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On The Extremes of Collection of External Numbers

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Abstract: The aim of this paper is to introduce and define the notion of external distance and the extreme of a collection of external numbers. More precisely under certain conditions we obtain the following results:

If $f : R \to R$ is an internal continuous increasing function and if β is an external number, then

$$\sup\{f(x) : x \le \beta\} \equiv \inf\{f(x) : x < \beta\},$$

$$\sup\{f(x) : x > \beta\} \equiv \inf\{f(x) : x \ge \beta\}$$

If $f : R \to R$ is an internal increasing function and J is an external interval of R, then

$$\sup\{f(t) - f(s) : s < t \text{ and } s, t \in J\} \equiv \underbrace{\sup}_{t\in J}\{\underbrace{\sup}_{s\in J}\{f(t) - f(s) : s < t\}\}.$$

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1. Introduction

It is known that the integral of a constant function f(x)=1on agiven interval (a,b) of R is equal to b-a which is the difference between the least upper bound and the greatest lower bound of the interval (a,b).

We will extend this fact to the external numbers. We define thenotion of the external distance and then the concepts of theleastupper bound and the greatest lower bound of an external interval.

We recall here that the external sets in R (see[2],[3],[4]) are not sets in the normal sense of the world ,but the collections of real numbers satisfying an external formula and not satisfying at least one theorem of conventional mathematics (see [2],[5]).

The external sets play an essential role in this paper, especially those external sets not satisfying the least upper bound theorem.

2. Preliminaries

Throughout this paper the following definitions and notations will be used.

Definition 2.1 **[2]**

Every set or formula which does not involve a new predicate "standard, infinitesimal, limited,…etc." is called an internal set or formula,otherwise,it is called an external set or formula .

Definition 2.2 **(see[3],[8])**

The set of all real numbers which are infinitely near to a real number a is called the monad of a,denoted m(a).So the external set of all infinitesimals is called the monad of 0, denoted by m(0).

Definition 2.3 **(see[2],[6],[7])**

The external set of all limited real numbersis called the principle galaxy. The notations Ä, L ,@ and U are used for collection of infinitesimal, limited, appreciable and unlimited numbers respectively (see [2], [6],[9],[10]) .
E denotes the set of all external numbers, the symbol $\equiv$ denotes identical, conv, Lconv, Uconv denotes convexity, lower convexity, upper convexity respectively.

The notation (∞,L) means that the interval does not contain any limited real numbers and the notation (-∞,L] means that the interval contains all limited real numbers.

Definition 2.4[1]

A cut in R is an ordered pair (A,B) of convex subsets of R such that $A \bigcup B = R$, $A \bigcap B = \varphi$ and A is dominated by B .The convex subset A is called a lower half line of R and the convex subset B is an upper half line of R.

The following theorem is a reformation of the classification of internal and external cuts of R using external numbers.

Theorem 2.1

If (M,N) is a cut of R, then there exists a unique external number α such that M=(-∞,α) or M=(∞,α].

Proof :

Let α be an external number, $A = \{a : a \le \alpha\}$, and $B = \{b : b \ge \alpha\}$, then A=(-∞,α] and B=(α,∞). Since $A \bigcup B = R$ and $A \bigcap B = \varphi$, then (A,B) is a cut of R.

Now we have to prove that α is unique, suppose that there exists an external number β different from α such that A=(-∞, β] , then either α > β or α<β. If α> β , then A∪B=(-∞, β] ∪ (α, ∞), if α< β , then A∩B=(-∞, β]∩(α,∞). In both cases we get contradiction, hence there exists a unique external number α such that A=(-∞,α) or A=(-∞ ,α].

Definition 2.5

A microscopic set is a convex subgroup of R which may be internal or external.

Definition 2.6

An external number is the algebraic sum of a real number and a microscopic set.

3. External Distance

Let E^+ denote the set of all positive external numbers α ≥ 0.

Definition 3.1

We define a mapping $\lambda:E \to E^+$ which is associated to any two fixed external numbers α =a+M and β=b+N an external number λ(α, β) in E^+. This mapping is called the external distance between α and β, and it is given by λ(α, β) =│b–a│ +(M+N). From the above definition we have the following remark:

Remark 3.2

1. λ(α, α) =M which is the microscopic part of α for all α in E.
2. λ(α, β)≥ 0 ∀α, β ∈E ,λ(α, ɤ) ≤(α, β)+ (β, ɤ)∀α, β,ɤ ∈E.
3. If λ(α, β)=the microscopic part of one of them ,then it does not necessary imply that α= β. For example if α=a+M,β =a+N, then λ(α, β)=│a-a│+(M+N)=M+N=the microscopic part of α or β while α ≠ β.

Proof (3):

Let α =a+M, β=b+N and ɤ =c+K ,then

λ(α,ɤ)=│c-a│+(M+K), λ(α,β)=│a-b│+(M+N) and,

λ(β, ɤ) =│c-b│+(N+K), It is clear that

│c-a│=│c+b-b-a│=│(b-a)+(c-b)│≤│b-a│+│c-b│ (1)

Since (M+K) ≤ (M+K)+N, then(M+K) ≤ (M+K)+2N this implies that

(M+K) ≤ (M+N)+(N+K) (2)

From (1) and (2), we get the result.

