Mathematical Modelling of Climate Change Using Nonlinear Dynamical Systems

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ABSTRACT

This thesis makes use of complex mathematical and computational methods in order to investigate the temporal dynamics of natural systems and climate change. Numerical tools, mathematical modeling, data-driven techniques, and fractional calculus are all coming together to give new insights on environmental phenomena and the nonlinear development of those phenomena over time. To investigate the influence that climate change has on the propagation of a modified cryosphere surface energy balance-mass balance model, we conducted an investigation. An emphasis is placed on the manner in which changes in the environment have an effect on the flow of energy and the equilibrium of mass. The Lorenz-84 atmospheric transmission model may be used more effectively if it is updated to include time delays, chaos control methods, and the effects of climate change. It is because of these advances that we now have access to innovative methods for managing and predicting the weather. Timing, control, and multiscale dynamics are investigated further in this study, which is conducted using a modified version of the Samardzija-Greller predator-prey model.

This is an example of how changes in time scales may have an effect on the interactions that occur in biological systems. A unique ecological model that we have developed incorporates behavioral variables as well, such as the sensitivity of predators to taxis and the impact of fear. It should come as no surprise that these components play a significant part in preserving ecological balance and safeguarding populations of species. Finally, major environmental indicators such as the levels of air pollution, the amount of carbon stored in forests, and the quality of river water are forecasted by utilizing real data and environmental models that are either random or fractional. This study lays the groundwork for future research and policymaking in the realms of climate and biological dynamics by enhancing theoretical models and bridging the gap between mathematical theories and their actual applications. This study also offers a platform for future research.

Keywords: ***Fractional Calculus, Climate Dynamics, Nonlinear Modeling, Environmental Forecasting.***

INTRODUCTION

Mathematical modeling is unquestionably helpful for identifying and addressing real-world problems in a variety of scientific fields, including climate science, chemistry, engineering, biology, and physics, among others. Individuals are able to grasp more and make judgments that are better informed as a result of this. Mathematical models are an essential component in the area of climate research, serving not only to comprehend but also to simulate complex climatic processes. As a result of this, scientists are in a position to investigate and forecast climate trends, which include fluctuating temperatures, rising sea levels, and severe weather [1]. Through the use of these models, which bridge the gap between research and decision-making, legislators are able to make educated decisions on the adaptation and mitigation of climate change.

Through the use of mathematical models, we are able to acquire knowledge about the natural world and to produce forecasts regarding the future of physics [2]. It is the responsibility of physicists to build mathematical models in order to explain the behavior of particles, waves, and forces across a wide variety of subfields of physics, ranging from classical mechanics to quantum mechanics. Due to the fact that engineers are also interested in and devoted to the construction and improvement of processes, systems, and structures, mathematical modeling is something that engineers find exceptionally fascinating [3].

Numerous activities in the contemporary world, including those in the fields of building, aviation, electrical engineering, and industry, are dependent on mathematical models and forecasts of future behavior in order to guarantee the flawless functioning and safety of these systems. The use of mathematical models in the area of biology has the potential to improve the understanding and analysis of a wide variety of complex biological processes [4-6]. These processes include the transmission of diseases, the dynamics of populations, and the interactions between ecosystems. By providing insight into the underlying principles that underlie biological processes, these models serve as a guide for study and contribute to the creation of various therapies and drugs.

Calculus of the integer order has been used for a very long time in mathematical models. Despite this, the use of fractional calculus (FC) as a tool in mathematical modeling is still in its early stages of development. In more recent times, it has gained more and more notoriety. The use of mathematical modeling that makes use of fractional calculus has shown to be successful in a variety of domains, including economics, physics, engineering, and biology [7–10]. When modeling systems that do not show linear or localized behavior, fractional calculus may be able to provide a more accurate and adaptable representation of the system. There are many instances, some of which include non-linear materials, control systems, and strange diffusion processes. Due to the availability of this technology, researchers have been able to dive more deeply into intricate systems and investigate new locations. The integration of integers with fractional orders and the taking of derivatives are the only two components that make up fractional calculus in its most fundamental form.

