International Journal of Enhanced Research in Science, Technology & Engineering
ISSN: 2319-7463, Vol. 5 Issue 9, September-2016

On the Flow of Dusty Gases with Pressure - dependent Viscosities through Porous Structures

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ABSTRACT

Marble's equations governing the flow of a viscous dusty gas are averaged over a representative elementary volume to develop equations governing the flow of a viscous dusty gas with pressure-dependent viscosities in variable-porosity media.

Keywords: Dusty Gase, Pressure-Dependent Viscosities, Porous Media.

1. INTRODUCTION

Flow of fluids with pressure-dependent viscosities in porous media and in free-space has received considerable attention in the literature due to the various applications this type of flow has in industry and in nature (cf. [1-10] and the references therein). Models of single-phase flow of a fluid with pressure dependent viscosity through porous structures have been reported in the literature and describe flow through porous media with different microstructures. Functional forms of dependence of viscosity on pressure have been reported by a number of researchers and used in the solutions to fluid flow problems (cf. [1-7] and the references therein).

Less studied, however, is the flow of dusty gases with pressure-dependent viscosities through porous sediments. These would find applications in practical problems involving very high pressures, such as recovery of ground water, contaminant transport in deep repositories, the study of flow of oil emulsions with a dispersed gas or water phase, the design of liquid-dust separators, and the study of lubrication problems involving a fluid phase dispersed in another. This is the subject matter of the current work whose objective is to develop a set of equations, using volume averaging, to describe the flow of a fluid-particle mixture through a porous medium assuming that each of the phases possesses a pressure-dependent viscosity.

Modelling dusty gas flow through porous media using the continuum approach is not a new undertaking, [11, 12, 13, 14]. In fact, a number of models based on Saffman's dusty gas model [15] and on Marble's dusty gas equations, among others [16,17], exist in the literature. Some of the available models take into account the porous microstructure and the type of boundary conditions to be used. Marble's dusty gas model [17] assumes a volume fraction for each of the phases involved and a viscosity associated with the dust-phase. This makes it possible to impose no-slip conditions on the dusty-phase velocity on solid boundaries. In Saffman's dusty gas model, [15] it is assumed that the dust particles possess a small dust particle number density (number of particles per unit volume). This model is characterized by the absence of a dust-phase viscosity, thus making it difficult to implement a no-slip boundary condition on solid walls. Dust particles may settle on solid boundaries, reflect back into the flow field, or set into motion other particles already settled on solid boundaries. Dust-phase velocities on solid boundaries remain quantities to be determined.

The current work employs the continuum approach to modelling dusty gas flow through porous media based on Marble's model. Intrinsic volume averaging rules are applied to Marble's equations over a control volume that is composed of a rigid porous material of variable porosity. It is assumed that there is no fragmentation of the solid matrix, while the flowing phases possess viscosities that are pressure-dependent.

2. GOVERNING EQUATIONS

The steady, incompressible fluid flow of a viscous dusty gas in free-space is governed by the following coupled set of balance equations, [17]:

Fluid-phase continuity

$$\nabla \bullet \gamma_f \vec{u} = 0 \qquad \ldots(1)$$

Fluid-phase momentum

$$\rho_f \gamma_f \vec{u} \bullet \nabla \vec{u} = -\nabla(\gamma_f p) + \nabla \bullet \vec{T}_f + \rho_d \gamma_d \frac{(\vec{v} - \vec{u})}{\tau} \qquad \ldots(2)$$

Dust-phase continuity

$$\nabla \bullet \gamma_d \vec{v} = 0 \qquad \ldots(3)$$

Dust-phase momentum

$$\rho_d \gamma_d \vec{v} \bullet \nabla \vec{v} = \nabla \bullet \vec{T}_d - \rho_d \gamma_d \frac{(\vec{v} - \vec{u})}{\tau} \qquad \ldots(4)$$

$$\gamma_f + \gamma_d = 1 \qquad \ldots(5)$$

$$\mu_f = \mu_f(p) \qquad \ldots(6)$$

$$\mu_d = \mu_d(p) \qquad \ldots(7)$$

wherein

$$\vec{T}_f = \mu_f \gamma_f (\nabla \vec{u} + \nabla \vec{u}^T) \qquad \ldots(8)$$

$$\vec{T}_d = \mu_d \gamma_d (\nabla \vec{v} + \nabla \vec{v}^T) \qquad \ldots(9)$$

and $\vec{u}$ is the fluid-phase velocity field, $\vec{v}$ is the particle-phase velocity field, p is the fluid (mixture) pressure, μ_f is the fluid-phase viscosity, μ_d is the particle-phase viscosity, ρ_f is the fluid-phase density, ρ_d is the particle-phase density, γ_d is the particle-phase volume fraction, γ_f is the fluid-phase volume fraction, and τ is the particle relaxation time (defined as the time required for a particle to adjust its path to that of fluid elements, and measured in terms of the ratio of particle density to coefficient of resistance of the particles). When relaxation time is small, $\tau \to 0$, $\vec{v} \to \vec{u}$.