Definition 3.3

If P is a collection of external numbers, we say that P is convex in E if ∀α, βϵ P,∀ δϵ E and α ≤ δ ≤ β then δ ϵ P.

Example 3.4

1. The collection of all external numbers defined by P_1={x:x ≤1+ε} is a convex set, the union of its elements in R gives the external interval (-∞,1+ε).

2. The collection of external numbers defined by $P_2=\{x+\varepsilon:x\leq 1+\varepsilon\}$ is not a convex set,because $\{0\}\epsilon P_2$, nevertheless, the union of its element in R gives the externalinterval $(-\infty,1+\varepsilon]$.
3. The collection of external numbers defined by$P_3=\{x \in R:x\leq 1+\varepsilon\}$is not a convex set because $\varepsilon\notin P_3$,while the union of its elements in R gives the external interval$(-\infty,1+\varepsilon]$.

4. Extremes of a Collection of External Numbers

Definition 4.1

Let T be a collection of external numbers, the lower convexity of T denoted by Lconv(T) is defined by

$$Lconv\,(T\,)=\{\gamma\in E\,:\exists\delta\in T and\ \gamma\leq\delta\ \}$$

The upper convexity of T denoted by Uconv(T) is defined by $Uconv\,(T\,)=\{\gamma\in E\,:\exists\delta\in T and\ \gamma\leq\delta\ \}$ and the convexity of T denoted by conv(T) is defined by

$$Conv\,(T\,)=Lconv\,(T\,)\cap Uconv\,(T\,).$$

Definition 4.2

Let T be a non-empty set of external numbers ,the least upper bound of T is upper boundary of conv(T), it is denoted by sup(T) and the greatest lower bound of T is the lower boundary of conv(T), it is denoted by inf(T).

It is clear that for any bounded set of external numbers the following relation holds

Sup(T)=inf(T) (3)

The following example gives the supremum and infimum of some sets

Example 4.3

(i) If $P=\{x: x\leq 1+\varepsilon\}$ and $P_1=\{x: x>1+\varepsilon\}$, the $\sup(P)=\inf(P_1)=1+\varepsilon$

(ii) $Inf\left\{\frac{1}{n}:st\left(n\right)\in N^{+}\right\}$ and $sup\left\{n:st\left(n\right)\in N\right\}$

(iii) $\text{Sup}\{x+\varepsilon L: x< 2+\varepsilon\}=2+\varepsilon$

(iv) $\text{Sup}\{1-\varepsilon^n-L\varepsilon^{n+1}: E>0, \text{st}(n)\in \text{N}\}=1-L\varepsilon^n$ infinitely large.

Remark 4.4

The concepts of the l.u.b and g.l.b of sets of external numbers are different from the common concepts of the l.u.b. and the g.l.b. of sets of real numbers. For example,

If $P=\{x:X< 1+\alpha\}$,then $\sup\{P)=1+\alpha$ yet $1\leq 1+\alpha\forall\ x\in P\ ,x\leq 1$

Theorem 4.5

If T is a non-empty set of external numbers not containing the external number R. Then

1. Lconv(T) is a lower half line of R
2. Uconv(T)is an upper half line of R
3. Conv(T) is an external interval whose upperboundary meets the upper boundary of Lconv(T) and whose lower boundary meets the lower boundary of Uconv(T).

Proof:

Suppose that sup(T)=a and inf(T)=b where a,b are two external numbers and a,b$\in$ T

1. Lconv(T)=$\{\gamma\ \in E:\exists\delta\in T\ and\ \gamma\leq\delta\}=\{\gamma\in E:\gamma\leq sup(T)\}=\{\gamma\in E:\gamma\leq a\}=(-\infty,a\,]$ is a lower half line of R.
2. Uconv(T)=$\{\gamma\in E:\exists\ \delta\ \in T\ and\ \gamma\geq\delta\}=\{\gamma\in E:\gamma\geq inf(T)\}=\{\gamma\in E:\gamma\geq b\}=[\,b,\infty)$ is an upper half line of R.
3. conv(T)=[Lconv(T)∩Uconv(T)] =$(-\infty,a]\cap[b,\infty)=[b,a]$ is an external interval whose upper boundary meets the upper boundary of Uconv(T).

If a,b$\notin$T, then Lconv(T)=$(-\infty$,a),Uconv(T)=(b,∞) and conv(T)=(b,a).

Example 4.6

Let T={$1+\varepsilon_1, 1+\varepsilon_2, 1+\varepsilon_3 : \varepsilon_1 \le \varepsilon_2 \le \varepsilon_3$ and ε_1, ε_2, ε_3 are $\in \alpha$}, we find Lconv(T), Uconv(T) and conv(T).

