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Mathematical Representations of Global Warming That Use Nonlinear Dynamical Systems

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ABSTRACT

Examining the climate change and natural system temporal dynamics, this thesis employs sophisticated mathematical and computational methodologies. New insights into environmental phenomena and their nonlinear evolution over time are being provided by a combination of numerical tools, mathematical modeling, data-driven methodologies, and fractional calculus. We took a look at how a modified cryosphere surface energy balance-mass balance model spreads in relation to climate change. In particular, how environmental changes influence energy flow and mass balance is stressed. Adding time delays, chaos control mechanisms, and climate change impacts to the Lorenz-84 atmospheric transmission model could make it more useful. Thanks to these developments, we can now take use of cutting-edge techniques for weather management and prediction.Using a modified version of the Samardzija-Greller predator-prey model, this work delves further into timing, control, and multiscale dynamics. Here we see how shifting temporal frames might impact the dynamics of biological systems. We have created a novel ecological model that takes into account behavioral factors like the effect of fear and how predators react to taxis. These components are crucial for maintaining ecological balance and protecting species populations, which is not surprising. Lastly, using real-world data and environmental models that are either fractional or random, we can predict key environmental indicators like air pollution levels, forest carbon storage, and river water quality. By improving theoretical models and closing the gap between mathematical theory and its practical applications, this study paves the way for further research and policymaking in the fields of biological dynamics and climate. Additionally, this study provides a foundation for studies.

Index Terms: ***Fractional Calculus, Climate Dynamics,***

Nonlinear Modeling, Environmental Forecasting.

INTRODUCTION

Many scientific disciplines rely on mathematical modeling to help them understand and solve real-world issues. This includes physics, climate science, chemistry, engineering, biology, and many more. People may understand more and make more well-informed decisions because of this. In order to understand and replicate complicated climatic phenomena, mathematical models play a crucial role in climate research. Because of this, climatic patterns such as changing temperatures, increasing sea levels, and extreme weather may be studied and predicted by scientists [1].

By using these models, which connect research with decision-making, lawmakers may make informed choices on climate change adaptation and mitigation. We can learn from the environment and make predictions about physics's future by using mathematical models [2]. In several branches of physics, from classical to quantum, scientists are tasked with developing mathematical models to describe the actions of particles, waves, and forces. Mathematical modeling is a very intriguing field for engineers because of their shared interest in and commitment to building and improving processes, systems, and structures [3].

Modern life is replete with activities that rely on mathematical models and predictions of behavior to ensure the smooth operation and safety of many systems. These sectors include construction, aviation, electrical engineering, and manufacturing, among many others. Mathematical models have the ability to enhance the comprehension and study of several intricate biological processes when applied to the field of biology [4-6]. Among these processes are the dynamics of populations, interactions between ecosystems, and the spread of diseases. These models aid in the development of new treatments and pharmaceuticals by shedding light on the fundamental ideas that underpin biological processes.

Many mathematical models have relied on calculus of the integer order for many years. The use of fractional calculus (FC) to mathematical modeling, however, remains in its infancy. Its fame has been steadily rising in recent years. Numerous fields have found success using mathematical models that include fractional calculus [7–10]. These fields include economics, physics, engineering, and biology. Fractional calculus may provide a more precise and flexible model for systems that do not exhibit linear or localized behavior. Among the many examples are control systems, non-linear materials, and peculiar diffusion processes. Researchers have been able to delve farther into complex systems and explore previously uncharted areas since this technology is readily available. In its most basic form, fractional calculus consists of two things: the integration of integers with fractional orders and the taking of derivatives.

