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Construction of Hexagonal-8-QAM constellation and Code for OFDM

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Abstract

A construction of hexagonal shaped 8-point QAM called H8QAM is shown. It is shown that H8QAM constellation outperforms the conventional rectangular constellation and 8-PSK. Further, we construct H8QAM sequences having low peak-to-mean envelope power ratios (PMEPR) from QPSK Golay sequences. Various upper bounds on peak envelope powers of these sequences are evaluated.

I. Introduction

Quadrature amplitude modulation (QAM) is a very popular constellation in the literature [1], [2], [3]. QAM occurs in many arrangements and the one that is most commonly encountered in literature is square arrangement. Non square QAMs and their advantages are known for a long time, we mention few of its relevant advantages. They can be differentially detected with non-coherent techniques which are computationally simpler than coherent detection [4], [2]. In this article, we focus on Hexagonal-8-QAM (H8QAM) constellation and its application in orthogonal frequency-division multiplexing (OFDM). H8QAM is choosen among other 8-point constellations because of its unique construction suited to develop suitable codes for OFDM.

OFDM has become a most favored technique for LTE (long term evolution) standards due to susceptibility to signal dispersion under multipath conditions. A major drawback of deploying OFDM is the high peak-to-mean envelope power ratio (PMEPR) of uncoded OFDM signals. High PMEPR leads to spectral growth of the OFDM signal in the form of intermodulation among subcarriers and out-of-band radiation. Therefore, the transmit amplifier used must have large linear range and are expensive and power consumption is high leading to short battery life.

Many solution have been proposed to control PMEPR in OFDM [5] and the reference there in. One approach is designing codes that not only provide error correction but also reduce the PMEPR [6], [7], [8]. It has been known since the work of Popovic and Boyd [9], [10] that the use of *Golay complementary sequences* (GCS) [11] as codewords to control the modulation of carrier signals results in OFDM with PMEPR of at most 2. Davis and Jedwab [12] made a major theoretical advance, discovering that the large sets of binary length 2^m Golay complementary pairs described in [11] can be obtained from certain second-order cosets of the classical first order Reed-Muller Code. Special cases of these codes were given in [13], [14], and the underlying theory was developed in [15]. However, the aformentioned codes are defined over the phase-shift keying (PSK) signal constellations. Codes based on other popular constellations such as square QAM were shown in [16], [17], [18]. Codes constructed from Golay complementary sequences are employed as pilot sequences by the European Telecommunications Standards Institute (ETSI) Broadband Radio Access Networks (BRAN) committee. Previously we presented the construction of such codes for non-square STAR-16-QAM as a scaled set sum of two quadrature phase-shift keying (QPSK) constellations is given in [4]. Here a sligthly modified construction is presented to obtain H8QAM and it is shown that such a construction give codes with low PMEPR of 3 (4.77dB).

The rest of the paper is organized as follows: Section 2 shows the construction of H8QAM constellation; Section 3 contains the symbol error rate analysis and comparison with various other 8-point constellations; Section 4 describes the OFDM system model and defines PMEPR; Section 5 takes up PMEPR for H8QAM sequences using Golay sequences and conclusion is in Section 6.