With the help of continuity equations (1) and (3), equations (2) and (4) are written in the following forms, respectively, where inertial terms are written in dyadic forms that are suitable for volume averaging:

$$\rho_f \nabla \bullet \gamma_f \vec{u}\vec{u} = -\nabla(\gamma_f p) + \nabla \bullet \vec{T}_f + \rho_d \gamma_d \frac{(\vec{v} - \vec{u})}{\tau} \qquad \ldots(10)$$

$$\rho_d \nabla \bullet \gamma_f \vec{v}\vec{v} = \nabla \bullet \vec{T}_d - \rho_d \gamma_d \frac{(\vec{v} - \vec{u})}{\tau}. \qquad \ldots(11)$$

Equations (1), (3), (10) and (11) are to be intrinsically averaged over a Representative Elementary Volume (REV), [18-22], using the averaging procedure and rules described in **Appendix I**.

Taking the averages of both sides of (1) and (3) and using rule (v) yields

$$< \nabla \bullet \gamma_f \vec{u} >= \nabla \bullet \varphi < \gamma_f \vec{u} >_\varphi + \frac{1}{V} \int_S \gamma_f \vec{u} \bullet \vec{n} dS = 0. \qquad \ldots(12)$$

$$< \nabla \bullet \gamma_d \vec{v} >= \nabla \bullet \varphi < \gamma_d \vec{v} >_\varphi + \frac{1}{V} \int_S \gamma_d \vec{v} \bullet \vec{n} dS = 0 \qquad \ldots(13)$$

where φ is the medium porosity, defined by equation (I.3) in **Appendix I**.

Using the divergence theorem, namely:

$$\int_S \vec{F} \bullet \vec{n} dS = \int_V \nabla \bullet \vec{F} dV \qquad \ldots(14)$$

and making use of continuity equations (1) and (3), the surface integrals in (12) and (13) vanish as follows:

$$\int_S \gamma_f \vec{u} \bullet \vec{n} dS = \int_V \nabla \bullet (\gamma_f \vec{u}) dV = 0. \qquad \ldots(15)$$

$$\int_S \gamma_d \vec{v} \bullet \vec{n} dS = \int_V \nabla \bullet (\gamma_d \vec{v}) dV = 0. \qquad \ldots(16)$$

Using (15) in (12), and (16) in (13), gives respectively

$$\nabla \bullet \varphi < \gamma_f \vec{u} >_\varphi = 0 \qquad \ldots(17)$$

$$\nabla \bullet \varphi < \gamma_d \vec{v} >_\varphi = 0. \qquad \ldots(18)$$

Equations (17) and (18) can be written with the help of (iv), respectively, as

$$\nabla \bullet \varphi < \gamma_f >_\varphi < \vec{u} >_\varphi + \nabla \bullet \varphi < \gamma_f^\circ \vec{u}^\circ >_\varphi = 0 \qquad \ldots(19)$$

$$\nabla \bullet \varphi < \gamma_d >_\varphi < \vec{v} >_\varphi + \nabla \bullet \varphi < \gamma_d^\circ \vec{v}^\circ >_\varphi = 0. \qquad \ldots(20)$$

Momentum equations (10) and (11) are averaged term by term, using rules (i), (iii) and (iv), to obtain

$$< \rho_d \gamma \frac{(\vec{v} - \vec{u})}{\tau} > = \frac{\rho_d}{\tau} \varphi < \gamma >_\varphi [< (\vec{v} >_\varphi - < \vec{u} >_\varphi] + \frac{\rho_d}{\tau} \varphi < \gamma^\circ (\vec{v}^\circ - \vec{u}^\circ) >_\varphi. \qquad \ldots(21)$$

