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CAS Wavelets for Solving General Two-Dimensional Partial Differential Equations of Higher Order with Application

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Abstract: In this paper, numerical techniques based on the CAS wavelets method is proposed of the numerical solution for two-dimensional partial differential equations of higher order. We derived the general formula for n of integrals to CAS wavelets analytically which called the operational matrix of integration, and we derived general formulas for solving two-dimensional partial differential equations of higher order using two and three-dimensional CAS wavelets method.
From application these formulas to solve two-dimensional non-linear coupled Benjamin-Bona-Mahony system, we gained the numerical stability even when the step size used was large, still the results were satisfactory, and the accuracy of the obtained solutions is quite high even if the number of calculation points is small, by increasing the number of collocation points; the error of the solution rapidly decreases.

Keywords: CAS wavelets, Two-Dimensional Partial Differential Equations of higher order, the Operational Matrix, BBM-BBM system.

1. Introduction

Wavelet analysis is a new numerical concept which allows one to represent a function in terms of a set of basic functions, called wavelets, which are localized in space, it is used for solving difficult problems in mathematics, physics, and engineering, with modern applications as diverse as wave propagation, data compression, image processing, pattern recognition, computer graphics.[1]
In recent years, there has been an increase among scientists and engineers in the application of wavelet technique as well as the numerical solution to solve non-linear problems, and in the numerical solutions of integral and differential equations. The main advantage of the wavelet technique is its ability to transform complex problems into a system of algebraic equations.[1]
The wavelets consist of a family of functions constructed from dilation and translation of a single function called the mother wavelet. When the dilation parameter a and the translation parameter b vary continuously, we have the following family of continuous wavelets:

$$\psi_{a,b}(t) = |a|^{\frac{-1}{2}} \psi\left(\frac{t-b}{a}\right), \quad a,b \in \Re, \quad a \neq 0, \tag{1}$$

If we restrict the parameters a and b to discrete values which are

$$a = a_0^{-k},\ b = n b_0\, a_0^{-k},\ a_0 > 1,\ b_0 > 0,$$

Where n and k are positive integers, then we have the following family of discrete wavelets

$$\psi_{k,n}(t) = |a_0|^{\frac{k}{2}} \psi\left(a_0^k\, t - n b_0\right), \tag{2}$$

Where $\psi_{k,n}(t)$ forms a wavelet basis for $L^2(\Re)$. In particular, when $a_0 = 2, b_0 = 1$, then $\psi_{k,n}(t)$ forms an orthonormal basis [4].
A few authors have studied the CAS wavelet in the numerical solutions. In (2006), Youse, S.A. and Banifatemi, A., [13], have used the CAS wavelets to approximate the solution of linear integral equations. Also, in [2,3,5], used this wavelet to solve optimal control systems by time-dependent coefficients, Volterra integral equations of the second kind, integro-differential equations, respectively. Also, this method can be used to obtain the numerical solutions of the fractional Fredholm integral equations [11]. In (2012), Barzkar, A., Assari, P., and Mehrpouya, M. A., [4], presented a computational method for solving Fredholm Hammerstein integral equations of the second kind based on the CAS wavelets. The method utilizes CAS wavelets constructed on the unit interval as basis in the Galerkin method and reduces the solution of the Hammerstein integral equation to the solution of a nonlinear system of algebraic equations. In (2013), Mingxu Yi, et al., [9], studied Numerical Solutions of Fractional Integro-differential equations of Bratu Type by using CAS Wavelets. In (2007-2009), Dougalis, et al., [6,7], studied the Boussinesq system of BBM-BBM type in two space dimensions.

We organized our paper as follows. In section 2, the CAS wavelets and function approximation is presented. Section 3 we will derive the general formula for the operational matrices for the CAS wavelets analytically. In section 4 derivation of general form for solving 2D-PDE using CAS wavelets method. Section 5 we explain the 2D non-linear BBM-BBM system and we use CAS wavelets to solve this system. Section 6 numerical results are presented. Concluding remarks are given in section 7.

2. CAS Wavelets

CAS wavelets $\psi_{n,m}(x)=\psi(k,\hat{n},m,x)$ have four arguments; $n=1,2,3,4,\ldots 2^k$, k can assume any positive integer, m is any integer, and x is the normalized time. The orthonormal CAS wavelets are defined on the interval $[0,1)$ by [4]:

$$\psi_{n,m}(x)=\begin{cases} 2^{\frac{k}{2}}\, CAS_m\left(2^k x-n+1\right), & for \quad \frac{n-1}{2^k}\le x<\frac{n}{2^k} \\ 0, & otherwise \end{cases} \tag{3}$$

where

$$CAS_m(x)=\cos(2m\pi x)+\sin(2m\pi x), \tag{4}$$

and $m\in\{-M,\ldots,M\}$. The CAS wavelets are orthonormal with respect to the weight function w(x)=1. For m=0, the CAS wavelets have the following form:

$$\psi_{n,0}(x)=2^{\frac{k}{2}}B_n(x)=2^{\frac{k}{2}}\begin{cases} 1, & for \quad \frac{n-1}{2^k}\le x<\frac{n}{2^k} \\ 0, & otherwise \end{cases}$$

where $\{B_n(x)\}_{n=1}^{2^k}$ are a basis set that are called the Block-Pulse functions (BPFs) over the interval [0,1].

Any function $f(x)\in L^2[0,1)$ may be expanded using CAS wavelets as [4]:

$$f(x)=\sum_{n=1}^{\infty}\sum_{m\in z}C_{n,m}\,\psi_{n,m}(x), \tag{5}$$

Where $C_{n,m}=<f(t),\psi_{n,m}(t)>$ are inner product.

If the infinite series in equation (5) is truncated, then equation (5) can be written as

$$f(x)=\sum_{n=1}^{2^k}\sum_{m=-M}^{M}C_{n,m}\,\psi_{n,m}(x)=C^T\,\psi(x), \tag{6}$$

Where C and $\psi(x)$ are $2^k(2M+1)*1$ matrices given by

$$C=\left[c_{1,(-M)},c_{1,(-M+1)},\ldots\ldots c_{1,M},c_{2,(-M)},\ldots\ldots c_{2^k,(-M)},\ldots\ldots c_{2^k,M}\right]^T$$

$$\psi(x)=\left[\psi_{1,(-M)}(x),\psi_{1,(-M+1)}(x),\ldots\ldots\psi_{1,M}(x),\psi_{2,(-M)}(x),\ldots\ldots\psi_{2^k,(-M)}(x),\ldots\ldots\psi_{2^k,M}(x)\right]^T$$

For convenience, in numerical solution, we rewrite equation (6) as follows [4]:

Let $i=n(2M+1)-M+m$, then

$$f(x)=\sum_{i=1}^{2^k(2M+1)}C_i\,\psi_{2^k(2M+1),i}(x), \tag{7}$$

Now, we define N-Dimensional CAS wavelets.