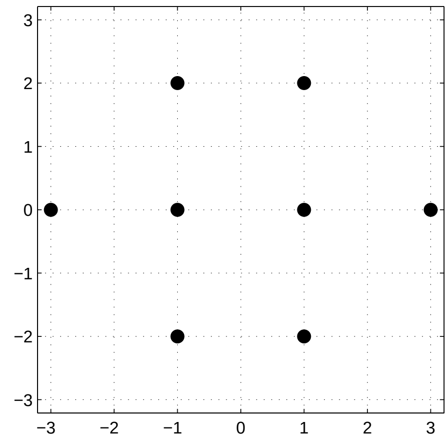

Fig. 1. Hexagonal-8-QAM constellation

II. Construction of H8QAM

The QPSK constellation set, denoted by Q, can be realized as $Q \stackrel{\text{def}}{=} j^x$ where $x \in \mathbf{Z}_4 = \{0,1,2,3\}$ and $j = \sqrt{-1}$. A unique n-tuple QPSK sequence $\mathbf{u} = (u_0, u_1, \cdots, u_{n-1})$ can be associated with a unique sequence $\mathbf{x} = (x_0, x_1, \cdots, x_{n-1})$, where $x_i \in \mathbf{Z}_4,\ 0 \le i \le n-1$, and $u_i = j^{x_i}$. We define $\hat{\mathbf{Z}}_4 = \{0,2\}$ as a subset of $\mathbf{Z}_4$. Another constellation QPSK set, denoted by $\hat{Q}$, can be realized as $\hat{Q} \stackrel{\text{def}}{=} j^{y_i}$ where $y \in \hat{\mathbf{Z}}_4 = \{0,2\}$. **Definition** 1: Let A and B be the sets of complex numbers. Then the *set sum* of A and B is defined as

$$A \oplus B \stackrel{\text{def}}{=} \{r + w | r \in A, w \in B\}. \tag{1}$$

Definition 2: Let h be a complex number and A be the set of complex numbers. Then the *product* of h and A is defined as

$$hA \stackrel{\text{def}}{=} \{hr | r \in A\}. \tag{2}$$

According to these definitions, the H8QAM constellation is constructed as follows:

$$\mathrm{H} \stackrel{\text{def}}{=} \frac{1}{\sqrt{5}}\hat{Q} \oplus \frac{2}{\sqrt{5}}Q \tag{3}$$

The corresponding constellation is given in Fig. 1. A sequence $\mathbf{u} = (u_0, u_1, \cdots, u_{n-1})$ of H8QAM, where $u_k \in H, k = 0, 1, \cdots, n-1$, can be uniquely associated with two sequences $\mathbf{x} = (x_0, x_1, \cdots, x_{n-1})$ and $\mathbf{y} = (y_0, y_1, \cdots, y_{n-1})$ such that $(x_k, y_k) \in \hat{\mathbf{Z}}_4 \times \mathbf{Z}_4$. Consequently, we can write

$$u_k = \frac{1}{\sqrt{5}} j^{x_k} + \frac{2}{\sqrt{5}} j^{y_k} \tag{4}$$

III. SER of Various 8-point Constellations

In order to determine the error performance for any QAM, we must specify the signal constellation. There are many possible configurations for $M = 8$ in the literature [1], [2]. We shall consider six constellations including the proposed constellation, H8QAM as in Fig. 1. Let d_{ij} represent the Euclidean distance between i-th and j-th signal point in the constellation, then minimum distance of constellation is defined as $d_{min} = \arg\min_{i \neq j} |d_{ij}|$. All constellations shown in Fig. 1 are constructed such that $d_{min} = 2$. Let $x + iy$ denote i-th point, then its power is calculated as, $|A_i|^2 = x^2 + y^2$. Assuming that all signal points are equally probable, the average transmitted signal power is given as:

$$P_{av} = \frac{1}{M} \sum_{i=1}^{M} |A_i|^2. \tag{5}$$

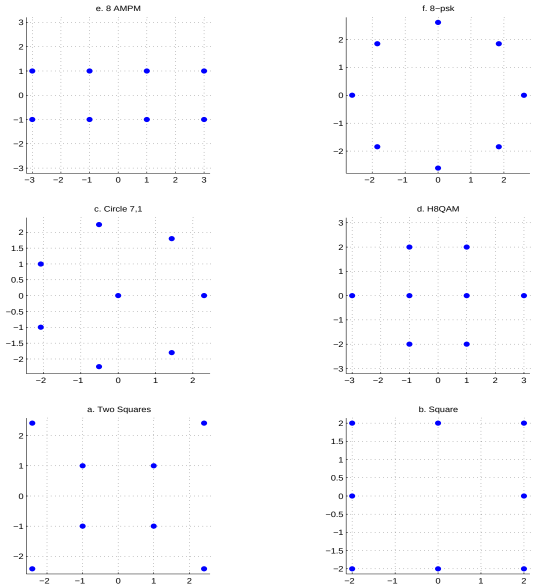

TABLE I
VARIOUS 8-POINT CONSTELLATIONS

The ratio of the average-signal-power to the average-noise-power can be expressed as follows (Eq.(6) in [1]):