Using rules (ii) and (iv)

$$< \nabla(\gamma_f p) > = \varphi \nabla < \gamma_f >_\varphi < p >_\varphi + \varphi \nabla < \gamma_f^\circ p^\circ >_\varphi + \frac{1}{V} \int_S (\gamma_f p)^\circ \vec{n} dS. \qquad \ldots(22)$$

Using rule *(v)*, we get:

$$< \nabla \bullet \vec{T}_f > = \nabla \bullet \varphi < \vec{T}_f >_\varphi + \frac{1}{V} \int_S \vec{T}_f \bullet \vec{n} dS \qquad \ldots(23)$$

$$< \nabla \bullet \vec{T}_d > = \nabla \bullet \varphi < \vec{T}_d >_\varphi + \frac{1}{V} \int_S \vec{T}_d \bullet \vec{n} dS. \qquad \ldots(24)$$

Using rules (i) and (v), we obtain

$$< \rho_f \nabla \bullet \gamma_f \vec{u}\vec{u} > = \rho_f \nabla \bullet \varphi < \gamma_f >_\varphi < \vec{u} >_\varphi < \vec{u} >_\varphi + \rho_f \nabla \bullet \varphi < \gamma^\circ{}_f \vec{u}^\circ \vec{u}^\circ >_\varphi + \frac{\rho_f}{V} \int_S \gamma_f \vec{u}\vec{u} \bullet \vec{n} dS \qquad \ldots(25)$$

$$< \rho_d \nabla \bullet \gamma_d \vec{v}\vec{v} > = \rho_d \nabla \bullet \varphi < \gamma_d >_\varphi < \vec{v} >_\varphi < \vec{v} >_\varphi + \rho_d \nabla \bullet \varphi < \gamma_d^\circ \vec{v}^\circ \vec{v}^\circ >_\varphi + \frac{\rho_d}{V} \int_S \gamma_d \vec{v}\vec{v} \bullet \vec{n} dS. \qquad \ldots(26)$$

Upon invoking rule (vi), the surface integrals on the right-hand-sides of equations (25) and (26) vanish. Momentum equations (10) and 11) thus take the following averaged forms, respectively, once equations (21)-(26) are used:

$$\rho_f \nabla \bullet \varphi < \gamma_f >_\varphi < \vec{u} >_\varphi < \vec{u} >_\varphi + \rho_f \nabla \bullet \varphi < \gamma^\circ{}_f \vec{u}^\circ \vec{u}^\circ >_\varphi$$
$$= -\varphi \nabla < \gamma_f >_\varphi < p >_\varphi - \varphi \nabla < \gamma_f^\circ p^\circ >_\varphi - \frac{1}{V} \int_S (\gamma_f p)^\circ \vec{n} dS + \nabla \bullet \varphi < \vec{T}_f >_\varphi + \frac{1}{V} \int_S \vec{T}_f \bullet \vec{n} dS \qquad \ldots(27)$$
$$+ \frac{\rho_d}{\tau} \varphi < \gamma >_\varphi [< (\vec{v} >_\varphi - < \vec{u} >_\varphi] + \frac{\rho_d}{\tau} \varphi < \gamma_d^\circ (\vec{v}^\circ - \vec{u}^\circ) >$$

$$\rho_d \nabla \bullet \varphi < \gamma_d >_\varphi < \vec{v} >_\varphi < \vec{v} >_\varphi + \rho_d \nabla \bullet \varphi < \gamma_d^\circ \vec{v}^\circ \vec{v}^\circ >_\varphi = \nabla \bullet \varphi < \vec{T}_d >_\varphi + \frac{1}{V} \int_S \vec{T}_d \bullet \vec{n} dS \qquad \ldots(28)$$
$$- \frac{\rho_d}{\tau} \varphi < \gamma >_\varphi [< (\vec{v} >_\varphi - < \vec{u} >_\varphi] - \frac{\rho_d}{\tau} \varphi < \gamma_d^\circ (\vec{v}^\circ - \vec{u}^\circ) >_\varphi$$

3. ANALYSIS OF DEVIATION TERMS AND SURFACE INTEGRALS

The deviation terms in the averaged continuity and momentum equations (19), (20), (27) and (28), and the surface integrals in (27) and (28), contain all the necessary information to quantify the effects of the phases involved on each other and the effects of the porous medium on the mixture constituents.