Definition: Any function $f(X)$, $X=(x_1,x_2,\cdots,x_N)$ be a n-time continuously and partial differentiable on the interval $I=[0,1]\times[0,1]\times\ldots[0,1]$ in R^N and bounded on each of these intervals, can be expanded by a CAS series of finite terms:

$$f_{2^{k_i},M_i}(X)=\sum_{n_N=1}^{2^{k_N}}\sum_{m_N=-M_N}^{M_N}\cdots\sum_{n_2=1}^{2^{k_2}}\sum_{m_2=-M_2}^{M_2}\sum_{n_1=1}^{2^{k_1}}\sum_{m_1=-M_1}^{M_1}C_{n_1,m_1,n_2,m_2,\ldots,n_N,m_N}\,CAS_{n_1,m_1}(x_1)CAS_{n_2,m_2}(x_2)\ldots CAS_{n_N,m_N}(x_N) \tag{8}$$

where $C_{j_1,k_1,j_2,k_2,\ldots,j_N,k_N}$ are the CAS coefficients defined by:

$$C_{n_1,m_1,n_2,m_2,\ldots,n_N,m_N}=\underbrace{\int_0^1\int_0^1\ldots\int_0^1}_{N-tims} f(x_1',x_2',\ldots,x_N')CAS_{n_1,m_1}(x_1')CAS_{n_2,m_2}(x_2')\ldots CAS_{n_N,m_N}(x_N')dx_1'\,dx_2'\ldots dx_N' \tag{9}$$

and $CAS_{n_i,m_i}(x_i)$, $i=1,2,\cdots N$, are the CAS wavelets defined in equation (3).

3. Analytical Integration of CAS Wavelets

In this section, we derive a general formula for n of integrals for the CAS wavelets defined in equation (3) analytically; these integrations, in turn, can be used in the numerical solution of differential equations of higher order. The CAS wavelets is defined in terms of trigonometric functions whose integration is periodical.
If we integrate equation (3) from (0) to (x), we obtain

$$P_{2^k(2M+1),1}(x)=\int_0^x \psi_{n,m}(x')\,dx'$$

Then

$$P_{2^k(2M+1),1}(x)=\begin{cases} 0 & 0\le x<\dfrac{n-1}{2^k} \\ \begin{cases} 2^{\frac{k}{2}}\dfrac{1}{2^{k+1}\pi m}\left(\sin\left(2\pi m\left(2^k x-n+1\right)\right)-\cos\left(2\pi m\left(2^k x-n+1\right)\right)\right) \\ -2^{\frac{k}{2}}\dfrac{-1}{2^{k+1}\pi m}, \end{cases} & \dfrac{n-1}{2^k}\le x<\dfrac{n}{2^k} \\ \left\{2^{\frac{k}{2}}\dfrac{1}{2^{k+1}\pi m}\left(\sin(2\pi m)-\cos(2\pi m)\right)-2^{\frac{k}{2}}\dfrac{-1}{2^{k+1}\pi m}\right., & \dfrac{n}{2^k}\le x<1 \end{cases}$$

Repeating the integration n times, we find

$$P_{i,v}(x)=\begin{cases} 0 & 0\le x<\dfrac{n-1}{2^k} \\ \begin{cases} 2^{\frac{k}{2}}\dfrac{(-1)^{a_v}}{\left(2^{k+1}\pi m\right)^v}\cos\left(2\pi m\left(2^k x-n+1\right)\right)+2^{\frac{k}{2}}\dfrac{(-1)^{b_v}}{\left(2^{k+1}\pi m\right)^v}\sin\left(2\pi m\left(2^k x-n+1\right)\right) \\ -\displaystyle\sum_{jj=0}^{v-1}2^{\frac{k}{2}}\dfrac{1}{jj!}\dfrac{(-1)^{a_v}}{\left(2^{k+1}\pi m\right)^{v-jj}}\left(x-\dfrac{n-1}{2^k}\right)^{jj}, \end{cases} & \dfrac{n-1}{2^k}\le x<\dfrac{n}{2^k} \\ \left\{\displaystyle\sum_{j=0}^{v-1}\dfrac{1}{j!}\left(x-\dfrac{n}{2^k}\right)^j\right. & \dfrac{n}{2^k}\le x<1 \\ \left.\cdot\left(2^{\frac{k}{2}}\dfrac{(-1)^{a_v}}{\left(2^{k+1}\pi m\right)^v}\cos(2\pi m)+2^{\frac{k}{2}}\dfrac{(-1)^{b_v}}{\left(2^{k+1}\pi m\right)^v}\sin(2\pi m)-\displaystyle\sum_{jj=0}^{v-1}2^{\frac{k}{2}}\dfrac{1}{jj!}\dfrac{(-1)^{a_v}}{\left(2^{k+1}\pi m\right)^{v-jj}}\left(\dfrac{1}{2^k}\right)^{jj}\right), \end{cases} \tag{10}$$

where

$$a_v=\begin{cases} 0 & if\quad v=3,4,7,8,11,12,\ldots \\ 1 & if\quad v=1,2,5,6,9,10,\ldots \end{cases}$$

and

$$b_v=\begin{cases} 0 & if\quad v=1,4,5,8,9,12,\ldots \\ 1 & if\quad v=2,3,6,7,10,11,\ldots \end{cases}$$

The case v=0 corresponds to CAS function $\psi_{n,m}(x)$ in equation (3). In order to solve boundary value problems we substitute x=B in equation (10). In certain cases where v=1, we find in interval $x\in[A,B]$:

$$q_1(i)=P_{2^k(2M+1),1}(x)=\begin{cases} 0 & 0\le x<\dfrac{n-1}{2^k} \\ \begin{cases} 2^{\frac{k}{2}}\dfrac{1}{2^{k+1}\pi m}\left(\sin\left(2\pi m\left(2^k B-n+1\right)\right)-\cos\left(2\pi m\left(2^k B-n+1\right)\right)\right) \\ -2^{\frac{k}{2}}\dfrac{-1}{2^{k+1}\pi m}, \end{cases} & \dfrac{n-1}{2^k}\le x<\dfrac{n}{2^k} \\ \left\{2^{\frac{k}{2}}\dfrac{1}{2^{k+1}\pi m}\left(\sin(2\pi m)-\cos(2\pi m)\right)-2^{\frac{k}{2}}\dfrac{-1}{2^{k+1}\pi m}\right., & \dfrac{n}{2^k}\le x<1 \end{cases}$$

4. General Two-Dimensional Partial Differential equation of Higher Order

Two-dimensional partial differential equation of higher order can be considered as follows [12]:

$$F\left(x_*, y_*, t, u, Du, D^2u, \ldots, D^{\lambda+\beta+\alpha}u\right) = f(x_*, y_*, t),$$
$$D^{\lambda+\beta+\alpha}u = \frac{\partial^{(\lambda+\beta+\alpha)}u(x_*, y_*, t)}{\partial t^{\lambda}\ \partial x_*^{\beta}\ \partial y_*^{\alpha}}, \tag{11}$$

such that $f(x_*, y_*, t)$ is known function or constant.

The independent variables t, x_* and y_* belong to the domain Ω; $t \in [A_1, B_1]$, $x_* \in [A_2, B_2]$, *and* $y_* \in [A_3, B_3]$ which has the boundary $\partial\Omega$. We have to calculate the function $u(x_*, y_*, t)$ which satisfies the required initial and boundary conditions.

4.1 Derivation of General Form for Solving 2D-PDE Using 2D Wavelets Method

The solution adopting the 2D CAS wavelets method starts by dividing the time interval $t \in [A_1, B_1]$ into N equal parts of length $\Delta t = (B_1 - A_1)/N$ and let $t_s = (s-1)\Delta t$, s=1,2,…,N.