$$\rho = \frac{1}{8\sigma^2}\sum_{i=1}^{M}|A_i|^2, \tag{6}$$

where σ^2 is the single-sided power spectral density of the white Gaussian noise. Closed form expressions for *symbol error rate* (SER) for the all constellations in Table. I do not exist, therefore to compare we use a union bound of SER (Eq. (8) in [1]):

$$P_e = \frac{\sigma}{8\sqrt{\pi}}\sum_{i=1}^{8}\sum_{\substack{k=1\\ i\neq k}}^{8}\frac{\exp[-|A_i - A_k|^2/(4\sigma^2)]}{|A_i - A_k|}. \tag{7}$$

The probability error performance curves obtained by (7) is very close to the exact SER when $P_e < 10^{-2}$ [1]. The probability of error depends on minimum distance between pairs of signal points and average transmitter power. As can be seen from the figure H8QAM constellation outperforms 8-psk, 8-AMPM and square 8 QAM. H8QAM

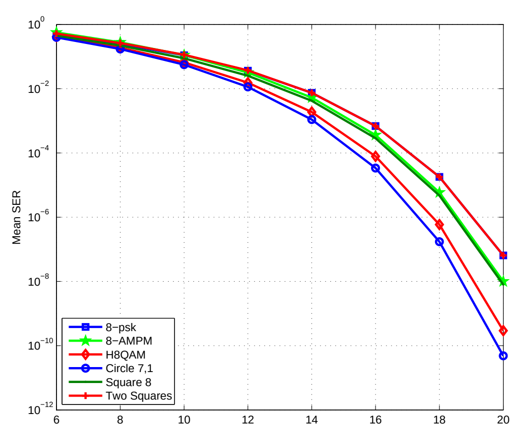

Fig. 2. Symbol Error Rate Curves

performance is outperformed by circle71QAM by a small margin. Out of six constellations given in Table. I of figures only two constellations can be constructed as a set sum of two QPSK constellations namely H8QAM and 8-AMPM, which is the key to developing low PMEPR codes. Comparing these two, H8QAM certainly has better error performance.

Now we discuss characteristic that influence constellation design in terms of SNR efficiency. Assuming that all points are equally likely in a given constellation, the key parameter that determine the SNR efficiency are the minimum squared distance (d^2_{min}) between its points, and its average power (P_{av}). We generally wish to maximize d^2_{min} for a given P_{av}, or to minimize P_{av} for a given d^2_{min}. Therefore, defining *constellation figure of merit*(CFM), as the ratio [3]:

$$\text{CFM} = \frac{d^2_{min}}{P_{av}} \tag{8}$$

By keeping d_{min} constant, in Table. II we give values of CFM for all costellations under consideration. Again,

Constellation design	Avg. Power (P_{av})	CFM (Λ)
8-psk	6.81	0.585
8-AMPM	6.00	0.667
H8QAM	5.00	0.800
Circle 7,1	4.63	0.860
Square 8	6.00	0.667
Two Squares	6.83	0.585

TABLE II
CFM FOR CONSTELLATIONS

H8QAM has a better CFM of 0.8 than 8-AMPM's CFM, which is 0.667. Thus, our motivation to select H8QAM among various 8-point constellations can be summarized as (a) H8QAM can be constructed by set sum of two QPSK constellations (b) Given (a) is satisfied, better CFM.