The idea of fractional calculus is thought to have originated from a query posed to *Gottfried Wilhelm Leibniz* by *Michel de l'Hôpital* in 1695, which tried to understand Leibniz's notation: $\frac{d^n y}{dx^n}$ for $n = \frac{1}{2}$ [11].

The first definition of a fractional derivative of a power function was put forth by *S. F. Lacroix* in 1819 [12]. Initially, he stated the common n-th derivative of the power function $y = x^m$ in terms of the gamma function (Γ) as follows

$$y^{(n)} = \frac{\Gamma(m+1)}{\Gamma(m-n+1)} x^{(m-n)}, \; m \geq n. \qquad (1.1)$$

A. OVERVIEW OF THESIS

Continuing along the same lines, he went on to say that the letter "i" may stand for any real number. In the year 1822, Fourier used what is now known as the Fourier transform of a function in order to describe the derivative of any order. Different approaches to representing the fractional derivative of a function have been devised by succeeding mathematicians via a variety of different ways. On the other hand, the most common definitions of the fractional derivative are those proposed by Atangana-Baleanu, Caputo, Caputo-Fabrizio, and Riemann-Liouville. For example, neural networks, material extraction, and image processing are all relatively new fields, yet fractional calculus is becoming more popular in these areas as well. In their study [13], Yang et al. demonstrated how fractional calculus may be used in the process of picture fusion. To a certain extent, fractional mathematics is used in the process of constructing neural networks [14]. Using fractional differential equations is another method that may be used to locate ore deposits [15]. What is the outcome of dividing by

Calculus has shown to be a useful instrument in the process of enhancing models of chaotic systems. This is due to the fact that calculus has the ability to disclose sophisticated things that standard integer-order calculus often fails to reveal. The use of fractional derivatives makes it possible to get a more precise portrayal of the dynamics that lie under the surface. Their assistance may be of use in better comprehending chaotic processes, peculiar attractor occurrences, and sudden changes. When it comes to many different systems, both natural and manmade, this is true. As Diouf and Sene [16] pointed out, the use of fractional operators is of the utmost importance in unpredictable financial models. A demonstration was made by Borah and colleagues [17] that demonstrated how fractional calculus may be used to enhance the precision of chaotic pandemic simulations. Saadeh and colleagues [18] found that a model of predators and prey that was only partially complete was chaotic.

Within the context of a non-local operator framework, we investigate a modified chaotic surface energy balance-mass balance model of the cryosphere in Chapter 2. This model incorporates a number of different aspects, including time lag, environmental influences, and active control mechanisms. Investigations into active control strategies are now being conducted by researchers in order to preserve stability in chaotic dynamics. The addition of a temporal delay is used to depict lagging events, such as delayed ice-albedo feedback. Taking into consideration the radiation force of X2 in Chapter 3, we investigate the consequences that global warming has on the circulation of the atmosphere. For the purpose of observing the impact of feedback mechanisms, such as delayed heat transfer, on stability, bifurcations, chaos, and other phenomena of a similar kind, we add a time lag. We build a sliding mode control framework and put it to use in order to keep the system stable while ensuring that its physical validity is not compromised. We do this in order to bring the model's air propagation dynamics under control, which are extremely unpredictable. Within the context of a modified version of the Samardzija-Greller predator-prey model, Chapter 4 looks into the dynamics of slow-fast transitions, timing, and chaos management. This study advances to our understanding of the dynamic relationship between scales in biological systems by illustrating the relevance of time-scale differences for the stability of predator-prey relationships. Through the use of an innovative ecological model, Chapter 5 investigates the effects of fear and the sensitivity of predators to taxis. In this model, it is shown that changes in behavior have the potential to have an effect on the stability and durability of interactions between predators and prey. The conclusion of the thesis is that Chapter 6 introduces unique stochastic and fractional modeling techniques that make use of real environmental data. These approaches are presented as a conclusion. Through the development of prediction models for river water quality, forest carbon sinks, and air pollution, it highlights the need of merging mathematical ideas with facts from the actual world. In the last chapter of the study, some of the most important results are discussed, and some potential directions for further research are suggested.