Continuing along the same lines, he went on to say that the letter "i" may stand for any real number. In the year 1822, Fourier used what is now known as the Fourier transform of a function in order to describe the derivative of any order. Different approaches to representing the fractional derivative of a function have been devised by succeeding mathematicians via a variety of different ways. On the other hand, the most common definitions of the fractional derivative are those proposed by Atangana-Baleanu, Caputo, Caputo-Fabrizio, and Riemann-Liouville. For example, neural networks, material extraction, and image processing are all relatively new fields, yet fractional calculus is becoming more popular in these areas as well. In their study [13], Yang et al. demonstrated how fractional calculus may be used in the process of picture fusion. To a certain extent, fractional mathematics is used in the process of constructing neural networks [14]. Using fractional differential equations is another method that may be used to locate ore deposits [15]. What is the outcome of dividing by

Calculus has shown to be a useful instrument in the process of enhancing models of chaotic systems. This is due to the fact that calculus has the ability to disclose sophisticated things that standard integer-order calculus often fails to reveal. The use of fractional derivatives makes it possible to get a more precise portrayal of the dynamics that lie under the surface. Their assistance may be of use in better comprehending chaotic processes, peculiar attractor occurrences, and sudden changes. When it comes to many different systems, both natural and manmade, this is true. As Diouf and Sene [16] pointed out, the use of fractional operators is of the utmost importance in unpredictable financial models. A demonstration was made by Borah and colleagues [17] that demonstrated how fractional calculus may be used to enhance the precision of chaotic pandemic simulations. Saadeh and colleagues [18] found that a model of predators and prey that was only partially complete was chaotic.

LITERATURE REVIEW

TheSamardzija-Greller Model: Recent advancements have generalized this model to include slow-fast dynamics. Literature (ResearchGate, 2025) indicates that predators often react on slower timescales than prey, and this "multiscale" difference determines whether an ecosystem remains stable or falls into a "limit cycle" (periodic oscillation) or total extinction.

The "Ecology of Fear" and Behavioral Taxis: Contemporary ecological modeling has moved beyond simple biomass transfer. Inclusion of "fear" (indirect effects where prey change behavior to avoid predators) and "taxis" (movement in response to environmental gradients) is now recognized as vital for population persistence under climate stress.

Climate-Induced Extinction Resilience: Studies (e.g., AIMS Press) show that spatial gradients can act as "beachheads" for species survival. If temperatures rise, species that can migrate along these gradients using "taxis" are significantly more likely to avoid the extinction thresholds predicted by simpler, non-spatial models.

PROCEDURE

That picture up there is a Duffing oscillator model, which is referred to as Model 2.1. Due to the fact that the ice-climate system is susceptible to perturbations in orbital forces and nonlinear bistable modes, both inside and across glaciers, this Duffing-like form serves as the foundation for the cryosphere model. Similar to the Duffing oscillator and its double-well potential, the cryosphere offers a helpful model for rapid transitions between cold and warm periods. This is because the cryosphere has two stable states that compete with one another and an unstable barrier that divides them. Homoclinic bifurcations, which may trigger large-scale regime transitions (to chaotic behavior), are also taken into consideration in the Duffing model as a consequence of periodic disturbances (or orbital fluctuations). This is because homoclinic bifurcations can lead to chaotic behavior. Furthermore, this enables us to make use of analytical techniques in order to investigate changes that lead to the formation of new states, in addition to providing a mathematical framework that is not only straightforward but also resilient for comprehending the irregularity and intensity of Quaternary glaciations. Consequently, a Duffing-type model may be used to facilitate an easy comprehension of the functioning of dynamic processes that occur in the cryosphere.

The Duffing oscillator is a well-known nonlinear dynamical system that has been the subject of substantial investigation among researchers in the fields of physics, mathematics, engineering, and biology [40]. It is of significant importance to many people that the Duffing oscillator model be used to make predictions and conduct research into occurrences that occur in the actual world. Through the use of a Duffing oscillator, Hu and Wen were able to illustrate the constituents of a mechanical breakdown signal [41]. In their discussion of the Duffing oscillator [42], Hesan Vahedi and his colleagues presented it as a unique passive islanding detection technique for the purpose of ensuring the safe functioning of distributed generation (DG). In contrast to other Duffing oscillator models that have been researched in a variety of contexts, the Cryosphere Model 2.1 has been mostly ignored up to this point. Consequently, we are interested in diving more into this methodology for projects that will be completed in the future.