IV. OFDM

The basic block diagram of the OFDM system is shown in Fig. 3. Let the number of subcarriers be n for the OFDM system. Consider a sequence $\mathbf{u} = (u_0, u_1, \cdots, u_{n-1})$ of length n consisting of elements of an M-point constellation. The collection of all M^n distinct sequences forms a code $\mathcal{C}$. The time domain OFDM signal after undergoing Inverse Fast Fourier Transform (IFFT) operation is given by

$$S_{\mathbf{u}}(t) = \sum_{i=0}^{n-1} u_i e^{j2\pi i \Delta f t}$$

for $0 \le t \le T$, where $\Delta f = f_{i+1} - f_i$ is an integer multiple of time period of OFDM symbol T. The value of Δf depends on the guard time and the cyclic prefix but we ignore this matter since it has no direct impact on the current work. This is chosen in order to maintain the orthogonality of subcarriers frequencies. The transmitted OFDM signal is the real part of the complex signal $S_{\mathbf{u}}(t)$. The real part of the transmitted complex envelope for a

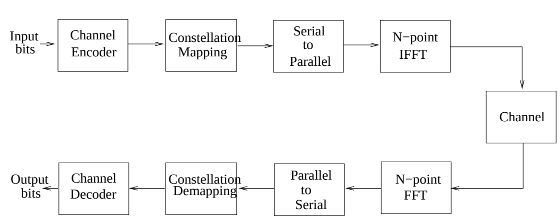

Fig. 3. Block diagram of OFDM system

given codeword $\mathbf{u}$ is given by $\Re(S_{\mathbf{u}}(t))$ and its instantaneous envelope power is equal to

$$\begin{aligned} P_{\mathbf{u}}(t) &= |S_{\mathbf{u}}(t)|^2 \\ &= \left(\sum_{i=0}^{n-1} u_i e^{j2\pi i\Delta f t}\right)\left(\sum_{k=0}^{n-1} u_k^* e^{-j2\pi k\Delta f t}\right) \\ &= \sum_{i=0}^{n-1}\sum_{k=0}^{n-1} u_i u_k^* e^{-j2\pi(i-k)\Delta f t}. \end{aligned} \tag{9}$$

Thus, the mean power of $S_{\mathbf{u}}(t)$ during a symbol period is

$$\frac{1}{T}\int_{0,T} P_{\mathbf{u}}(t)dt = ||\mathbf{u}||^2 \stackrel{\text{def}}{=} \sum_{k=0}^{n-1} |\mathbf{u}|^2.$$

We define *peak envelope power* (PEP) of a sequence or codeword as the maximum instantaneous power of $S_{\mathbf{u}}(t)$ within T denoted by

$$\text{PEP}(\mathbf{u}) = \max_{0 \leq t \leq T} P_{\mathbf{u}}(t)$$

and the PMEPR of a code is

$$\text{PMEPR}(\mathcal{C}) = \max_{\mathbf{u} \in \mathcal{C}} \text{PEP}(\mathbf{u})/P_{av}(\mathcal{C})$$

where $P_{av}(\mathcal{C})$ is the mean envelope power of an OFDM signal in one symbol period averaged over all OFDM signals generated from $\mathcal{C}$.

$$\begin{aligned} P_{av}(\mathcal{C}) &= \frac{1}{T}\sum_{\mathbf{u}\in\mathcal{C}} p(\mathbf{u}) \int_{0,T} P_{\mathbf{u}}(t)dt \\ &= \frac{1}{T}\sum_{\mathbf{u}\in\mathcal{C}} p(\mathbf{u})||\mathbf{u}||^2 \end{aligned}$$

where $p(\mathbf{u})$ is the probability of transmitting the codeword $\mathbf{u}$. Since all codewords are assumed equally probable and for QPSK constellation in which all symbols have unit energy i.e., $||\mathbf{u}||^2 = n$. Therefore we get