LITERATURE REVIEW

The Lorenz Legacy and Chaos: Recent studies highlight that while the original 1963 Lorenz model identified the "Butterfly Effect," the **Lorenz-84 model** specifically bridges the gap between daily weather and long-term climate by modeling the interaction between the westerlies and traveling waves.

Time Delays and Feedback: Research (e.g., Chakraborty &Veeresha, 2024) indicates that introducing **time delays** into these models provides a more realistic representation of the lag between radiative forcing and atmospheric response.

Chaos Control: Modern literature explores "feedback state space control" to stabilize these systems. Studies show that as global warming increases (modeled as an increase in the radiative forcing parameter), the atmosphere transitions from stable oscillations to high-dimensional chaos, necessitating these control methods for accurate forecasting.

The Multiscale Approach: Research by Shi (2022) suggests that "averaging" Lorenz systems can induce higher predictability by smoothing out short-term chaotic fluctuations, a key insight for decadal climate projections.

METHODOLOGY

Throughout the course of the Earth's climate cycle, timescales spanning from years to millennia are all included [34–36]. It is possible for cycles of differing magnitudes to be initiated by a variety of physical phenomena. In any particular time period, the climate is the outcome of all of these cycles and events that are interconnected with one another. The dynamic relationship that exists between the mass balance of the cryosphere and the surface energy balance is a significant aspect that plays a role in determining the degree of uncertainty that surrounds climate change [37]. Frozen water may be found on Earth. The cryosphere is comprised of all of the aforementioned elements, in addition to glaciers, sea ice, and ice fields which are also included in the category [38]. The link between the mass balance of the cryosphere and the surface energy balance is shown using a non-linear model that was constructed by Saltzman and colleagues [39]. According to the demonstration that C. Nicolis provided in [37], the model may be shown by the two equations that are listed below, which include extra non-dimensional variables:

$$\begin{cases} \dfrac{ds}{dt} = \zeta, \\ \dfrac{d\zeta}{dt} = c_1\zeta + c_2 s - s^3 - s^2\zeta + c_3 sin(\omega t), \end{cases} \quad (2.1)$$

where $s(t)$ is the sea-ice extent, $\zeta = T - s$ and T is the mean ocean surface temperature,

c_1 and c_2 are certain constants, and $c_3 sin(\omega t)$ stands for solar forcing.

That picture up there is a Duffing oscillator model, which is referred to as Model 2.1. Due to the fact that the ice-climate system is susceptible to perturbations in orbital forces and nonlinear bistable modes, both inside and across glaciers, this Duffing-like form serves as the foundation for the cryosphere model. Similar to the Duffing oscillator and its double-well potential, the cryosphere offers a helpful model for rapid transitions between cold and warm periods. This is because the cryosphere has two stable states that compete with one another and an unstable barrier that divides them. Homoclinic bifurcations, which may trigger large-scale regime transitions (to chaotic behavior), are also taken into consideration in the Duffing model as a consequence of periodic disturbances (or orbital fluctuations). This is because homoclinic bifurcations can lead to chaotic behavior. Furthermore, this enables us to make use of analytical techniques in order to investigate changes that lead to the formation of new states, in addition to providing a mathematical framework that is not only straightforward but also resilient for comprehending the irregularity and intensity of Quaternary glaciations. Consequently, a Duffing-type model may be used to facilitate an easy comprehension of the functioning of dynamic processes that occur in the cryosphere.

