Fixed Point Theorems for Weakly Compatible Mappings in Metric Spaces

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ABSTRACT

In this paper, we prove a common fixed point theorem from the class of compatible continuous mappings to a larger class of mappings having weakly compatible mappings without appeal to continuity which generalizes the result of Fisher[2], Jungck[4], Lohani and Badshah[6].

Keywords: compatible mappings, weakly compatible mappings, fixed point, metric space.

1. INTRODUCTION

In 1998, Jungck & Rhoades [3] introduced the concept of weakly compatible maps in metric spaces and proved a common fixed point theorem for these mappings by generalizing previous known results given by many authors in various ways.

In this paper, we prove a fixed point theorem for weakly compatible maps without appeal to continuity. We prove a common fixed point theorem, from the class of compatible continuous maps to a larger class of maps having weakly compatible maps without appeal to continuity, which generalizes the result of Fisher [2], Jungck[4], Lohani and Badshah [6].

2. PRELIMINARIES

Now we give some definitions which are used in this paper.

Definition. A pair of maps A, S: (X, d) $\rightarrow$ (X, d) is **compatible pair** if

$$\lim_{n\to\infty} d(ASx_n, SAx_n) = 0,$$

Definition. A pair of maps A,S: $(X, d) \rightarrow (X, d)$ is **weakly compatible pair** if they commute at coincidence points i.e. $Ax = Sx$ implies $ASx = SAx$.

Example.

Let X=[0, 3] be equipped with the usual metric space d(x, y) = $|x - y|$ Define A,S: [0, 3] $\rightarrow$ [0, 3] by

$$A(x) = \left| \begin{array}{lll} x & if & x \in [0,1) \\ 3 & if & x \in [1,3] \end{array} \right| \quad \text{and} \quad S(x) = \left| \begin{array}{lll} 3 - x & if & x \in [0,1) \\ 3 & if & x \in [1,3] \end{array} \right|$$

Then for $x = 3,\ ASx = SAx$, showing that A, S are weakly compatible maps on [0, 3].

Example.

Let X=[2, 20] and d be the usual metric on X. Define mappings A,S: $X \to X$ by $Ax = x$ if $x = 2$ or > 5, $Ax = 6$ if $2 < x \leq 5$, $Sx = x$ if x=2, $Sx = 12$ if $2 < x \leq 5$, $Sx = x - 3$ if $x > 5$.
The mappings A and S are non-compatible and sequence $\{x_n\}$ defined by $x_n = 5 + (1/n), n \geq 1$. Then $Sx_n \to 2, Ax_n \to 2$, $SAx_n \to 2$ and $ASx_n \to 6$. But they are weakly compatible since they commute at coincidence point at x=2.

Example.

Let X=R and define $A, S : R \to R$ by $Ax = x/3, x \in R$ and $Sx = x^2, x \in R$. Here 0 and 1/3 are two coincidence points for the maps A and S. Note that A and S commute at 0, i.e. AS(0)=SA(0)=0, but AS(1/3)=A(1/9)=1/27 and SA(1/3) =S(1/9)=1/81 and so A and S are not weakly compatible maps on R.

Remark.

Weakly compatible maps need not be compatible.

3. MAIN RESULT

We need the following lemma to prove our main result.

Lemma. Let A, B, S and T be self mappings from a metric space (X,d) into itself satisfying the following conditions.

$$A(x) \subset S(x) \text{ and } B(x) \subset T(x) \qquad (3.1.1)$$

$$d(Ax, By) \leq \alpha \frac{d(Sy, By)(1 + d(Tx, Ax))}{[1 + d(Tx, Sy)]} + \beta [d(Tx, By) + d(Sy, Ax)] + \gamma d(Tx, Sy) \text{ for all x, y in X.} \qquad (3.1.2)$$