$$P_{av}(\mathcal{C}) = n. \tag{10}$$

V. PMEPR of H8QAM

In this section, we find PMEPR of OFDM signal corresponding to H8QAM constellation. The transmitted OFDM signal corresponding to the sequence $\mathbf{u}$ of H8QAM is given by

$$\begin{aligned} S_{\mathbf{u}}(t) &= \sum_{i=0}^{n-1} u_i e^{j2\pi i\Delta f t} \\ &\stackrel{\text{def}}{=} S_{\mathbf{x},\mathbf{y}}(t) = \frac{1}{\sqrt{5}} S_{\mathbf{x}}(t) + \frac{2}{\sqrt{5}} S_{\mathbf{y}}(t), \end{aligned} \tag{11}$$

where

$$S_{\mathbf{x}}(t) \stackrel{\text{def}}{=} \sum_{i=0}^{n-1} j^{x_i} e^{j2\pi i\Delta f t}$$

and

$$S_{\mathbf{y}}(t) \stackrel{\text{def}}{=} \sum_{i=0}^{n-1} j^{y_i} e^{j2\pi i\Delta f t}$$

such that $(x_i, y_i) \in \widehat{\mathbf{Z}}_4 \times \mathbf{Z}_4$. The instantaneous envelope power of the transmitted signal given by (11) is given as

$$\begin{aligned} P_{\mathbf{u}}(t) &\stackrel{\text{def}}{=} P_{\mathbf{x},\mathbf{y}}(t) = |S_{\mathbf{x},\mathbf{y}}(t)|^2 \\ &= |\frac{1}{\sqrt{5}} S_{\mathbf{x}}(t) + \frac{2}{\sqrt{5}} S_{\mathbf{y}}(t)|^2 \end{aligned} \tag{12}$$

Consider two complex valued sequences $\mathbf{x} = (x_0, x_1, \cdots, x_{n-1})$ and $\mathbf{y} = (y_0, y_1, \cdots, y_{n-1})$ of length n satisfying

$$C_{\mathbf{x}}(\eta) + C_{\mathbf{y}}(\eta) = (||\mathbf{x}||^2 + ||\mathbf{y}||^2)\delta(\eta) \tag{13}$$

where $C_{\mathbf{x}}(\eta) = \sum_i x_i x_{i+\eta}^*$ is the aperiodic autocorrelation of sequence $\mathbf{x}$ at delay shift η and $\delta(\eta)$ is the Kronecker function. Then, $\mathbf{x}$ and $\mathbf{y}$ are called *Golay complementary sequences*(GCS) [11]. We say that $\mathbf{x}$ is a *Golay sequence* (GS) if there exists a sequence $\mathbf{y}$ that is complementary to $\mathbf{x}$. Popovic made this important observation [19] and also reported in [15], [17], [20].

Lemma 1: Let $\mathbf{x}$ and $\mathbf{y}$ of length n be two sequences form a GCS such that any combination of $(x_k, y_k)$where$k = 0, 1, \cdots, n-1$ can be used to generate a H8QAM symbol as in (4). Consider the following two summations:

$$S_{\mathbf{x}}(t) = \sum_{k=0}^{n-1} j^{x_k} e^{-j2\pi k \Delta f t},$$

$$S_{\mathbf{y}}(t) = \sum_{k=0}^{n-1} j^{y_k} e^{-j2\pi k \Delta f t}.$$

Then for any t, the following inequality hold:

$$|S_{\mathbf{x}}(t)| \leq \sqrt{2n}, |S_{\mathbf{y}}(t)| \leq \sqrt{2n} \tag{14}$$

The proof of lemma 1 is based on the key property:$|S_{\mathbf{x}}(t)|^2 + |S_{\mathbf{y}}(t)|^2 = 2n$.