To observe the minute changes in the behavior of the System 2.1, we have considered the fractional order system in the Caputo sense as follows

$$\begin{cases} {}^{C}_{t_0}D_t^{\alpha}[s(t)] = \zeta, \\ {}^{C}_{t_0}D_t^{\alpha}[\zeta(t)] = c_1\zeta + c_2 s - s^3 - s^2\zeta + c_3 sin(\omega t), \end{cases} \quad (2.2)$$

The Duffing oscillator is a well-known nonlinear dynamical system that has been the subject of substantial investigation among researchers in the fields of physics, mathematics, engineering, and biology [40]. It is of significant importance to many people that the Duffing oscillator model be used to make predictions and conduct research into occurrences that occur in the actual world. Through the use of a Duffing oscillator, Hu and Wen were able to illustrate the constituents of a mechanical breakdown signal [41]. In their discussion of the Duffing oscillator [42], Hesan Vahedi and his colleagues presented it as a unique passive islanding detection technique for the purpose of ensuring the safe functioning of distributed generation (DG). In contrast to other Duffing oscillator models that have been researched in a variety of contexts, the Cryosphere Model 2.1 has been mostly ignored up to this point. Consequently, we are interested in diving more into this methodology for projects that will be completed in the future.

where α denotes the fractional order of the system and $0 < \alpha \leq 1$. Figure 2.1 shows the behavior of the System 2.2.

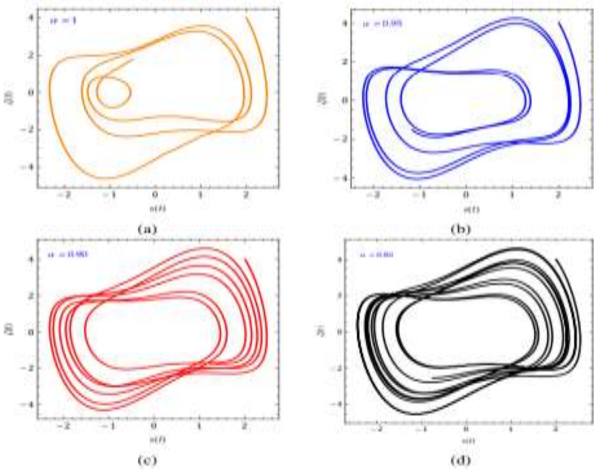
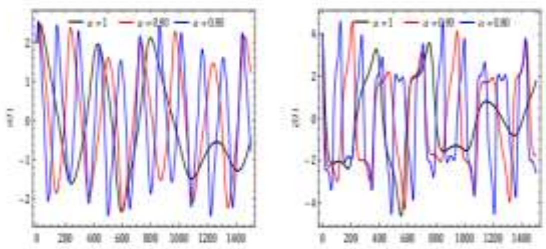


Figure 2.1: Behaviour of the System 2.2 for (a) $\alpha = 1$, (b) $\alpha = 0.95$, (c) $\alpha = 0.9$ and (d) $\alpha = 0.8$, for $c_1 = 1.01$, $c_2 = 0.1$, $c_3 = 1.4$ and $\omega = 0.9$.

When it comes to tackling a variety of problems, mathematical techniques are often considered to be the strategy of choice [43–45]. There are a number of major reasons why mathematical methods are superior than alternatives. Some of these reasons include speed, clarity and precision, fairness, generalizability, and original creation. On the other hand, applying logic to problems that arise in the real world is not always a straightforward procedure. Not only can numerical techniques produce correct results by exactly approximating the answer, but they also provide visual solutions that simplify mathematical models that would otherwise be unintelligible. This is where numerical approaches shine. The following are some of the most well-known numerical approaches reported in published works:

The approaches that were suggested by Euler, Adams-Bashforth, Runge-Kutta (RK), and Adams-Bashforth-Moulton [46-48]. Higher-order Runge-Kutta procedures are helpful for complicated systems because they give stiff ordinary differential equations that are both correct and reliable from a mathematical standpoint. However, putting it into practice might prove to be difficult and costly in terms of resources. In spite of the fact that it is only used in a limited capacity in the literature, the Runge-Kutta method of the seventh order is an outstanding numerical approach to initial value problems. As part of this investigation, the fractional order system will be solved by using a modified version of the RK-7 method.\



(a) (b)

Figure 2.2: Time series plots of (a) $s(t)$ and (b) $\zeta(t)$ for the System 2.2 for $c_1 = 1.01, c_2 = 0.1, c_3 = 1.4$ and $\omega = 0.9$.