In equations (19) and (20), the terms $<\gamma_d^\circ \vec{v}^\circ>_\varphi$ and $<\gamma_f \vec{u}^\circ)>_\varphi$, and the term $<\gamma_d^\circ(\vec{v}^\circ-\vec{u}^\circ)>_\varphi$ in equations (27) and (28), which can be expressed in the form:

$$
\begin{aligned}
&<\gamma_d^\circ(\vec{v}^\circ-\vec{u}^\circ)>_\varphi=<\gamma_d^\circ\vec{v}^\circ>_\varphi-<\gamma_d^\circ\vec{u}^\circ>_\varphi \\
&=<\gamma_d^\circ>_\varphi<\vec{v}^\circ>_\varphi-<\gamma_d^\circ>_\varphi<\vec{u}^\circ>_\varphi+<\gamma_d^{\circ\circ}\vec{v}^{\circ\circ}>_\varphi-<\gamma_d^{\circ\circ}\vec{u}^{\circ\circ}>_\varphi+...
\end{aligned} \quad ...(29)
$$

involve products of deviations of average phase velocities and the average deviations of the phase volume fractions, in addition to average deviations of products of deviations. They represent dispersions of dust and fluid particles due to fluctuations in the dust-phase and fluid-phase average velocity vectors. In the absence of high velocity and porosity gradients, the average fluctuations are small and their products are negligible.

Accordingly, $<\gamma_d^\circ(\vec{v}^\circ-\vec{u}^\circ)>_\varphi$ is dropped from (27) and (28), and continuity equations (19) and (20) take the following averaged forms, respectively:

$$\nabla\bullet\varphi<\gamma_f>_\varphi<\vec{u}>_\varphi=0 \quad ...(29)$$

$$\nabla\bullet\varphi<\gamma_d>_\varphi<\vec{v}>_\varphi=0. \quad ...(30)$$

The deviation terms $<\gamma_f^\circ\vec{u}^\circ\vec{u}^\circ>_\varphi$ and $<\gamma_d^\circ\vec{v}^\circ\vec{v}^\circ>_\varphi$ appearing in equations (27) and (28) are related to hydrodynamic dispersion of the average phase velocities, [19]. It is the sum of mechanical dispersion (due to tortuosity of the flow path in the porous microstructure) and molecular diffusion of the phase vorticities. In the absence of high velocity and porosity gradients, the average fluctuations are small and their products are negligible.

Likewise, the deviation term in equation (27) involving $<\gamma_f^\circ p^\circ>_\varphi$ can be expanded as

$$<\gamma_f^\circ p^\circ>_\varphi=<\gamma_f^\circ>_\varphi<p^\circ>_\varphi+<\gamma_f^{\circ\circ}>_\varphi<p^{\circ\circ}>_\varphi+<\gamma_f^{\circ\circ\circ}>_\varphi<p^{\circ\circ\circ}>_\varphi+... \quad ...(31)$$

which involves sums of products of small deviations and products of their deviations, smoothed over an REV, hence negligible.

The term $\frac{1}{V}\int_S(\gamma_f p)^\circ\vec{n}dS$ in (27) represents fluctuations of the product of fluid-phase volume fraction and pressure, at the fluid-porous medium interface. Le Bars and Grae Worster [23] argue that $\frac{1}{V}\int_S p^\circ\vec{n}dS$ is small, hence can be neglected. Since deviations in the fluid-phase volume fraction are also small, the product of the deviations $\gamma_f^\circ p^\circ$ and $\frac{1}{V}\int_S(\gamma_f p)^\circ\vec{n}dS$ can be ignored.

Following Le Bars and Grae Worster [23], the terms $\frac{1}{V}\int_S\vec{T}_f\bullet\vec{n}dS$ and $\frac{1}{V}\int_S\vec{T}_d\bullet\vec{n}dS$ can be recognized as interfacial viscous stress exchanges corresponding to the microscopic momentum exchanges of the fluid-dust mixture with the solid porous matrix. They depend on the morphology of the porous matrix, the viscosities of the fluid- and dust-phases (hence depend on pressure of the mixture), the relative velocity of the fluid-phase and the solid-phase, and on the phase volume fractions. By letting

$$\vec{I}_f=\nabla\vec{u}+\nabla\vec{u}^T \quad ...(32)$$

$$\vec{I}_d=\nabla\vec{v}+\nabla\vec{v}^T \quad ...(33)$$

equations (23) and (24) can be written respectively as

$$<\nabla\bullet\vec{T}_f>=\nabla\bullet\varphi<\mu_f\gamma_f\vec{I}_f>_\varphi+\frac{1}{V}\int_S\mu_f\gamma_f\vec{I}_f\bullet\vec{n}dS \quad \ldots(34)$$

$$<\nabla\bullet\vec{T}_d>=\nabla\bullet\varphi<\mu_d\gamma_d\vec{I}_d>_\varphi+\frac{1}{V}\int_S\mu_d\gamma_d\vec{I}_d\bullet\vec{n}dS\,. \quad \ldots(35)$$