Where $\alpha, \beta, \gamma \geq 0, 0 \leq \alpha + 2\beta + \gamma < 1$. Then for any arbitrary point x_n in X, by (3.1.1), therefore, there exists a point $x_1 \in X$ such that $Sx_1 = Ax_0$ and for this point x_1, We can choose a point $x_2 \in X$ such that $Bx_1 = TX_2$ and so on. Inductively, we can define a sequence $\{y_n\}$ in X such that

$$y_{2n} = SX_{2n+1} = AX_{2n} \text{ and } y_{2n+1} = TX_{2n+2} = BX_{2n+1} \text{ for } n = 0,1,2 .... \qquad (3.1.3)$$

Then the sequence $\{y_n\}$ defined by (3.1.3) is a Cauchy sequence in X.

Proof: From (3.1.2), we have

$$d(y_{2n}, y_{2n+1}) = d(Ax_{2n}, Bx_{2n+1})$$
$$\leq \alpha \frac{d(Sx_{2n+1}, Bx_{2n+1})[1 + d(Tx_{2n}, Ax_{2n})]}{[1 + d(Tx_{2n}, Sx_{2n+1})]}$$
$$+ \beta [d(Tx_{2n}, Bx_{2n+1}) + d(Sx_{2n+1}, Ax_{2n})] + \gamma d(Tx_{2n}, Sx_{2n+1})$$
$$d(y_{2n} y_{2n+1}) \leq \alpha \frac{d(y_{2n}, y_{2n+1})[1 + d(y_{2n+1}, y_{2n})]}{[1 + d(y_{2n+1}, y_{2n})]}$$
$$+ \beta [d(y_{2n+1}, y_{2n+1})] + d(y_{2n}, y_{2n})$$

$$+ \gamma d(y_{2n+1}, y_{2n})$$

On simplification we have

$$d(y_{2n}, y_{2n+1}) \le \frac{(\gamma + \beta) d(y_{2n}, y_{2n+1})}{1 - \alpha - \beta}$$

$d(y_{2n}, y_{2n+1}) \le h\, d(y_{2n}, y_{2n+1})$ where $h = \dfrac{\gamma + \beta}{1 - \alpha - \beta} < 1$

Now $d(y_n, y_{n+1}) \le hd(y_{n+1}, y_n) \le \ldots.. \le h^2 d(y_0, y_1)$

For every integer $t > 0$, we get

$$d(y_n, y_{n+t}) \le d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + \ldots. + d(y_{n+t-1}, y_{n+t})$$

$$\le (1 + h + h^2 + \ldots\ldots + h^{n-1}) d(y_n, y_{n+1})$$

$$d(y_n, y_{n+1}) \le \frac{h^n}{1-h} d(y_0, y_1)$$

Letting $n \to \infty$, we have $d(y_n, y_{n+t}) \to 0$. Therefore $\{y_n\}$ is a Cauchy sequence in x.

Now, we prove our main result by using this lemma.

Theorem. Let (A,T) and (B, S) be weakly compatible pairs of self maps of a complete metric space (X, d) satisfying (3.1.1) and (3.1.2). Then A, B, S and T have a unique common fixed point in X.

Proof: By Lemma 3.1, $\{Y_n\}$ is a Cauchy sequence in X. Since X is complete, therefore, there exists a point z in X such that $\lim_{n\to\infty} y_n = z$.

Also, $\lim_{n\to\infty} Sx_{2n+1} = \lim_{n\to\infty} Ax_{2n} = \lim_{n\to\infty} Tx_{2n+2} = \lim_{n\to\infty} Bx_{2n+1} = z$.