Since the wavelets are defined for x,y∈[0,1], we must first normalize with regard to $x_* \in [A_2, B_2]$, *and* $y_* \in [A_3, B_3]$ changing the variables as follows:

$$x = \frac{1}{L_x}(x_* - A_2) \quad , \quad L_x = B_2 - A_2,$$
$$y = \frac{1}{L_y}(y_* - A_3) \quad , \quad L_y = B_3 - A_3,$$

We divide the space intervals $x \in [0,1]$, *and* $y \in [0,1]$ into M_2 and M_3 equal parts. We assume that the solution could be in the form:

$$\frac{\partial^{(\lambda+\beta+\alpha)}u(x,y,t)}{\partial t^{\lambda}\ \partial x^{\beta}\ \partial y^{\alpha}} = \sum_{l=1}^{M_3}\sum_{j=1}^{M_2} C_{j,l}\ \Psi_j(x)\,\Psi_l(y), \tag{12}$$

where the elements $C_{j,l}$ are constants in the subinterval $t \in (t_s, t_{s+1}]$, $\Psi_i(t)$ and $\Psi_j(x)$ are defined in equation (3).

Integration (12) $\lambda - times$ in relation to (t) from (t_s) to (t), we obtain

$$\underbrace{\int_{t_s}^{t}\int_{t_s}^{t}\cdots\int_{t_s}^{t}}_{\lambda-times} \frac{\partial^{(\lambda+\beta+\alpha)}u(x,y,t)}{\partial t^{\lambda}\ \partial x^{\beta}\ \partial y^{\alpha}}(dt)^{\lambda} = \underbrace{\int_{t_s}^{t}\int_{t_s}^{t}\cdots\int_{t_s}^{t}}_{\lambda-times} \sum_{l=1}^{M_3}\sum_{j=1}^{M_2} C_{j,l}\ \Psi_j(x)\,\Psi_l(y)(dt)^{\lambda}$$

Then

$$\frac{\partial^{(\beta+\alpha)}u(x,y,t)}{\partial x^{\beta}\ \partial y^{\alpha}} = \frac{(t-t_s)^{\lambda}}{(\lambda)!}\sum_{l=1}^{M_3}\sum_{j=1}^{M_2} C_{j,l}\ \Psi_j(x)\,\Psi_l(y) + \sum_{ii=0}^{\lambda-1}\frac{(t-t_s)^{ii}}{(ii)!}\frac{\partial^{(ii+\beta+\alpha)}u(x,y,t_s)}{\partial t^{ii}\ \partial x^{\beta}\ \partial y^{\alpha}} \tag{13}$$

by integrating equation (13) $\beta - times$ with regard to (x) from (0) to (x), we obtain

$$\begin{aligned}\frac{\partial^{(\alpha)}u(x,y,t)}{\partial y^{\alpha}} &= \frac{(t-t_s)^{\lambda}}{(\lambda)!}\sum_{l=1}^{M_3}\sum_{j=1}^{M_2} C_{j,l}\ P_{\beta,j}(x)\,\Psi_l(y) + \sum_{jj=0}^{\beta-1}\frac{x^{jj}}{(jj)!}\frac{\partial^{(jj+\alpha)}u(0,y,t)}{\partial x^{jj}\ \partial y^{\alpha}} \\ &+ \sum_{ii=0}^{\lambda-1}\frac{(t-t_s)^{ii}}{(ii)!}\frac{\partial^{(ii+\alpha)}u(x,y,t_s)}{\partial t^{ii}\ \partial y^{\alpha}} - \sum_{ii=0}^{\lambda-1}\frac{(t-t_s)^{ii}}{(ii)!}\sum_{jj=0}^{\beta-1}\frac{x^{jj}}{(jj)!}\frac{\partial^{(ii+jj+\alpha)}u(0,y,t_s)}{\partial t^{ii}\ \partial x^{jj}\ \partial y^{\alpha}}\end{aligned} \tag{14}$$

Integrating equation (14) $\alpha - times$ with regard to (y) from (0) to (y), we obtain

$$\begin{aligned}u(x,y,t) &= \frac{(t-t_s)^{\lambda}}{(\lambda)!}\sum_{l=1}^{M_3}\sum_{j=1}^{M_2} C_{j,l}\ P_{\beta,j}(x)\,P_{\alpha,l}(y) + \sum_{ll=0}^{\alpha-1}\frac{y^{ll}}{(ll)!}\frac{\partial^{(ll)}u(x,0,t)}{\partial y^{ll}} \\ &+ \sum_{jj=0}^{\beta-1}\frac{x^{jj}}{(jj)!}\frac{\partial^{(jj)}u(0,y,t)}{\partial x^{jj}} - \sum_{jj=0}^{\beta-1}\frac{x^{jj}}{(jj)!}\sum_{ll=0}^{\alpha-1}\frac{y^{ll}}{(ll)!}\frac{\partial^{(jj+ll)}u(0,0,t)}{\partial x^{jj}\ \partial y^{ll}} \\ &+ \sum_{ii=0}^{\lambda-1}\frac{(t-t_s)^{ii}}{(ii)!}\frac{\partial^{(ii)}u(x,y,t_s)}{\partial t^{ii}} - \sum_{ii=0}^{\lambda-1}\frac{(t-t_s)^{ii}}{(ii)!}\sum_{ll=0}^{\alpha-1}\frac{y^{ll}}{(ll)!}\frac{\partial^{(ii+ll)}u(x,0,t_s)}{\partial t^{ii}\ \partial y^{ll}} \\ &- \sum_{ii=0}^{\lambda-1}\frac{(t-t_s)^{ii}}{(ii)!}\sum_{jj=0}^{\beta-1}\frac{x^{jj}}{(jj)!}\frac{\partial^{(ii+jj)}u(0,y,t_s)}{\partial t^{ii}\ \partial x^{jj}} \\ &+ \sum_{ii=0}^{\lambda-1}\frac{(t-t_s)^{ii}}{(ii)!}\sum_{jj=0}^{\beta-1}\frac{x^{jj}}{(jj)!}\sum_{ll=0}^{\alpha-1}\frac{y^{ll}}{(ll)!}\frac{\partial^{(ii+jj+ll)}u(0,0,t_s)}{\partial t^{ii}\ \partial x^{jj}\ \partial y^{ll}}\end{aligned} \tag{15}$$

In this formula, the integrals $P_{\beta,j}(x)$ and $P_{\alpha,l}(y)$ are calculated depending on the equation (10). The other terms in equation (15) are calculate from the initial and boundary conditions and according to the type of the initial and boundary conditions.