Solution:

Lconv(T)=$\{\gamma \in E: \exists\delta \in T \; and \gamma \le \delta\} = \{\gamma \in E: \gamma \le 1+\varepsilon_3\}$

Uconv(T)=$\{\gamma \in E: \exists\delta \in T and \gamma \ge \delta\} = \{\gamma \in E : \gamma \ge 1+\varepsilon_1\}$

Conv(T)=Lconv(T) ∩ uconv(T) ={ $\gamma \in E : 1+\varepsilon_1 \le \gamma \le 1+\varepsilon_3$}

The following theorem extends the classical characterization of the supremum and justifies this terminology.

Theorem 4.7

If P is a nom empty collection of external numbers and if β is an external number, then β =sup(P) iff one of the following holds:

1. (β∩Lconv(P)=Φ) and ($\forall \gamma < \beta \; \exists \; \delta \in P: \delta > \gamma\}$ (i)
2. (β ∈ Lconv(P)) and ($\forall\delta \in P, \delta \le \beta$) (ii)

Proof:

1. Let P be a non-empty collection of external numbers, then

   (ß∩Lconv(P)=φ) and ($\forall \gamma < \beta, \exists \; \delta \in p: \delta > \gamma$)iff$\bigcup_{s\epsilon p}(-\infty, \delta)$=(-∞, β), i.e. β =sup(P).

   Suppose that β ∩Lconv(P)= Φand($\forall \gamma < \beta \exists \delta \in \rho: \delta > \gamma$) thus it is clear that β =sup (ρ).

   Conversely, suppose that β =sup (ρ) i.e. $\bigcup_{\delta\epsilon p}(-\infty, \delta) = (-\infty, \beta)$, it is clear that $\beta \cap Lconv(p) = \Phi$.Now let $\gamma < \beta$ this implies that $\gamma \in (-\infty, \beta) \Longrightarrow \gamma \in \bigcup_{\delta\epsilon p}(-\infty, \delta\,]$, *then* $\exists \; \delta_0 \in P$ such that $\gamma\epsilon(-\infty, \delta_0\,]$. This implies that $\delta_0 > \gamma$. Hence ($\forall \gamma < \beta, \exists \; \delta \in p: \delta > \gamma$).

2. $\big(\beta \in Lconv(p)\big)$ *and* ($\forall\delta \in P, \delta \le \beta$) iff $\bigcup_{\delta\epsilon p}(-\infty, \delta) = (-\infty, \beta)$

Similarly we can proof this part.

Remark 4.8

From relations (i) and (ii) we conclude that an external number β is the l.u.b. of a collection of external numbers P iff one of the following statements holds:

1. βstrictly dominates all elements of P, and any external number strictly dominated by β is strictly dominated by an element of P.

Or

2. $\beta$is totally included in the conv(P) and dominates all elements of P. Similarly, an external number γ is the g.l.b. of the set P of external numbers iff one of the following holds:

1. $(\gamma \cap uconv(P) = \Phi) and \; (\forall\beta > \gamma, \exists\delta \in p: \delta < \beta)$

Or

2. ($\gamma \subset uconv$(P))and $\forall \; \delta \in p, \delta \ge \gamma$).

Concerning the l.u.b.and the g.l.b. of external numbers, we have the following theorem:

Theorem 4.9

Every non empty subset of E has unique l.u.b. and unique g.l.b.

Proof follows directly from Theorem (1.2)

Proposition 4.10

If f:R⟶R is an internal, continuous and increasing function, and if β is an external number, then

Sup{f(x):x ≤ β=Inf{f(x):x> β} (A)

Sup{f(x):x<β}=Inf{f(x):x ≥ β} (B)

Proof:

It is clear that (-∞,f(β)] is a lower half line of R. If (–∞,f(β)] isan internal interval ,then the result is trivial. Suppose that (-∞,f(β)] is an external lower half line of R, then{(-∞,f(β)], (f(β),∞)} is an external cut of R. By theorem

(1.2).The upper boundary of(-∞,f(ß)] is exactly the lower boundary of $(f(\beta), \infty)$, hence the first relation is proved. The relation (B) can be proved in a similar way.

Proposition 4.11

If f:R⟶R is an internal increasing function and I is an external interval of R, then

Sup{f(t)-f(s):s<t and s,t∈ I}=$\underbrace{sup}_{t\in I}$\{\{$\underbrace{sup}_{s\in I}$\{f(t)-f(s) \}:s<t\}\}.

Proof:

Let $B = \underset{s\in L}{Sup}\{f(t)-f(s)\}$ and $B = b + N = \underset{s\in L}{Sup}\{B_t\}$, then we have f or $\alpha \le t \in L$, $B_t \subseteq \alpha$, So

$B \subseteq \alpha$ (1)

Now, let T be a real number which strictly larger than M, then∃(s_r,t_r)∈ $L \times L$: $f(t_r)$-$f(s_r)$>a-T, consequently

a-T≤$\underbrace{sup}_{s\in L}$\{f($t_r$)-f(s)\}$\le \beta$, since α=sup{a-T:T>M} we get

$\alpha \subseteq B$ (2)

From (1) and (2) we get α =β.

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