Using averaging rule (iv), and ignoring products of deviations, we can write:

$$\nabla\bullet\varphi<\mu_f\gamma_f\vec{I}_f>_\varphi=\nabla\bullet(\varphi<\mu_f>_\varphi<\gamma_f>_\varphi<\vec{I}_f>_\varphi) \quad \ldots(36)$$

$$\nabla\bullet\varphi<\mu_d\gamma_d\vec{I}_d>_\varphi=\nabla\bullet(\varphi<\mu_d>_\varphi<\gamma_d>_\varphi<\vec{I}_d>_\varphi)\,. \quad \ldots(37)$$

The surface integrals in (34) and (35) can be written, respectively, as:

$$\frac{1}{V}\int_S\vec{T}_f\bullet\vec{n}dS=\frac{1}{V}\int_S\mu_f\gamma_f\vec{I}_f\bullet\vec{n}dS \quad \ldots(38)$$

$$\frac{1}{V}\int_S\vec{T}_d\bullet\vec{n}dS=\frac{1}{V}\int_S\mu_d\gamma_d\vec{I}_d\bullet\vec{n}dS\,. \quad \ldots(39)$$

The integral terms on the Right-Hand-Sides of (34) and (35) are proportional to $\frac{1}{V}\int_S\mu_f\gamma_f\frac{\partial\vec{u}}{\partial n}dS$ and $\frac{1}{V}\int_S\mu_f\gamma_f\frac{\partial\vec{v}}{\partial n}dS$, respectively, and can be expressed as functions of the porosity, velocity of the porous matrix, average velocity, average viscosity, and average phase fraction, namely, [24-28],

$$\frac{1}{V}\int_S\mu_f\gamma_f\frac{\partial\vec{u}}{\partial n}dS=F_f(\varphi,<\mu_f>_\varphi,<\gamma_f>_\varphi,\vec{v}_s,<\vec{u}>_\varphi) \quad \ldots(40)$$

$$\frac{1}{V}\int_S\mu_f\gamma_f\frac{\partial\vec{v}}{\partial n}dS=F_d(\varphi,<\mu_d>_\varphi,<\gamma_d>_\varphi,\vec{v}_s,<\vec{v}>_\varphi) \quad \ldots(41)$$

where $\vec{v}_s$ is the velocity of the solid porous matrix.

As a first approximation, and following [22, 25], these can be expressed in terms of the sum of a shear force integral that accounts for the viscous drag effects which dominate in the Darcy regime (that is, for small Reynolds number) and an inertial force integral that accounts for inertial drag effects which dominate in the Forchheimer regime (that is, for high Reynolds number flow). The Right-Hand-Sides of (36) and (37) can be expressed in forms proportional to the relative velocity of the phases involved and the velocity of the porous solid matrix, namely $\vec{v}_s-<\vec{u}>_\varphi$ and $\vec{v}_s-<\vec{v}>_\varphi$. Assuming that the porous matrix is stationary, hence $\vec{v}_s=\vec{0}$, equations (36) and (37) take the forms:

$$F_f(\varphi,<\mu_f>_\varphi,<\gamma_f>_\varphi,<\vec{u}>_\varphi)=-\varphi(f_1+f_2)<\mu_f>_\varphi<\gamma_f>_\varphi<\vec{u}>_\varphi \quad \ldots(42)$$

$$F_d(\varphi,<\mu_d>_\varphi,<\gamma_d>_\varphi,<\vec{v}>_\varphi)=-\varphi(f_1+f_2)<\mu_d>_\varphi<\gamma_d>_\varphi<\vec{v}>_\varphi \quad \ldots(43)$$

where f_1 is the *velocity-independent* viscous shear geometric factor that depends on the geometry of the porous medium and gives rise to the Darcy resistance, and f_2 the *velocity-dependent* inertial geometric factor that gives rise to the Forchheimer inertial term.