4.2 Derivation of General Form for Solving 2D-PDE Using 3D Wavelets Method

The solution of the general form of 2D-PDE (11) by the 3D wavelets method starting by normalizing with regard to the time interval $t_* \in [A_1, B_1]$ and the space intervals $x_* \in [A_2, B_2]$, *and* $y_* \in [A_3, B_3]$ assuming that:

$$t = \frac{1}{L_t}(t_* - A_1), \quad , \quad L_t = B_1 - A_1, \quad x = \frac{1}{L_x}(x_* - A_2) \quad , \quad L_x = B_2 - A_2,$$

$$y = \frac{1}{L_y}(y_* - A_3) \quad , \quad L_y = B_3 - A_3,$$

Later, we divide the time interval $t \in [0,1]$ and the space intervals $x \in [0,1]$, *and* $y \in [0,1]$ into M_1, M_2 and M_3 equal parts, respectively.
We assume that the solution as follows:

$$\frac{\partial^{(\lambda+\beta+\alpha)} u(x,y,t)}{\partial t^{\lambda}\, \partial x^{\beta}\, \partial y^{\alpha}} = \sum_{l=1}^{M_{3_1}} \sum_{j=1}^{M_2} \sum_{i=1}^{M_1} C_{i,j,l}\ \Psi_i(t)\, \Psi_j(x)\, \Psi_l(y), \tag{16}$$

where the elements $C_{i,j,l}$ are constants.

We integrate (16) $\lambda - times$ with regard to (t) from (0) to (t), we get:

$$\frac{\partial^{(\beta+\alpha)} u(x,y,t)}{\partial x^{\beta}\, \partial y^{\alpha}} = \sum_{l=1}^{M_{3_1}} \sum_{j=1}^{M_2} \sum_{i=1}^{M_1} C_{i,j,l}\ P_{\lambda,i}(t)\, \Psi_j(x)\, \Psi_l(y) + \sum_{ii=0}^{\lambda-1} \frac{(t)^{ii}}{(ii)!} \frac{\partial^{(ii+\beta+\alpha)} u(x,y,0)}{\partial t^{ii}\, \partial x^{\beta}\, \partial y^{\alpha}} \tag{17}$$

By integrating (17) $\beta - times$ with regard to (x) and $\alpha - times$ with regard to (y), we obtain

$$\begin{aligned} u(x,y,t) = & \sum_{l=1}^{M_{3_1}} \sum_{j=1}^{M_2} \sum_{i=1}^{M_1} C_{i,j,l}\ P_{\lambda,i}(t)\, P_{\beta,j}(x)\, P_{\alpha,l}(y) + \sum_{ll=0}^{\alpha-1} \frac{y^{ll}}{(ll)!} \frac{\partial^{(ll)} u(x,0,t)}{\partial y^{ll}} \\ & + \sum_{jj=0}^{\beta-1} \frac{x^{jj}}{(jj)!} \frac{\partial^{(jj)} u(0,y,t)}{\partial x^{jj}} - \sum_{jj=0}^{\beta-1} \frac{x^{jj}}{(jj)!} \sum_{ll=0}^{\alpha-1} \frac{y^{ll}}{(ll)!} \frac{\partial^{(jj+ll)} u(0,0,t)}{\partial x^{jj}\, \partial y^{ll}} \\ & + \sum_{ii=0}^{\lambda-1} \frac{(t)^{ii}}{(ii)!} \frac{\partial^{(ii)} u(x,y,0)}{\partial t^{ii}} - \sum_{ii=0}^{\lambda-1} \frac{(t)^{ii}}{(ii)!} \sum_{ll=0}^{\alpha-1} \frac{y^{ll}}{(ll)!} \frac{\partial^{(ii+ll)} u(x,0,0)}{\partial t^{ii}\, \partial y^{ll}} \\ & - \sum_{ii=0}^{\lambda-1} \frac{(t)^{ii}}{(ii)!} \sum_{jj=0}^{\beta-1} \frac{x^{jj}}{(jj)!} \frac{\partial^{(ii+jj)} u(0,y,0)}{\partial t^{ii}\, \partial x^{jj}} + \sum_{ii=0}^{\lambda-1} \frac{(t)^{ii}}{(ii)!} \sum_{jj=0}^{\beta-1} \frac{x^{jj}}{(jj)!} \sum_{ll=0}^{\alpha-1} \frac{y^{ll}}{(ll)!} \frac{\partial^{(ii+jj+ll)} u(0,0,0)}{\partial t^{ii}\, \partial x^{jj}\, \partial y^{ll}} \end{aligned} \tag{18}$$

5. Application of CAS Wavelets Method for Solving Non-linear 2D Coupled Benjamin-Bona-Mahony System

In this section, we apply the CAS wavelets method for solving non-linear coupled Benjamin-Bona-Mahony (BBM-BBM) system which has the form [10]:

$$\begin{aligned} & \eta_{t_*} + \nabla . V + \nabla . \eta V - b\, \Delta \eta_{t_*} = 0 \\ & V_{t_*} + \nabla\, \eta + \frac{1}{2} \nabla\, |V|^2 - d\, \Delta V_{t_*} = 0 \end{aligned} \tag{19}$$

for $X_* = [x_*, y_*] \in \Omega$, $t_* > 0$, *where* Ω is a bounded open set in IR^2, the initial data:

$$\eta(X_*,0) = \eta_0(X_*) \quad , \quad V(X_*,0) = V_0(X_*) \qquad t_* \geq 0 \quad , \quad X_* \in \partial\Omega \tag{20}$$

and zero Dirichlet homogenous boundary conditions [10]:

$$\eta(X_*,t_*) = 0 \quad , \quad V(X_*,t_*) = 0 \qquad t_* \geq 0 \quad , \quad X_* \in \partial\Omega \tag{21}$$

Since the wavelets are defined for $x \in [0,1]$, we must first normalize the system (19) and the initial-boundary condition (20) and (21) with regard to $X_* = [x_*, y_*]$ and the domain $\Omega = [x_{\min}, x_{\max}] \times [y_{\min}, y_{\max}]$, we change the variables:

$$x = \frac{1}{L_x}(x_* - x_{\min}) \quad , \qquad L_x = x_{\max} - x_{\min} \ , \ y = \frac{1}{L_y}(y_* - y_{\min}) \ , \qquad L_y = y_{\max} - y_{\min}$$

then the system (19) become

$$\begin{aligned}
&\eta_t + \frac{1}{L_x}u_x + \frac{1}{L_y}v_y + \frac{1}{L_x}\eta u_x + \frac{1}{L_x}\eta_x u + \frac{1}{L_y}\eta v_y + \frac{1}{L_y}\eta_y v - \frac{b}{L_x^2}\eta_{xxt} - \frac{b}{L_y^2}\eta_{yyt} = 0\\
&u_t + \frac{1}{L_x}\eta_x + \frac{1}{L_x}u u_x + \frac{1}{L_x}v v_x - \frac{d}{L_x^2}u_{xxt} - \frac{d}{L_y^2}u_{yyt} = 0\\
&v_t + \frac{1}{L_y}\eta_y + \frac{1}{L_y}u u_y + \frac{1}{L_y}v v_y - \frac{d}{L_x^2}v_{xxt} - \frac{d}{L_y^2}v_{yyt} = 0
\end{aligned} \tag{22}$$

with the initial and boundary conditions:

$$\eta(X,0) = \eta_0(X) \quad , \quad V(X,0) = V_0(X) \qquad t \geq 0 \quad , \quad X \in \partial\Omega \tag{23}$$

$$\eta(X,t) = 0 \quad , \quad V(X,t) = 0 \qquad t \geq 0 \quad , \quad X \in \partial\Omega \tag{24}$$

we have two cases in solving the system (22):