Consider the generalized form of (12):

$$P_{\mathbf{x},\mathbf{y}}(t) = |S_{\mathbf{x},\mathbf{y}}(t)|^2 = |\frac{1}{\sqrt{5}} S_{\mathbf{x}}(t) + \frac{2}{\sqrt{5}} S_{\mathbf{y}}(t)|^2 \tag{15}$$

For any sequence $\mathbf{x} \in \mathbf{Z}_4$, we define $\mathbf{x} + 2 = (x_0 + 2, x_1 + 2, \cdots, x_{n-1} + 2)$. Accordingly, we have

$$\begin{aligned} S_{\mathbf{x}+2}(t) &= \sum_{k=0}^{n-1} j^{x_k} j^2 e^{-j2\pi k \Delta f t} \\ &= -S_{\mathbf{x}}(t) \end{aligned} \tag{16}$$

Based on the results obtained in Lemma 1, we derive a theorem for peak envelope power of $P_{\mathbf{x},\mathbf{y}}(t)$. **Theorem** 1: If $\mathbf{x}$ and $\mathbf{y}$ form a GCS, then $P_{\mathbf{x},\mathbf{y}}(t) \leq 4n$.

Proof: Expanding (15) and using (16), we get

$$P_{\mathbf{x},\mathbf{y}}(t) = \frac{1}{5}|S_{\mathbf{x}}(t)|^2 + \frac{4}{5}|S_{\mathbf{y}}(t)|^2 + \frac{2}{5}[S_{\mathbf{x}}(t) S_{\mathbf{y}(t)}^* + S_{\mathbf{x}}^*(t) S_{\mathbf{y}}(t)] \tag{17}$$

$$P_{\mathbf{x},\mathbf{y}+2}(t) = \frac{1}{5}|S_{\mathbf{x}}(t)|^2 + \frac{4}{5}|S_{\mathbf{y}}(t)|^2 - \frac{2}{5}[S_{\mathbf{x}}(t) S_{\mathbf{y}(t)}^* + S_{\mathbf{x}}^*(t) S_{\mathbf{y}}(t)] \tag{18}$$

Adding (17) and (18), we get

$$P_{\mathbf{x},\mathbf{y}}(t) + P_{\mathbf{x},\mathbf{y}+2}(t) = 2\big(\frac{1}{5}|S_{\mathbf{x}}(t)|^2 + \frac{4}{5}|S_{\mathbf{y}}(t)|^2\big) \tag{19}$$

applying the inequalities of (14) and since $P_{\mathbf{x},\mathbf{y}+2}(t) \geq 0$, we have

$$P_{\mathbf{x},\mathbf{y}}(t) \leq 4n. \tag{20}$$

H8QAM is constructed using (3) and each sequence $\mathbf{x}$ and $\mathbf{y}$ are statistically independent and contain symbols from QPSK constellation. All symbols of QPSK constellation have unit energy and are equiprobable. Using (10), we get $P_{av}(S_{\mathbf{x}}(t)) = n$ and $P_{av}(S_{\mathbf{y}}(t)) = n$. Therefore average power of each OFDM symbol over all codewords of $\mathcal{H}$, $P_{av} = \frac{1}{5}P_{av}(S_{\mathbf{x}}(t)) + \frac{2}{5}P_{av}(S_{\mathbf{y}}(t)) = 3n/\sqrt{5} \approx 1.33n$. As a result, when sequences $\mathbf{x}$ and $\mathbf{y}$ form GCS the PMEPR of H8QAM is given by $\frac{P_{\mathbf{x},\mathbf{y}}(t)}{P_{av}} \leq \frac{4n}{1.33n}$. Thus, when Golay complementary sequences are used, the PMEPR of H8QAM is upper-bounded by 3 (4.77dB).

VI. Conclusion

In this article, we present construction of a Hexagonal 8-point constellation whose symbols can be found as a sum of two QPSK symbols. We compared this constellation various other 8-point constellations in terms of SER performance and CFM. This kind of constructions helps us develop a code $\mathcal{H}$ using GCS that will lead to a PMEPR upper-bounded by 3 (4.77dB).

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