Expressions for f_1 and f_2 require a mathematical description of the porous matrix and its microstructure, (cf. [18-28] where detailed analysis of porous microstructures for various types of media is provided). In the current work, we express (38) and (39) in terms of the following equivalent and customary expressions of Darcy resistance (expressions (44) and (45), below), and Forchheimer term, expressions (46) and (47) below, written here for each of the flowing phases and take into account the average phase fractions:

$$-f_1<\mu_f>_\varphi<\gamma_f>_\varphi\varphi<\vec{u}>_\varphi=-\frac{<\gamma_f>_\varphi}{\eta}\varphi<\mu_f>_\varphi\varphi<\vec{u}>_\varphi \quad \ldots(44)$$

$$-f_1 < \mu_d >_\varphi < \gamma_d >_\varphi \varphi < \vec{v} >_\varphi = -\frac{< \gamma_d >_\varphi}{\eta} \varphi < \mu_d >_\varphi \varphi < \vec{v} >_\varphi \quad \ldots(45)$$

$$-f_2 < \mu_f >_\varphi < \gamma_f >_\varphi \varphi < \vec{u} >_\varphi = -\frac{\rho_f \delta < \gamma_f >_\varphi}{\sqrt{\eta}} \varphi < \vec{u} >_\varphi \left|\varphi < \vec{u} >_\varphi\right| \quad \ldots(46)$$

$$-f_2 < \mu_d >_\varphi < \gamma_d >_\varphi \varphi < \vec{v} >_\varphi = -\frac{\rho_d \delta < \gamma_d >_\varphi}{\sqrt{\eta}} \varphi < \vec{v} >_\varphi \left|\varphi < \vec{v} >_\varphi\right| \quad \ldots(47)$$

In terms of the factor f_1, hydrodynamic permeability, η is given by, (*cf.* [22]), $\eta = \frac{\varphi}{f_1}$. In order to illustrate the connection between permeability, porosity and the geometric factor, **Table 1** given f_1 using Ergun's equation and the Kozeny-Carman relation, and shows the expression for the hydrodynamic permeability.

Table 1. Geometric Factor and Hydrodynamic Permeability using Ergun's Equation and Kozeny-Carman Relation.

	Geometric Factor f_1	**Hydrodynamic Permeability** $\eta = \frac{\varphi}{f_1}$
Ergun's Equation	$\frac{150(1-\varphi)^2}{\varphi^2 d_p{}^2}$	$\frac{\varphi^3 d_p{}^2}{150(1-\varphi)^2}$
Kozeny-Carman Relation	$\frac{180(1-\varphi)^2}{\varphi^2 d_m{}^2}$	$\frac{\varphi^3 d_m{}^2}{180(1-\varphi)^2}$

d_p *is the average pore diameter in a channel-like porous material.*
d_m *is the median diameter of spherical particle constituents of the solid porous matrix*

Using (36), (37), (44)-(47), and ignoring deviation terms discussed above, equations (27) and (28) take the following averaged forms, respectively:

$$\rho_f \nabla \bullet \varphi < \gamma_f >_\varphi < \vec{u} >_\varphi < \vec{u} >_\varphi = -\varphi \nabla < \gamma_f >_\varphi < p >_\varphi + \nabla \bullet (\varphi < \mu_f >_\varphi < \gamma_f >_\varphi < \vec{I}_f >_\varphi)$$
$$-\frac{< \gamma_f >_\varphi}{\eta} \varphi < \mu_f >_\varphi \varphi < \vec{u} >_\varphi - \frac{\rho_f \delta < \gamma_f >_\varphi}{\sqrt{\eta}} \varphi < \vec{u} >_\varphi \left|\varphi < \vec{u} >_\varphi\right| + \frac{\rho_d}{\tau} \varphi < \gamma >_\varphi [< (\vec{v} >_\varphi - < \vec{u} >_\varphi]$$
…(48)

$$\rho_d \nabla \bullet \varphi < \gamma_d >_\varphi < \vec{v} >_\varphi < \vec{v} >_\varphi = \nabla \bullet (\varphi < \mu_d >_\varphi < \gamma_d >_\varphi < \vec{I}_d >_\varphi) - \frac{< \gamma_d >_\varphi}{\eta} \varphi < \mu_d >_\varphi \varphi < \vec{v} >_\varphi$$
$$-\frac{\rho_d \delta < \gamma_d >_\varphi}{\sqrt{\eta}} \varphi < \vec{v} >_\varphi \left|\varphi < \vec{v} >_\varphi\right| - \frac{\rho_d}{\tau} \varphi < \gamma >_\varphi [< (\vec{v} >_\varphi - < \vec{u} >_\varphi]$$
…(49)