5.1 Case 1:

In this case, we use 2D CAS wavelets method to solve BBM-BBM system by comparing the system (22) with the general form of two-dimensional PDE (11), we substitute $\beta = \alpha = 2$ *and* $\lambda = 1$ in the equation (15), and obtain

$$\begin{aligned}
u(x,y,t) = {} & (t-t_s)\sum_{l=1}^{M_3}\sum_{j=1}^{M_2} C_{j,l}\, P_{2,j}(x)\, P_{2,l}(y) + u(x,y,t_s)\\
& + y\left(u_y(x,0,t) - u_y(x,0,t_s)\right) - y\left(u_y(0,0,t) - u_y(0,0,t_s)\right)\\
& - x\left(u_x(0,0,t) - u_x(0,0,t_s)\right) + x\left(u_x(0,y,t) - u_x(0,y,t_s)\right)\\
& - xy\left(u_{xy}(0,0,t) - u_{xy}(0,0,t_s)\right) + \left(u(x,0,t) - u(x,0,t_s)\right)\\
& + \left(u(0,y,t) - u(0,y,t_s)\right) - \left(u(0,0,t) - u(0,0,t_s)\right)
\end{aligned} \tag{25}$$

Taking x=1 and y=1 at the boundary condition, we get

$$\begin{aligned}
u(x,y,t) = {} & (t-t_s)\sum_{l=1}^{M_3}\sum_{j=1}^{M_2} C_{j,l}\, P_{2,j}(x)\, P_{2,l}(y) - x(t-t_s)\sum_{l=1}^{M_3}\sum_{j=1}^{M_2} C_{j,l}\, q_2(j)\, P_{2,l}(y)\\
& - y(t-t_s)\sum_{l=1}^{M_3}\sum_{j=1}^{M_2} C_{j,l}\, P_{2,j}(x)\, q_2(l) + xy\,(t-t_s)\sum_{l=1}^{M_3}\sum_{j=1}^{M_2} C_{j,l}\, q_2(j)\, q_2(l)\\
& + u(x,y,t_s) + (1-y)\left(u(x,0,t) - u(x,0,t_s)\right) + (1-x)\left(u(0,y,t) - u(0,y,t_s)\right)\\
& - (1-x-y+xy)\left(u(0,0,t) - u(0,0,t_s)\right) + x\left(u(1,y,t) - u(1,y,t_s)\right)\\
& - x(1-y)\left(u(1,0,t) - u(1,0,t_s)\right) + y\left(u(x,1,t) - u(x,1,t_s)\right)\\
& - y(1-x)\left(u(0,1,t) - u(0,1,t_s)\right) - xy\left(u(1,1,t) - u(1,1,t_s)\right)
\end{aligned} \tag{26}$$

where $q_2(j) = P_{2,j}(1) \qquad j = 1,2,\cdots,M_2,$.

Similarly, we find $\eta(x,y,t)$, $v(x,y,t)$ by the wavelets form with changing the coefficients $C_{j,l}$ to the coefficients $E_{j,l}$ and $D_{j,l}$ in regard to $\eta(x,y,t)$ and $v(x,y,t)$ respectively.

Now, we substitute $u(x,y,t)$, $\eta(x,y,t)$, and $v(x,y,t)$ with their derivatives in the system (22) and divide the domain $\Omega = [0,1]\times[0,1]$ into collocation points M_2 and M_3 equal parts:

$$x_r = \frac{(r-0.5)}{M_2}, \qquad r = 1,2,\cdots,M_2 \;,\; y_k = \frac{(k-0.5)}{M_3}, \qquad k = 1,2,\cdots,M_3$$

and replacing t by t_{s+1} , $\Delta t = t_{s+1} - t_s$; the system (22) becomes:

$$\begin{aligned}
&\sum_{l=0}^{M_3}\sum_{j=0}^{M_2} E_{j,l}\, R^{(1)}_{j,l,r,k} + \sum_{l=0}^{M_3}\sum_{j=0}^{M_2} C_{j,l}\, R^{(2)}_{j,l,r,k} + \sum_{l=0}^{M_3}\sum_{j=0}^{M_2} D_{j,l}\, R^{(3)}_{j,l,r,k} = F^{(1)}(r,k,s),\\
&\sum_{l=0}^{M_3}\sum_{j=0}^{M_2} E_{j,l}\, R^{(4)}_{j,l,r,k} + \sum_{l=0}^{M_3}\sum_{j=0}^{M_2} C_{j,l}\, R^{(5)}_{j,l,r,k} + \sum_{l=0}^{M_3}\sum_{j=0}^{M_2} D_{j,l}\, R^{(6)}_{j,l,r,k} = F^{(2)}(r,k,s),\\
&\sum_{l=0}^{M_3}\sum_{j=0}^{M_2} E_{j,l}\, R^{(7)}_{j,l,r,k} + \sum_{l=0}^{M_3}\sum_{j=0}^{M_2} C_{j,l}\, R^{(8)}_{j,l,r,k} + \sum_{l=0}^{M_3}\sum_{j=0}^{M_2} D_{j,l}\, R^{(9)}_{j,l,r,k} = F^{(3)}(r,k,s),
\end{aligned} \tag{27}$$

where $s \in \{1,2,\cdots,N\}, r \in \{1,2,\cdots,M_2\}, k \in \{1,2,\cdots,M_3\}$ and

$$R^{(1)}_{j,l,r,k} = \left(P_2(j,r) - x_r\, q_2(j) - \frac{b}{L_x^2}\Psi(j,r)\right)(P_2(l,k) - y_k\, q_2(l)) - \frac{b}{L_y^2}(P_2(j,r) - x_r\, q_2(j))\Psi(l,k),$$

$$R^{(2)}_{j,l,r,k} = R^{(4)}_{j,l,r,k} = \frac{\Delta t}{L_x}(P_1(j,r) - q_2(j))(P_2(l,k) - y_k\, q_2(l)),$$

$$R^{(3)}_{j,l,r,k} = R^{(7)}_{j,l,r,k} = \frac{\Delta t}{L_y}(P_2(j,r) - x_r\, q_2(j))(P_1(l,k) - q_2(l)),$$

$$R^{(5)}_{j,l,r,k} = R^{(9)}_{j,l,r,k} = \left(P_2(j,r) - x_r\, q_2(j) - \frac{d}{L_x^2}\Psi(j,r)\right)(P_2(l,k) - y_k\, q_2(l)) - \frac{d}{L_y^2}(P_2(j,r) - x_r\, q_2(j))\Psi(l,k),$$

$$R^{(6)}_{j,l,r,k} = R^{(8)}_{j,l,r,k} = 0,$$

$$F^1(r,k,s) = F_1(x_r, y_k, t_s),$$

$$F^2(r,k,s) = F_2(x_r, y_k, t_s),$$

$$F^3(r,k,s) = F_3(x_r, y_k, t_s),$$

note that

$$q_2(j) = P_{2,j}(1), \quad \Psi(j,r) = \Psi_j(x_r), \quad P_v(j,r) = P_{v,j}(x_r), \qquad j = 1,2,\cdots,M_2, \quad r = 1,2,\cdots,M_2, \quad v = 1,2$$