A. BIFURCATIONANALYSIS

Through the use of bifurcation analysis, this part assesses whether or not System 2.4 displays disorganized behavior. Bifurcation maps are an effective tool in the field of nonlinear dynamics [53] that may be used to visualize the reaction of a system to changes in the characteristics of the system. The comprehension of complicated dynamics, the testing of models, the construction of systems, and the prediction of significant events are all aided by these maps, which disclose the locations of bifurcation points and stability zones. The values of J3, a bifurcation parameter in our new system, were varied between (i) [1.1, 1.5] and (ii) [-4, 0] in order to generate Figures 2.3 (a), (b), and Figures 2.3 (c), (d). These figures were created by adjusting the value of J3. Figures 2.3 (a) and 2.3 (b) depict the branching graphs for System 2.4 for 300 cycles and 500 cycles, respectively. These graphs are illustrated in the accompanying figures. Period-doubling splits may be seen at a number of sites, including θ3 = 1.2, 1.3, and 1.38, among others.

B. DELAYEDMODEL

The next part will discuss the progression of time delay and how it occurs. The version 2.5. As a result of the prolonged nature of the change in mean ocean temperature and the possible considerable influence that it may have on the cryosphere, the second equation of Model 2.5 contains a time delay. There are delays in the effect on ocean temperature because the ocean needs time to respond to external forces such as solar radiation or climate change. This causes the ocean to take longer to react than it would otherwise. This is because the ocean system has its own internal resistance, which is the reason for this phenomenon. If you neglect this time delay, you will be oversimplifying your model, which will prevent you from getting a decent picture of how changes in ocean temperature effect the cryosphere. By taking into account this latency, the model is able to more properly depict the interaction and complexity of the climate system, which is found on Earth. It is possible that we will acquire the knowledge and the phenomenon that occurs when the behavior of the cryosphere is influenced by changes in the average temperature of the water. This model is what we refer to as the delayed model:

when changes in mean ocean temperature influence the behavior of the Cryosphere.

We propose the delayed model as

$$\begin{cases} {}_{t_0}^{C}D_t^{\alpha}[s(t)] = \zeta(t), \\ {}_{t_0}^{C}D_t^{\alpha}[\zeta(t)] = p_1\zeta(t-\theta) + p_2 s(t) - s(t)^3 - s(t)^2\zeta(t-\theta) + p_3 sin(p_4 t) + R. \end{cases} \tag{2.20}$$

where θ is the time delay.

IV IMPLICATIONS OF SLIDING MODE CONTROL

By using the third equation, a second feedback loop is effectively constructed. Additionally, it makes use of the sine phase (**α**) and is reliant on the value of S. These feedback loops demonstrate that a change in one variable may have an effect on the other variables.

by virtue of the fact that the climate system displays a wide range of behaviors as a consequence of the complex interactions that it undergoes. The memory-dependent and non-local behavior of dynamical systems is accounted for by fractional calculus, which is the reason why we are able to record these feedback processes. By using fractional derivatives, this model may be able to provide a better understanding of the long-range interactions and memory effects that influence the movement of large-scale eddies as well as the reaction of the westerly wind current to a variety of driving factors. Using the Caputo fractional derivative, we update System 3.1 so that it more accurately reflects the way in which it is developing over time.

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$$\begin{cases} {}_{t_0}^{C}D_t^{\alpha}[W(t)] = -\zeta^2 - \eta^2 - aW + aP, \\ {}_{t_0}^{C}D_t^{\alpha}[\zeta(t)] = W\zeta - bW\eta - \zeta + Q, \\ {}_{t_0}^{C}D_t^{\alpha}[\eta(t)] = bW\zeta + W\eta - \eta, \end{cases} \tag{3.2}$$

where α represents the fractional order of the system and $0 < \alpha \leq 1$.

The Caputo fractional derivative is preferred over all other fractional derivatives for the following reasons:

A condition that is referred to be an initial value case is one in which the value of the function and its derivatives remain stationary at a single point. Caputo fractional derivatives perform very well in this scenario. Because of this, they are perfect for mimicking systems to which the beginning points are already known.