In System 2.1, the consequences of climate change are not taken into consideration. The major objective of this study is to monitor any possible changes in the behavior of the model as a result of the addition of climate change and global warming implications. The relevance of including the effects of climate change into the model for the surface energy balance-mass balance of the cryosphere is recognized in this study, which distinguishes it from previous similar works. In order to get a deeper comprehension of the ways in which it is possible for changes in external conditions to influence stability, we may include a radiation force of $\boldsymbol{V}O2$ and see the impact that it has on the behavior of the model.

A. BifurcationAnalysis

Through the use of bifurcation analysis, this part assesses whether or not System 2.4 displays disorganized behavior. Bifurcation maps are an effective tool in the field of nonlinear dynamics [53] that may be used to visualize the reaction of a system to changes in the characteristics of the system. The comprehension of complicated dynamics, the testing of models, the construction of systems, and the prediction of significant events are all aided by these maps, which disclose the locations of bifurcation points and stability zones. The values of J3, a bifurcation parameter in our new system, were varied between (i) [1.1, 1.5] and (ii) [-4, 0] in order to generate Figures 2.3 (a), (b), and Figures 2.3 (c), (d). These figures were created by adjusting the value of J3. Figures 2.3 (a) and 2.3 (b) depict the branching graphs for System 2.4 for 300 cycles and 500 cycles, respectively. These graphs are illustrated in the accompanying figures. Period-doubling splits may be seen at a number of sites, including θ3 = 1.2, 1.3, and 1.38, among others.

IV MODEL OF ATMOSPHERIC PROPAGATION

The use of fractional calculus is a good technique to use in order to begin the process of grasping chaos in models of complex systems. Chaos systems are difficult for traditional models to grasp because of their complicated characteristics and the fact that they cannot be predicted. In spite of this, we now have a more accurate picture thanks to the incorporation of non-integer order derivatives into fractional calculus.These fractional operators may be of assistance in the accurate and comprehensive description of complicated systems [58-60]. This is because they take into consideration the seemingly insignificant details that are overlooked by traditional calculus. As far as partial groups were concerned, Hartley and colleagues [61] were of the opinion that Chua's method was too disorganized. Through the use of a fractional paradigm, G. M. Zaslavsky conducted research on the possibility of chaotic processes [62]. Already, Li and Peng had shown via the use of fractional order that Chen's method was disorderly [63].

Edward N. Lorenz came up with a simpler mathematical model of the circulation of the atmosphere in the year 1984. A simplified representation of the topography and weather patterns of Earth is included into it. Because of this, the Lorenz-84 climate model has gained a lot of notoriety. The Hadley cell circulation and the globe-encircling circulation are two more names for this movement among others.

Within the atmosphere, one of the most important functions of wind is to facilitate the movement of heat from the equator to the poles. It travels as a result of the temperature difference that exists between the poles and the equator. For the purpose of generating this motion, the Lorenz-84 model makes use of a wind component that is defined on a global scale. Based on the Lorenz-84 model, it is said that the circulating wind is responsible for the movement of heat to the poles because of the temperature difference between the equator and the poles. It is the wind that is responsible for transporting warm air from the tropics to the poles. This is because heat is always moving. As a consequence of heat movement, the temperature difference that exists between high and low latitudes is diminished. According to what Lorenz said in [64], the model was characterized by

a network of large-scale eddies that transport heat to the poles, where M represents the symmetric thermal forcing in response to differences in temperature between latitudes, Q represents the asymmetric thermal forcing in response to differences in temperature between continents and oceans, and U represents the strength of the cyclical westerly wind current.

$$\begin{cases} \frac{dW}{dt} = -\zeta^2 - \eta^2 - aW + aP, \\ \frac{d\zeta}{dt} = W\zeta - bW\eta - \zeta + Q, \\ \frac{d\eta}{dt} = bW\zeta + W\eta - \eta, \end{cases} \quad (3.1)$$