and

$$\begin{aligned}
F_1(x,y,t) = {} & \frac{-1}{L_x}(\eta(x,y,t_s)u_x(x,y,t_s) + \eta_x(x,y,t_s)u(x,y,t_s)) \\
& \frac{-1}{L_y}(\eta(x,y,t_s)v_y(x,y,t_s) + \eta_y(x,y,t_s)v(x,y,t_s)) \\
& -((1-y)\eta_t(x,0,t) + (1-x)\eta_t(0,y,t) - (1-x-y+xy)\eta_t(0,0,t) \\
& + x\eta_t(1,y,t) - x(1-y)\eta_t(1,0,t) + y\eta_t(x,1,t) - y(1-x)\eta_t(0,1,t) - xy\ \eta_t(1,1,t)) \\
& -\frac{1}{L_x}(u_x(x,y,t_s) + (1-y)(u_x(x,0,t) - u_x(x,0,t_s)) - (u(0,y,t) - u(0,y,t_s)) \\
& + (1-y)(u(0,0,t) - u(0,0,t_s)) + (u(1,y,t) - u(1,y,t_s)) - (1-y)(u(1,0,t) - u(1,0,t_s)) \\
& + y(u_x(x,1,t) - u_x(x,1,t_s)) + y(u(0,1,t) - u(0,1,t_s)) - y\ (u(1,1,t) - u(1,1,t_s))) \\
& -\frac{1}{L_y}(v_y(x,y,t_s) - (v(x,0,t) - v(x,0,t_s)) + (1-x)(v_y(0,y,t) - v_y(0,y,t_s)) \\
& + (1-x)(v(0,0,t) - v(0,0,t_s)) + x(v_y(1,y,t) - v_y(1,y,t_s)) + x(v(1,0,t) - v(1,0,t_s)) \\
& + (v(x,1,t) - v(x,1,t_s)) - (1-x)(v(0,1,t) - v(0,1,t_s)) - x\ (v(1,1,t) - v(1,1,t_s))) \\
& + \frac{b}{L_x^2}((1-y)\eta_{xxt}(x,0,t) + y\eta_{xxt}(x,1,t)) + \frac{b}{L_y^2}((1-x)\eta_{yyt}(0,y,t) + x\eta_{yyt}(1,y,t))
\end{aligned} \tag{28a}$$

$$\begin{aligned}
F_2(x,y,t) = {} & \frac{-1}{L_x}(u(x,y,t_s)u_x(x,y,t_s) + v(x,y,t_s)v_x(x,y,t_s)) \\
& -((1-y)u_t(x,0,t) + (1-x)u_t(0,y,t) - (1-x-y+xy)u_t(0,0,t) \\
& + xu_t(1,y,t) - x(1-y)u_t(1,0,t) + yu_t(x,1,t) - y(1-x)u_t(0,1,t) - xy\ u_t(1,1,t)) \\
& -\frac{1}{L_x}(\eta_x(x,y,t_s) + (1-y)(\eta_x(x,0,t) - \eta_x(x,0,t_s)) - (\eta(0,y,t) - \eta(0,y,t_s)) \\
& + (1-y)(\eta(0,0,t) - \eta(0,0,t_s)) + (\eta(1,y,t) - \eta(1,y,t_s)) - (1-y)(\eta(1,0,t) - \eta(1,0,t_s)) \\
& + y(\eta_x(x,1,t) - \eta_x(x,1,t_s)) + y(\eta(0,1,t) - \eta(0,1,t_s)) - y\ (\eta(1,1,t) - \eta(1,1,t_s))) \\
& + \frac{d}{L_x^2}((1-y)u_{xxt}(x,0,t) + yu_{xxt}(x,1,t)) + \frac{d}{L_y^2}((1-x)u_{yyt}(0,y,t) + xu_{yyt}(1,y,t))
\end{aligned} \tag{28b}$$

$$\begin{aligned}
F_3(x,y,t) = {} & \frac{-1}{L_x}(u(x,y,t_s)u_y(x,y,t_s) + v(x,y,t_s)v_y(x,y,t_s)) \\
& -((1-y)v_t(x,0,t) + (1-x)v_t(0,y,t) - (1-x-y+xy)v_t(0,0,t) \\
& + xv_t(1,y,t) - x(1-y)v_t(1,0,t) + yv_t(x,1,t) - y(1-x)v_t(0,1,t) - xy\ v_t(1,1,t)) \\
& -\frac{1}{L_y}(\eta_y(x,y,t_s) - (\eta(x,0,t) - \eta(x,0,t_s)) + (1-x)(\eta_y(0,y,t) - \eta_y(0,y,t_s)) \\
& + (1-x)(\eta(0,0,t) - \eta(0,0,t_s)) + x(\eta_y(1,y,t) - \eta_y(1,y,t_s)) + x(\eta(1,0,t) - \eta(1,0,t_s)) \\
& + (\eta(x,1,t) - \eta(x,1,t_s)) - (1-x)(\eta(0,1,t) - \eta(0,1,t_s)) - x\ (\eta(1,1,t) - \eta(1,1,t_s))) \\
& + \frac{d}{L_x^2}((1-y)v_{xxt}(x,0,t) + yv_{xxt}(x,1,t)) + \frac{d}{L_y^2}((1-x)v_{yyt}(0,y,t) + xv_{yyt}(1,y,t))
\end{aligned} \tag{28c}$$

We transform the system (27) from the fourth-order matrices into a second-order matrices as follows. To simplify the calculation, we take $M_2 = M_3 = M$.

Let $\eta = M(j-1)+i \quad , \quad \mu = M(r-1)+s$

Now, the system (27) leads to the following:

$$\sum_{\eta=1}^{3M^2} S(\mu,\eta)B(\eta) = F(\mu), \qquad for \quad 1 \le \mu \le 3M^2$$

Now, we get the following system of linear equations

$$S \cdot B = F \tag{29}$$

Here B and F are $1\times 3M^2$ vectors and S is a $(3M^2)\times(3M^2)$ matrix defined as follows:

$$B = [B_1 \quad B_2 \quad B_3]^T,$$

$$F = [F_1 \quad F_2 \quad F_3]^T,$$

$$S = \begin{bmatrix} S_1 & S_2 & S_3 \\ S_4 & S_5 & S_6 \\ S_7 & S_8 & S_9 \end{bmatrix},$$

After solving the linear system (29) and finding the wavelets coefficients $E_{j,l}, C_{j,l}$ *and* $D_{j,l}$, we find the solutions for $\eta(x,y,t)$, $u(x,y,t)$ and $v(x,y,t)$ from equation (26).

5.2 Case 2:

In this case, we apply the 3D CAS wavelets method for solving non-linear BBM-BBM system (22) by using equation (18) with the boundary conditions at x=1, y=1, we obtain,

$$\begin{aligned} u(x,y,t) = & \sum_{l=1}^{M_3}\sum_{j=1}^{M_2}\sum_{i=1}^{M_1} C_{i,j,l}\, P_{1,i}(t)\, P_{2,j}(x)\, P_{2,l}(y) - x\sum_{l=1}^{M_3}\sum_{j=1}^{M_2}\sum_{i=1}^{M_1} C_{i,j,l}\, P_{1,i}(t)\, q_2(j)\, P_{2,l}(y) \\ & - y\sum_{l=1}^{M_3}\sum_{j=1}^{M_2}\sum_{i=1}^{M_1} C_{i,j,l}\, P_{1,i}(t)\, P_{2,j}(x)\, q_2(l) + x\,y\sum_{l=1}^{M_3}\sum_{j=1}^{M_2}\sum_{i=1}^{M_1} C_{i,j,l}\, P_{1,i}(t)\, q_2(j)\, q_2(l) \\ & + u(x,y,0) + (1-y)(u(x,0,t)-u(x,0,0)) + (1-x)(u(0,y,t)-u(0,y,0)) \\ & - (1-x-y+x\,y)(u(0,0,t)-u(0,0,0)) + x(u(1,y,t)-u(1,y,0)) \\ & - x(1-y)(u(1,0,t)-u(1,0,0)) + y(u(x,1,t)-u(x,1,0)) \\ & - y(1-x)(u(0,1,t)-u(0,1,0)) - x\,y\,(u(1,1,t)-u(1,1,0)) \end{aligned} \tag{30}$$

Similarly, we find $\eta(x,y,t)$ and $v(x,y,t)$ with changing the coefficients $C_{i,j,l}$ to the coefficients $E_{i,j,l}$ and $D_{i,j,l}$ with regard to $\eta(x,y,t)$ and $v(x,y,t)$, respectively.