The following is consistent with derivatives of integer order: Caputo fractional derivatives converge to the same integer order derivative when the fractional order approaches an integer. This occurs when the fractional order approaches an integer. Because of this, the shift from integer order calculus to fractional order calculus is completed in a rather seamless manner.

When compared to alternative interpretations of the word, such as the Riemann-Liouville fractional derivative, Caputo fractional derivatives have a tendency to have a more smoothbehavior. Because it generates models that are more physically correct and continuous, the smoothness characteristic is the one that is often favored.

As a result of its more robust connection to the Fourier transform, the Caputo fractional derivative is preferable for convolution-related tasks. This is shown by the fact that the convolution theorem has been proven.

In order to describe dynamical systems, we decided to adopt the fractional order **Ω** between 0 and 1 for the Caputo fractional derivative. This is because it is consistent with integer order derivatives from a theoretical standpoint and has evident algebraic features that are relevant in a variety of scenarios.Furthermore, the bulk of the results that will be reported here to establish the well-posedness of the proposed model hold true for fractional orders **Ω** that lie within the range of 0 to 1. Taking into account the fact that the temperature forcings M and Q change over the course of time has resulted in a comprehensive level of investigation into this model. Broer, Vitolo, and Simó [65] conducted an investigation on the development of the Lorenz-84 model in the year 2002. They did so under the premise that both F and Q are periodic. In the follow-up study that was conducted in 2005 [66], it was revealed that the model that was exposed to yearly forcing had attractor patterns that were similar to those of Hénon. One of the primary focuses of Bódai and Tél's [67] research was to investigate the impact of yearly driving on the model. Using snapshot attractor methods, they have seen the disorder that exists throughout the system. Musafirov [68] discovered a collection of perturbed systems that were suited for the Lorenz-84 climate model and had reflecting functions that were similar to one another. A sliding mode chaos management strategy that was introduced by Premakumari and colleagues [69] is included into the Lorenz-84 model. Investigations have also been conducted to determine how the connected Lorenz-84 model develops when other models are present simultaneously. Several components of the Lorenz-84 model have been examined by Lennaert Van Veen and colleagues [70], in addition to the box model developed by Stommel.

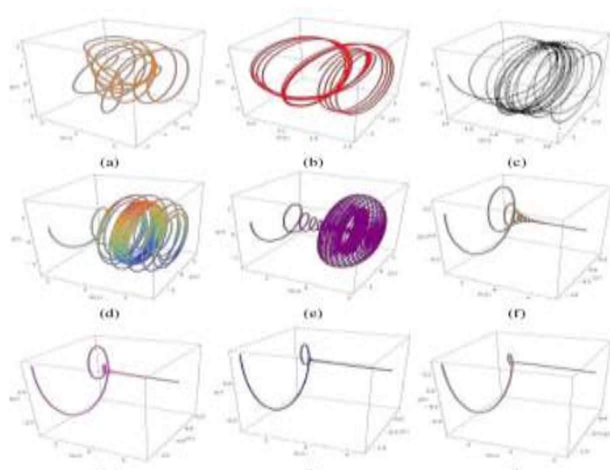

Figure 3.1: Behavior of the System 6.13 for α-values (a) 1, (b) 0.97, (c) 0.95, (d) 0.90, (e) 0.87, (f) 0.85, (g) 0.80, (h) 0.75 and (i) 0.70 for $a = 0.25$, $b = 4$, $P = 8$ and $Q = 1$.

A. LYAPUNOVEXPONENTANALYSIS

In the process of showing the existence of chaos in a dynamic system, Lyapunov exponents are an extremely important consideration. These numbers [73] provide a description of the exponential rate at which routes in a system that are close together move away from one another or toward one another over the course of time. Chaotic systems are characterized by the presence of at least one positive Lyapunov exponent [54]. This is because chaotic systems are very dependent on the beginning conditions. When the Lyapunov exponent is positive, paths that are immediately next to one another scatter in a short amount of time. As may be seen in the illustrations below, chaotic systems display behavior that is both intricate and very unpredictable.