A.SAMARDZIJA-GRELLER MODEL

In the early part of the 20th century, two of the most influential scholars in the field, Alfred Lotka and Vito Volterra, used predator-prey models in their breakthrough research [77, 78]. The dynamics that exist between different groups of hunters and their prey were intended to be shown by their model. Due to the fact that its core assumptions were very straightforward and predictable, this traditional model was not as effective in explaining the cyclical interactions that occur between predators and prey. The researchers finally came to the conclusion that simpler models did not adequately represent the complexity of biological systems, and as a result, they endeavored to construct models that were more precise and comprehensive. A significant event that occurred during the latter part of the twentieth century was the implementation of the Samardzija-Greller plan. The notion that would eventually become this technique was first offered by Nikola Samardzija and Larry D. Greller in the year [79]. It contained random dynamics within the setting of the interactions between three species, each of which was acting as a predator and a prey. Through the application of chaos theory to a "one prey-two predators" scenario, the Samardzija-Greller model revealed that unexpected and non-linear behaviors might occur in a biological system that was more realistic. This remarkable development shed even more light on the complexities and unpredictability of interactions between predators and their prey. Samardzija and Greller [79] came up with a model that they then developed.

B. MODELMODIFICATION

Through the use of the updates to Model 4.1 that were mentioned before, we made considerable modifications to the predator models. One of the improvements that we have made is the capability to either categorize predators as interspecific rivals or intraspecific competitors.The following real-life circumstances are taken into consideration by these changes, which is why they are so important:

Because individuals of the same species have a tendency to share resources and ecological niches, there is competition among members of the same species for mates, food, and habitat space. Due to the fact that predators engage in fierce competition for the same prey individuals, it becomes more challenging for one or both of the predator species to successfully seek food when there is only one species present. Due to this, both species of predators are forced to modify their hunting techniques and the sources of food they consume.

V. ECOLOGICAL MODELS

The predator-prey model has been used by ecologists for a very long time. In the early 1900s, Lotka [77] and Volterra [78] contributed to the development of predator-prey models in a number of ways. A balance between the availability of resources, breeding, and hunting has been proven to be maintained in ecosystems by the processes that have been shown by the mathematical models. Furthermore, they have given us with essential information on the dynamic nature of the groups of predators and prey being observed. An fascinating aspect of these ideas is the fear effect [109], which describes the behavior of prey species when they are in the presence of other species who are attempting to attack them. Prey animals often go into a state of intense terror when they are in the company of what may be considered predators. These emotions may manifest themselves in a variety of ways, including increased alertness, changing behaviors when searching for food, and changes in the sorts of settings that animals prefer to inhabit.

It is possible for prey animals to minimize their susceptibility to being eaten by predators by reducing the amount of activity they engage in or by avoiding places that are often visited by predators. The fear effect is very important because of the potentially detrimental effects it may have on both prey and predators on the same animal. In the event that the behavior of the target species changes as a consequence of the fear effect, this may have an influence on the generation rates of their offspring as well as the dynamics of their populations. It is possible that the fear effect will have an influence on the amount of energy that predators consume and their ability to hunt. Furthermore, predator-prey models have shed light on cooperative defense, which is a kind of protection in which prey species may coordinate their behaviors in order to more effectively fend off or elude predators [110, 111]. Interactions between different species of prey might potentially improve their chances of surviving.

A. BIFURCATIONANALYSIS

We examine the behavior of the nonlinear System 5.1 at important points and the ways in which it varies in this section. In order to accomplish this goal, it is required to do research on the impact of fractional-order dynamics on system cycles as well as transitions from stable to unstable and unexpected patterns. By carrying out these tests, we will have a better grasp of the capabilities of our system as well as the applications that it may potentially be used for.

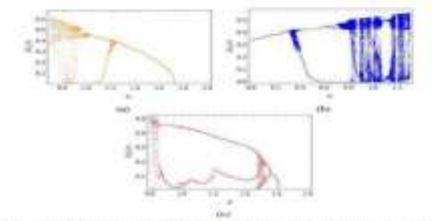


Figure 5.1: Bifurcations in the Model 5.1 for $\alpha = 1$ due to bifurcation parameters (a) e_4, (b) e_1 and (c) β, for $M_1(0) = 0.3$, $M_2(0) = 1.3$, $N(0) = 0.5$.