We substitute $u(x,y,t)$, $\eta(x,y,t)$, and $v(x,y,t)$ with their derivatives in the system (22), and divide the domain $\Omega = [0,1]\times[0,1]$ and the time interval $t \in [0,1]$ into collocation points M_1, M_2 and M_3 equal parts in the following manner:

$t_s = \dfrac{(s-0.5)}{M_1}, \quad s = 1,2,\cdots,M_1, \quad x_r = \dfrac{(r-0.5)}{M_2}, \quad r = 1,2,\cdots,M_2$ and $y_k = \dfrac{(k-0.5)}{M_3}, \quad k = 1,2,\cdots,M_3$

Then

$$\begin{aligned} & \sum_{l=1}^{M_3}\sum_{j=1}^{M_2}\sum_{i=1}^{M_1} E_{i,j,l}\, R^{(1)}_{i,j,l,s,r,k} + \sum_{l=1}^{M_3}\sum_{j=1}^{M_2}\sum_{i=1}^{M_1} C_{i,j,l}\, R^{(2)}_{i,j,l,s,r,k} + \sum_{l=1}^{M_3}\sum_{j=1}^{M_2}\sum_{i=1}^{M_1} D_{i,j,l}\, R^{(3)}_{i,j,l,s,r,k} = F^{(1)}(r,k,s), \\ & \sum_{l=1}^{M_3}\sum_{j=1}^{M_2}\sum_{i=1}^{M_1} E_{i,j,l}\, R^{(4)}_{i,j,l,s,r,k} + \sum_{l=1}^{M_3}\sum_{j=1}^{M_2}\sum_{i=1}^{M_1} C_{i,j,l}\, R^{(5)}_{i,j,l,s,r,k} + \sum_{l=1}^{M_3}\sum_{j=1}^{M_2}\sum_{i=1}^{M_1} D_{i,j,l}\, R^{(6)}_{i,j,l,s,r,k} = F^{(2)}(r,k,s), \\ & \sum_{l=1}^{M_3}\sum_{j=1}^{M_2}\sum_{i=1}^{M_1} E_{i,j,l}\, R^{(7)}_{i,j,l,s,r,k} + \sum_{l=1}^{M_3}\sum_{j=1}^{M_2}\sum_{i=1}^{M_1} C_{i,j,l}\, R^{(8)}_{i,j,l,s,r,k} + \sum_{l=1}^{M_3}\sum_{j=1}^{M_2}\sum_{i=1}^{M_1} D_{i,j,l}\, R^{(9)}_{i,j,l,s,r,k} = F^{(3)}(r,k,s), \end{aligned} \tag{31}$$

where $s \in \{1,2,\cdots,M_1\}, r \in \{1,2,\cdots,M_2\}, k \in \{1,2,\cdots,M_3\}$ and

$$R^{(1)}_{i,j,l,s,r,k} = \Psi(i,s)\left(P_2(j,r) - x_r\, q_2(j) - \frac{b}{L_x^2}\Psi(j,r)\right)(P_2(l,k) - y_k\, q_2(l))$$
$$-\frac{b}{L_y^2}\Psi(i,s)(P_2(j,r) - x_r\, q_2(j))\Psi(l,k),$$
$$R^{(2)}_{i,j,l,s,r,k} = R^{(4)}_{i,j,l,s,r,k} = \frac{1}{L_x} P_1(i,s)(P_1(j,r) - q_2(j))(P_2(l,k) - y_k\, q_2(l)),$$
$$R^{(3)}_{i,j,l,s,r,k} = R^{(7)}_{i,j,l,s,r,k} = \frac{1}{L_y} P_1(i,s)(P_2(j,r) - x_r\, q_2(j))(P_1(l,k) - q_2(l)),$$
$$R^{(5)}_{i,j,l,s,r,k} = R^{(9)}_{i,j,l,s,r,k} = \Psi(i,s)\left(P_2(j,r) - x_r\, q_2(j) - \frac{d}{L_x^2}\Psi(j,r)\right)(P_2(l,k) - y_k\, q_2(l))$$
$$-\frac{d}{L_y^2}\Psi(i,s)(P_2(j,r) - x_r\, q_2(j))\Psi(l,k),$$
$$R^{(6)}_{i,j,l,s,r,k} = R^{(8)}_{i,j,l,s,r,k} = 0,$$
$$F^1(r,k,s) = F_1(x_r, y_k, t_s),$$
$$F^2(r,k,s) = F_2(x_r, y_k, t_s),$$
$$F^3(r,k,s) = F_3(x_r, y_k, t_s),$$

It is clear that the wavelet coefficients $E_{i,j,l}, C_{i,j,l}$ *and* $D_{i,j,l}$ can be obtained by solving the linear system (31). For simplicity, we take $M_1 = M_2 = M_3 = M$ and transform the system (31) into a form with second-order matrices in the following:

$$\eta = (M)^2(j-1) + M(l-1) + i \quad , \quad \mu = (M)^2(r-1) + M(k-1) + s,$$

Now the system (31) can be rewritten in the following form:

$$\sum_{\eta=1}^{3M^3} S(\mu,\eta)B(\eta) = F(\mu) \qquad for \quad 1 \le \mu \le 3M^3,$$

which results in the following system of linear equations:

$$S \cdot B = F \tag{32}$$

where B and F are $1 \times 3M^3$ vectors and S is a $3M^3 \times 3M^3$ matrix. Solving the linear system (32) and finding the wavelets coefficients $E_{i,j,l}, C_{i,j,l}$ *and* $D_{i,j,l}$, we find the solutions for $\eta(x,y,t)$, $u(x,y,t)$ and $v(x,y,t)$ from equation (30).

6. Numerical Experiments:

In what follows, we present the results of two-dimensional BBM-BBM system (22) which has been used in studying of Tsunami wave generation and propagation. The significance of studying the propagation of a Tsunami wave Back to being from the complex phenomenon's and its natural disasters which represent a major risk for populations..

