and $\overrightarrow{OC}$ respectively [2].

$$\cos AOC= \overrightarrow{OA}\cdot\overrightarrow{OC}=x_A x_C + y_A y_C + z_A z_C \tag{10}$$

$$\cos BOC = \overrightarrow{OB}\cdot\overrightarrow{OC} = x_B x_C + y_B y_C + z_B z_C \tag{11}$$

The angle BOC (or the length BC) is determined by Napier's rules used for spherical triangles. As the angles of AB(a_3), AC and μ are known, the length BC is calculated with the equation below;

$$\cos BC= \cos a_3 \cos AC+ \sin a_3 \sin AC\cos\mu \tag{12}$$

So the coordinates (x_C, y_C, z_C) of the coupler point C at any position of the link 2 can easily be determined by solving the three linear equations numbered (9), (10) and (11).

$$x_C = \frac{y_B\, c_a - y_A\, c_b - \tan\alpha\, \cos\beta(z_A\, c_b - z_B c_a)}{u}$$
$$y_C = \frac{x_A c_b - x_B\, c_a - \tan\alpha\, \sin\beta\, (z_A c_b - z_B\, c_a)}{u}$$
$$z_C = \frac{-\tan\alpha[\sin\beta(-y_A c_b + y_B c_a) - \cos\beta(x_A\, c_b - x_B\, c_a)]}{u} \tag{13}$$

Where,

$c_a=\cos AOC$, $c_b=\cos BOC$,

$$u = y_B\, x_A - x_B\, y_A + \tan\alpha\;\; [\sin\beta(\, y_A z_B - z_A\, y_B) - \cos\beta(\, x_B\, z_A - x_A\, z_B)]$$

The coupler curve of any point can be traced by calculating the coordinates in any mathematical software for a whole motion of the mechanism (13). This method can also be implemented easily to the spherical mechanisms having more links. In the examples below, coupler curves of two spherical four bar mechanisms (one of is crank-rocker and the other is drag link) and a spherical six bar mechanism are demonstrated.

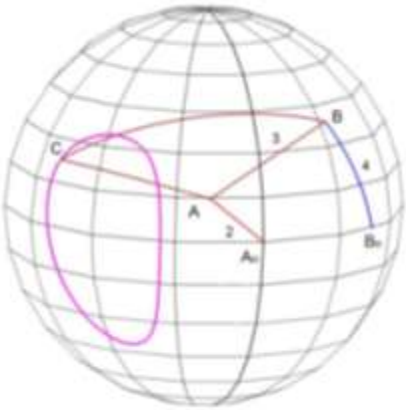


Figure 6. Coupler curve of spherical crank-rocker (a_1 = 40°, a_2 = 20°, a_3 = 45°, a_4 = 35°, AC = 50°, μ = BAC = 130°)

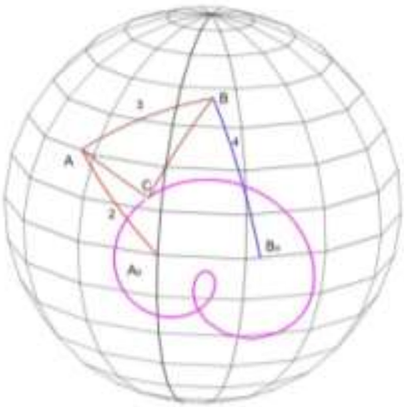


Figure 7. Coupler curve of spherical drag-link (a_1 = 30°, a_2 = 40°, a_3 = 42°, a_4 = 50°, AC = 30°, μ=BAC= – 60°)

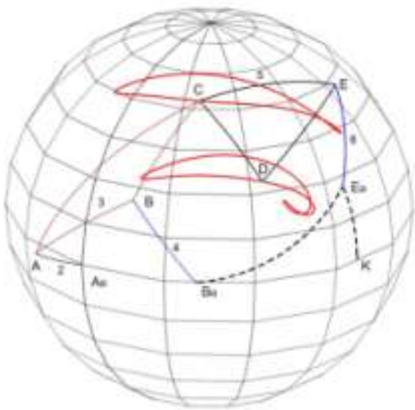


Figure 8. Coupler curves of C and D points of spherical six bar mechanism (a_1 = 35°, a_2 = 20°, a_3 = 42°, a_4 = 30°, AC= 72°, μ=BAC= 22°, a_5 =CE= 45°, a_6 =33°, μ_2 = ECD =75°, CD = 30°, B_oE_o = 85°, ε =E_oB_oK =30°)

CONCLUSION

A set of linear equations with three unknowns was derived using spherical and three dimensional geometry for coordinates of a coupler point belonging to a spherical mechanism. The coordinates of the coupler point were determined by solving these linear equations and the coupler curve was drawn along whole cycle of the mechanism. Thus, relatively a simpler analytical method can be used for drawing of coupler curve in spherical mechanism have been shown. Considering the mathematical difficulties in drawing the coupler curve by coupler curve equations in closed form, the method is believed to ensure simplicity.

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