4. FINAL FORMS OF GOVERNING EQUATIONS

The averaged continuity and momentum equations take the forms of equations (29), (30), (48) and (49). In light of the above approximations of deviation terms and surface integrals, continuity equations (19) and (20), and momentum equations (27) and (28), take the following final averaged forms in which we have used the following notation:

$\vec{q}_f = \varphi < \vec{u} >_\varphi$, $\vec{q}_d = \varphi < \vec{v} >_\varphi$, $p^* =< p >_\varphi$, $\vec{I}_f = \varphi < \vec{T}_f >_\varphi$, $\vec{I}_d = \varphi < \vec{T}_d >_\varphi$, $\gamma_f^* =< \gamma_f >_\varphi$, $\mu_f^* = \varphi < \mu_f >_\varphi$, $\gamma_d^* =< \gamma_d >_\varphi$ and $\mu_d^* = \varphi < \mu_d >_\varphi$:

$$\nabla \bullet \gamma_f^* \vec{q}_f = 0 \quad \ldots(50)$$

$$\nabla \bullet \gamma_d^* \vec{q}_d = 0 \quad \ldots(51)$$

$$\rho_f \nabla \bullet \gamma_f^* \frac{\vec{q}_f \vec{q}_f}{\varphi} = -\varphi \nabla \gamma_f^* p^* + \nabla \bullet \mu_f^* \gamma_f^* \left\{ \nabla \left(\frac{\vec{q}_f}{\varphi} \right) + \left(\nabla \left(\frac{\vec{q}_f}{\varphi} \right) \right)^T \right\} - \frac{\gamma_f^* \mu_f^*}{\eta} \vec{q}_f - \frac{\rho_f \delta \gamma_f^*}{\sqrt{\eta}} \vec{q}_f |\vec{q}_f| + \frac{\rho_d}{\tau} \gamma_d^* [\vec{q}_d - \vec{q}_f] \quad \ldots(52)$$

$$\rho_d \nabla \bullet \gamma_d^* \frac{\vec{q}_d \vec{q}_d}{\varphi} = \nabla \bullet \gamma_d^* \mu_d^* \left\{ \nabla \left(\frac{\vec{q}_d}{\varphi} \right) + \left(\nabla \left(\frac{\vec{q}_d}{\varphi} \right) \right)^T \right\}) - \frac{\gamma_d^* \mu_d^*}{\eta} \vec{q}_d - \frac{\rho_d \delta \gamma_d^*}{\sqrt{\eta}} \vec{q}_d |\vec{q}_d| - \frac{\rho_d}{\tau} \gamma_d^* [\vec{q}_d - \vec{q}_f] \quad \ldots(53)$$

where in obtaining (52) and (53) we have used

$$\vec{I}_f = \varphi < \vec{T}_f >_\varphi = \mu_f^* \gamma_f^* \{ \nabla \frac{\vec{q}_f}{\varphi} + \nabla \frac{\vec{q}_f^T}{\varphi} \} \quad \ldots(54)$$

$$\vec{I}_d = \varphi < \vec{T}_d >_\varphi = \mu_d^* \gamma_d^* \{ \nabla \frac{\vec{q}_d}{\varphi} + \nabla \frac{\vec{q}_d^T}{\varphi} \}. \quad \ldots(55)$$

Dependence of viscosity on pressure can be expressed using various relations (see for example [5, 10]). In the current work, and in the absence of Forchheimer effects, we can express the Darcy resistance as follows:

$$\frac{1}{V} \int_S \vec{T}_f \bullet \vec{n} dS = -\gamma_f^* \alpha_f (p^*) \vec{q}_f \quad \ldots(56)$$

$$\frac{1}{V} \int_S \vec{T}_d \bullet \vec{n} dS = -\gamma_d^* \alpha_d (p^*) \vec{q}_d \quad \ldots(57)$$

where $\alpha_f(p^*)$ and $\alpha_d(p^*)$ are functions of pressure in the sense defined in [5] for the flow of fluids with pressure-dependent viscosities in porous media. The functions $\alpha_f(p^*)$ and $\alpha_d(p^*)$ can be chosen as equal and/or take combinations of the forms suggested in [5] for the case of single phase flow through porous media.