We present a series of numerical experiments aimed at checking the accuracy of the numerical scheme which was derived in the previous sections and illustrating the behavior of the solutions for the non-linear system with the error norm δ_e [8]:

$$\delta_e = \frac{1}{N}\left\| u^{exact} - u^{num} \right\|_2 = \frac{1}{N}\sqrt{\sum_{j=0}^{N}\left| u_j^{exact} - u_j^{num} \right|^2}$$

we took zero Dirichlet homogenous boundary conditions for η, u *and* v on the whole boundary in the square $[0,1]\times[0,1]$ with exact solutions: [10]

$$\eta(x,y,t) = e^t . \sin(\pi x).(y-1).y$$
$$u(x,y,t) = e^t . x.\cos(3\pi x/2).\sin(\pi y)$$
$$v(x,y,t) = e^t . \sin(\pi x).\cos(3\pi y/2).y$$

Table (1), show a comparison between the exact solutions and the numerical solutions using the 2D and 3D CAS wavelets method.

Table (1)

Numerical Solution of Two-Dimensional BBM-BBM system using CAS wavelets method when $b=1/6$, $d=1/6$, $\Delta t=1/\mathrm{M}$, $t=0.95$.

x	y	2D CAS $\eta_{Num}(x,y,t)$	3D CAS $\eta_{Num}(x,y,t)$	$\eta_{Exact}(x,y,t)$
0.05	0.05	-7.45835101e-003	-7.45763003e-003	-7.43769954e-003
0.15	0.15	-5.80395857e-002	-5.80475826e-002	-5.79388044e-002
0.25	0.25	-1.32844676e-001	-1.32839774e-001	-1.32708534e-001
0.35	0.35	-2.02928978e-001	-2.02902500e-001	-2.02896644e-001
0.45	0.45	-2.44522533e-001	-2.44505063e-001	-2.44685204e-001
0.55	0.55	-2.44418198e-001	-2.44373968e-001	-2.44685204e-001
0.65	0.65	-2.02616706e-001	-2.02558146e-001	-2.02896644e-001
0.75	0.75	-1.32483262e-001	-1.32435003e-001	-1.32708534e-001
0.85	0.85	-5.77885793e-002	-5.77416020e-002	-5.79388044e-002
0.95	0.95	-7.41817542e-003	-7.40892604e-003	-7.43769954e-003
x	y	2D CAS $u_{Num}(x,y,t)$	3D CAS $u_{Num}(x,y,t)$	$u_{Exact}(x,y,t)$
0.05	0.05	7.62788736e-003	7.62818181e-003	7.61283717e-003
0.15	0.15	5.19074537e-002	5.19228575e-002	5.18317794e-002
0.25	0.25	6.77355540e-002	6.77933207e-002	6.77138100e-002
0.35	0.35	-2.47269467e-002	-2.45382925e-002	-2.44909034e-002
0.45	0.45	-2.32950600e-001	-2.32758043e-001	-2.32450306e-001
0.55	0.55	-4.64186430e-001	-4.64085376e-001	-4.63618740e-001
0.65	0.65	-5.78477612e-001	-5.78521389e-001	-5.77917662e-001
0.75	0.75	-4.90809080e-001	-4.90939033e-001	-4.90426796e-001
0.85	0.85	-2.50977512e-001	-2.51036388e-001	-2.50854956e-001
0.95	0.95	-3.47378613e-002	-3.47353334e-002	-3.47259295e-002
x	y	2D CAS $v_{Num}(x,y,t)$	3D CAS $v_{Num}(x,y,t)$	$v_{Exact}(x,y,t)$
0.05	0.05	7.62838205e-003	7.62867802e-003	7.61283717e-003
0.15	0.15	5.19107134e-002	5.19260226e-002	5.18317794e-002
0.25	0.25	6.77420861e-002	6.77989916e-002	6.77138100e-002
0.35	0.35	-2.47199244e-002	-2.45338328e-002	-2.44909034e-002
0.45	0.45	-2.32942337e-001	-2.32754092e-001	-2.32450306e-001
0.55	0.55	-4.64174478e-001	-4.64080104e-001	-4.63618740e-001
0.65	0.65	-5.78461379e-001	-5.78512346e-001	-5.77917662e-001
0.75	0.75	-4.90793719e-001	-4.90927956e-001	-4.90426796e-001
0.85	0.85	-2.50965506e-001	-2.51025959e-001	-2.50854956e-001
0.95	0.95	-3.47366745e-002	-3.47340809e-002	-3.47259295e-002

In Table (2), we compute the norm of the error δ_e between the exact solution and the numerical solution adopting the CAS wavelets method with different values for Subdivisions wavelets M and k=1.

Table (2)

The error δ_e for the numerical solution of the two-dimensional BBM-BBM system using CAS wavelets with $\Delta t=1/2M, t=0.96875, b=1/6$, $d=1/6$, $L_x=L_y=1$.

The method	**M**	δ_e **for** $\eta(x,y,t)$	δ_e **for** $u(x,y,t)$	δ_e **for** $v(x,y,t)$
2D CAS	**M=1**	**1.9428e-005**	**4.7788e-005**	**4.6768e-005**
	M=2	**1.3152e-005**	**3.0403e-005**	**2.9790e-005**
3D CAS	**M=1**	**3.5259e-006**	**8.5844e-006**	**8.4392e-006**
	M=2	**1.4710e-006**	**3.1699e-006**	**3.1087e-006**

In figure (1), the maximum error for $\eta-wave$, which approximates the last time t=1, to illustrate the impact of error accumulation on the solution by 2D CAS wavelets method.

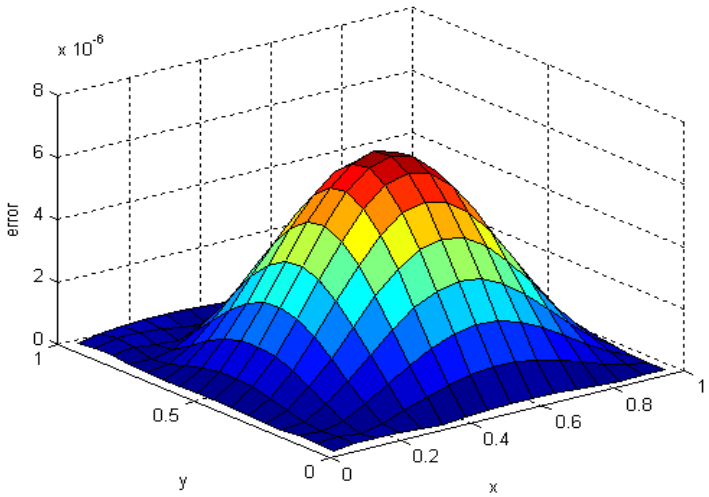

Figure (1): The maximum error at $t = 0.9375$ by using 2D CAS wavelets method.

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Conclusion

In this paper, we develop an accurate and efficient the CAS wavelets method for solving two-dimensional partial differential equations of higher order. The benefits of the wavelet approaches are sparse matrices of representation, fast transformation, and possibility of implementation of fast algorithms.

We derived the general formula for the CAS wavelets to find the numerical solution for two-dimensional partial differential equations of higher order and applied this formula to solve the non-linear BBM-BBM system. It's worth mentioning that the wavelets solution provides excellent results even for small values of (M and k) as note in Table (1). Also when $2^k(2M+1) = 36$, $2^k(2M+1) = 44$, …, we can obtain the results closer to the exact values. For two-dimensional system, when the values M and k are not sufficiently large value, this means that Δt is not sufficiently a small value which is equal to $1/2^k(2M+1)$, we obtained more precision by the 3D CAS wavelets than 2D CAS wavelets.

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