$B(X) \subset T(X)$ so, there exists a point $u \in X$ such that $z = Tu$, then using (3.1.2), we obtain

$$d(Au, z) \le d(Au, Bx_{2n+1}) + d(Bx_{2n+1}, z)$$

$$\le \alpha \frac{d(Sx_{2n+1}, Bx_{2n+1})[1 + d(Tu, Au)]}{[1 + d(Tu, Sx_{2n+1})]}$$

$$+ \beta\{d(Tu, Bx_{2n+1}) + d(Sx_{2n+1}, Au)\} + \gamma d[Tu, Sx_{2n+1}]$$

Taking limit as $n \to \infty$ yields $d(Au, z) \le \beta d(Au, z)$, a contradiction, since $\alpha, \beta, \gamma \ge 0, \le \alpha + 2\beta + \gamma < 1$. Therefore $Au = Tu = z$.

Since $A(x) \subset S(x)$, there exists a point $v \in X$ such that $z = Sv$ .Then again using (3.1.2), we get

$$d(z, Bv) = d(Au, Bv) \le \alpha \frac{d(Sv, Bv)[1 + d(Tu, Au)]}{[1 + d(Tu, Sv)]}$$

$$+ \beta[d(Tu, Bv) + d(Sv, Au)] + \gamma[d(Tu, Sv)]$$

$d(z, Bv) \le (\alpha + \beta) d(z, Bv)$, a contradiction

Therefore, $z = Bv$. Thus $Au = Tu = Bv = Sv = z$

Since pair of maps A and T are weakly compatible, then $ATu = TAu$, i.e. Az=Tz. Now we show that z is a fixed point of A. If $Az \ne z$, then by (3.1.2),

$$d(Az, z) = d(Az, Bv) \le \alpha \frac{d(Sv, Bv) + d(Tz, Az)}{[1 + d(Tz, Sv)]}$$
$$+ \beta[d(Tz, Bv) + d(Sv, Az) + \gamma(Tz, Sv)]$$
$$= \beta[d(Az, z) + d(z, Az) + \gamma(Az, z)]$$
$$= (2\beta + \gamma)d(Az, z), \text{ which yields } Az = z.$$

Therefore $Tz = Az = z.$

Similarly , pairs of maps B and S are weakly compatible, we have Bz=Sz=z, since

$$d(z, Bz) \le d(Az, Bz) \le \alpha \frac{d(Sz, Bz)[1 + d(Tz, Az)]}{[1 + d(Tz, Sz)]}$$
$$+ \beta[d(Tz, Bz) + d(Sz, Az)] + \gamma d(Tz, Sz)$$
$$= \beta[d(z, Bz) + d(z, Bz)] + \gamma d(Bz, z).$$
$$= (2\beta + \gamma)d(Bz, z), \text{ which yields } Bz = z.$$

Therefore z=Tz=Sz=Az=Bz and z is a common fixed point of A, B, S and T.
Uniqueness follows easily from (3.1.2)
The following example illustrates our theorem.

Example.

Let X=[0, 1] and d be usual metric, i.e. d(x, y)= |x–y|
Define maps A,B, S, T : $X \to X$ as follows:
$Sx = \theta, Ax = x, Tx = x/32, Bx = x$

Pairs (T, B) and (S,A) are weakly compatible. Now

$$d(Sx, Ty) = |y/32| \le [x/2 + 59y/384 + 31xy/192] \text{ for all } x, y \in X$$
$$\le 1/6 \left| 31/32 \frac{|y|(1+|x|)}{1+|x-y|} \right| + 1/12 |x - y/32| + |y| + 1/4 |x - y|$$
$$= 1/6 \frac{d(By, Ty)[1 + d(Ax, Sx)]}{[1 + d(Ax, By)]}$$
$$+ 1/12 [d(Ax, Ty) + d(By, Sx)] + 1/4 d(Ax, By)$$

Hence all the assumptions of the theorem are satisfied with $\alpha = 1/6, \beta = 1/12, \gamma = 1/4$ and zero is the unique common fixed point of A, B, S and T.

Remark

Our theorem improves the result of Fisher [2], Jungck[4], Lohani and Badshah [6] in two aspects. Firstly our theorem does not require the mappings to be continuous; secondly we prove the result for weakly compatible mappings instead of compatible mappings.

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