5. CONCLUSION

In this work we used the method of intrinsic volume averaging to develop equations of viscous dusty gas flow through porous media, where the viscosities of the flowing phases are taken as functions of pressure. The developed equations allow for the imposition of no-slip on both the fluid and dust velocities on solid boundaries.

6. APPENDIX I

Averaging Rules

A Representative Elementary Volume, REV, is a control volume, *V*, composed of a fluid-phase and a (stationary) solid-phase, [18, 19]. Fluid is contained in the pore space, V_φ, and the solid-phase is contained in the porous matrix solid of volume V_s, in the same proportion as the whole porous medium. An REV is therefore a control volume whose porosity is the same as that of the whole porous medium. The porosity, φ, is defined as the ratio of the pore volume, V_φ, to the bulk volume, $V = V_\varphi + V_s$, of the medium, namely

$$\varphi = \frac{V_\varphi}{V} \qquad \text{…(I.1)}$$

Porosity can also be defined in terms of a fluid-phase function, χ_φ, as follows. Define χ_φ at a point $\vec{x}$ in *V* as [23]:

$$\chi_\varphi(\vec{x}) = \begin{cases} 1; & \vec{x} \in V_\varphi \\ 0; & \vec{x} \in V_s \end{cases} \qquad \text{…(I.2)}$$

Porosity is then defined as

$$\varphi = \frac{1}{V}\iiint_V \chi_\varphi dV = \frac{1}{V}\iiint_{V_\varphi} 1dV = \frac{V_\varphi}{V}. \qquad \text{…(I.3)}$$

Length scales associated with the REV are microscopic and macroscopic length scales, *l* and *L* respectively. An REV is chosen such that

$$l^3 << V << L^3 \qquad \text{…(I.4)}$$

wherein the microscopic length scale, l, could be the average pore diameter, and the macroscopic length scale, L, could be a depth of a channel.

In order to develop the equations of flow through a porous structure we define the volume average (volumetric phase average) of a fluid quantity *F* per unit volume, as:

$$< F > = \frac{1}{V}\iiint_V \chi_\varphi F dV = \frac{1}{V}\iiint_{V_\varphi} F dV \equiv \frac{1}{V}\int_{V_\phi} F dV. \qquad \text{…(I.5)}$$

The intrinsic phase average (that is, the volumetric average of *F* over the effective pore space, V_ϕ) is defined as:

$$< F >_\varphi = \frac{1}{V_\varphi}\iiint_V \chi_\varphi F dV = \frac{1}{V_\varphi}\iiint_{V_\varphi} F dV \equiv \frac{1}{V_\varphi}\int_{V_\phi} F dV. \qquad \text{…(I.6)}$$

Relationship between the volumetric phase average and the intrinsic phase average can be seen from equations (I.5), and (I.6), and the definition of porosity, (I.3), as:

$$< F > = \varphi < F >_\varphi. \qquad \text{…(I.7)}$$

The deviation of an averaged quantity from its true (microscopic) value is denoted by F° and is given by the quantity

$$F^\circ = F - < F >_\varphi. \qquad \text{…(I.8)}$$

The following averaging theorems have been established, [18, 19]. Letting *F* and *H* be volumetrically additive scalar quantities, $\vec{F}$ a vector quantity, and *c* a constant (whose average is itself), then:

(i)… $< cF > = c < F > = c\varphi < F >_\varphi$.

(ii)… $< \nabla F > = \varphi \nabla < F >_\varphi + \frac{1}{V}\int_S F^\circ \vec{n} dS$

where *S* is the surface area of the solid matrix in the REV that is in contact with the fluid, and $\vec{n}$ is the unit normal vector pointing into the solid, and a surface integral of the form $\iint_S \vec{n} dS$ has been abbreviated as $\int_S \vec{n} dS$.

(iii)… $< F \mp H > = < F > \mp < H > = \varphi < F \mp H >_\varphi = \varphi < F >_\varphi \mp \varphi < H >_\varphi$

(iv)… $< FH > = \varphi < FH >_\varphi = \varphi < F >_\varphi < H >_\varphi + \varphi < F^\circ H^\circ >_\varphi$

(v)… $< \nabla \bullet \vec{F} > = \nabla \bullet \varphi < \vec{F} >_\varphi + \frac{1}{V}\int_S \vec{F} \bullet \vec{n} dS.$

(vi)… Due to the no-slip condition, a surface integral is zero if it contains the fluid velocity vector explicitly.

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