Table 4.1: Lyapunov exponents of 4.3 for varying α, for $P[0] = 1$, $Q[0] = 1.4$, $R[0] = 1$ and for $a = 2.4$, $b = 3$, $c = 2$, $d_1 = d_2 = e_1 = e_2 = 0.01$.

α	LE1	LE2	LE3
1	0.0135997	0.0196859	-0.0166701
0.95	0.0224352	0.00912667	-0.0200168
0.90	0.0184108	-0.0142709	-0.00346664
0.85	0.0207477	-0.000716995	-0.0318338
0.80	0.00026046	0.0120852	-0.0312502
0.75	-0.000235545	-0.0182104	-0.0446468

DISCUSSION

This research emphasizes that the environment is a **coupled system**. Changes in the atmospheric transmission (Lorenz-84) directly impact the cryosphere, which in turn alters the habitats modeled in our ecological sections. The use of nonlinear dynamical systems theory provides the necessary "lens" to see these interconnected tipping points before they are reached in reality.

CONCLUSION

Numerous models of ecosystems and climates are used in a substantial manner throughout this thesis. It was necessary to devote a considerable lot of attention to the nonlinear fractional dynamics because of their relevance. The existence of key characteristics of stable, unstable, chaotic, and split behavior was demonstrated via fractional frames. It's possible that this may result in much improved control systems and forecasting capabilities.These studies shed light on the usefulness of complex mathematical models in gaining an understanding of the present biological and environmental concerns, especially those that are associated with global warming, when time delay and human impact are taken into consideration.While the findings of this research will be helpful in theoretical examinations of fractional dynamical systems, they will also contribute to our capacity to grasp environmental problems such as climate change and to create solutions to reduce their effects.

Within the second part of the study, the consequences of climate change were investigated using a modified model of the energy balance and mass balance of the cryosphere. When a measure for f2 radiative forcing was provided, there were discernible movements from stability to chaos and instability as a function of fractional order and forcing levels. These alterations occurred as a result of the inclusion of the measure. In accordance with complex approaches such as bifurcation analysis and chaos management, the addition of time to behavior that is unstable may result in the behavior becoming more stable. The relevance of incorporating the effects of climate change into prediction models for the purpose of enhancing their accuracy and dependability is brought to light by these results. This is due to the fact that the cryosphere must continue to preserve its predictability.

In Chapter 3, a modified fractional Lorenz-84 climate model was investigated to determine the impacts of time delay, sliding mode control, and an increase in the average world temperature.It is possible for a system to experience fractional-order changes in the presence of radiation force, which may lead the system to oscillate between stable and unstable states. When a time delay is included into the system, the stability of the system is considerably increased. Consequently, the behavior becomes asymptotically stable for specific fractional orders, and the chaos is minimized as a consequence of this finding. This illustrates that the improved time-delayed model gives a more accurate picture of the mobility of the atmosphere as well as the interplay of pressure, temperature, and wind speed on a wide scale.

The fourth part of the presentation focused on analyzing the dynamics of a modified version of the Samardzija-Greller predator-prey model. The concepts of synchronization, stability, chaos, and control were investigated via the use of fractional calculus. In our demonstration, we showed that the synchronization rates are dependent on the order, which indicates that higher orders speed up the convergence process. The use of numerical models contributes to the development of chaos and synchrony management approaches by giving information on the dynamics of ecosystem change over time. Furthermore, our research brought to light the cyclicality and unpredictability of the fractional slow-fast system, as well as highlighted the complicated multiscale spatiotemporal dynamics of the system. During the discussion of the fractional Hopf split that took place in Chapter 5, a three-species ecological model that included fear effects and predator-taxis sensitivity was presented. What this demonstrates is the significant part that behavioral dynamics play in the dynamics of predators and prey environments. It was the major emphasis of this study to investigate the influence that these components have on the safety of the environment.New models of the environment were developed in the sixth chapter by making use of data from the actual world and various modeling techniques, including fractional and random modeling. In order to create more accurate projections about air pollution, forest carbon sinks, and river water quality, it was necessary to combine data-driven methodologies with complex mathematical algorithms. This clearly established the models' capacity to be used in real-world situations.

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