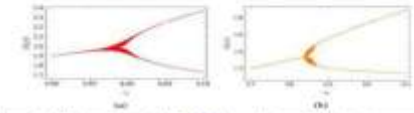


Figure 5.2: Bifurcations in the Model 5.1 for $\alpha = 1$ when the bifurcation parameters (a) e_1 and (b) e_2 are varied, for $M_1(0) = M_2(0) = N(0) = 0.1$.

DISCUSSION

This research emphasizes that the environment is a **coupled system**. Changes in the atmospheric transmission (Lorenz-84) directly impact the cryosphere, which in turn alters the habitats modeled in our ecological sections. The use of nonlinear dynamical systems theory provides the necessary "lens" to see these interconnected tipping points before they are reached in reality.

CONCLUSION

Numerous models of ecosystems and climates are used in a substantial manner throughout this thesis. It was necessary to devote a considerable lot of attention to the nonlinear fractional dynamics because of their relevance. The existence of key characteristics of stable, unstable, chaotic, and split behavior was demonstrated via fractional frames. It's possible that this may result in much improved control systems and forecasting capabilities.These studies shed light on the usefulness of complex mathematical models in gaining an understanding of the present biological and environmental concerns, especially those that are associated with global warming, when time delay and human impact are taken into consideration.While the findings of this research will be helpful in theoretical examinations of fractional dynamical systems, they will also contribute to our capacity to grasp environmental problems such as climate change and to create solutions to reduce their effects.

Within the second part of the study, the consequences of climate change were investigated using a modified model of the energy balance and mass balance of the cryosphere. When a measure for f2 radiative forcing was provided, there were discernible movements from stability to chaos and instability as a function of fractional order and forcing levels. These alterations occurred as a result of the inclusion of the measure. In accordance with complex approaches such as bifurcation analysis and chaos management, the addition of time to behavior that is unstable may result in the behavior becoming more stable. The relevance of incorporating the effects of climate change into prediction models for the purpose of enhancing their accuracy and dependability is brought to light by these results. This is due to the fact that the cryosphere must continue to preserve its predictability.

In Chapter 3, a modified fractional Lorenz-84 climate model was investigated to determine the impacts of time delay, sliding mode control, and an increase in the average world temperature.It is possible for a system to experience fractional-order changes in the presence of radiation force, which may lead the system to oscillate between stable and unstable states. When a time delay is included into the system, the stability of the system is considerably increased. Consequently, the behavior becomes asymptotically stable for specific fractional orders, and the chaos is minimized as a consequence of this finding. This illustrates that the improved time-delayed model gives a more accurate picture of the mobility of the atmosphere as well as the interplay of pressure, temperature, and wind speed on a wide scale. The fourth part of the presentation focused on analyzing the dynamics of a modified version of the Samardzija-Greller predator-prey model. The concepts of synchronization, stability, chaos, and control were investigated via the use of fractional calculus. In our demonstration, we showed that the synchronization rates are dependent on the order, which indicates that higher orders speed up the convergence process.

The use of numerical models contributes to the development of chaos and synchrony management approaches by giving information on the dynamics of ecosystem change over time. Furthermore, our research brought to light the cyclicality and unpredictability of the fractional slow-fast system, as well as highlighted the complicated multiscale spatiotemporal dynamics of the system. During the discussion of the fractional Hopf split that took place in Chapter 5, a three-species ecological model that included fear effects and predator-taxis sensitivity was presented. What this demonstrates is the significant part that behavioral dynamics play in the dynamics of predators and prey environments. It was the major emphasis of this study to investigate the influence that these components have on the safety of the environment.New models of the environment were developed in the sixth chapter by making use of data from the actual world and various modeling techniques, including fractional and random modeling. In order to create more accurate projections about air pollution, forest carbon sinks, and river water quality, it was necessary to combine data-driven methodologies with complex mathematical algorithms. This clearly established the models' capacity to be used in real-world situations.

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