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Alpha Power Rayleigh Distribution and Its Application to Life Time Data

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ABSTRACT

In this paper we present a new two parameter Rayleigh distribution motivated by Alpha Power Transformation known as Alpha Power Rayleigh distribution. The new distribution is more flexible and has some interesting properties. The properties of the proposed distributions are discussed and explicit expressions are derived. Parameter estimation is accomplished using maximum likelihood estimation. The applications of Alpha Power Rayleigh distribution are emphasized using two real life examples.

Keywords: Alpha Power Transformation, Alpha Power Rayleigh distribution, entropy, order statistics, reliability.

I. INTRODUCTION

Rayleigh distribution is one of the distribution of prominent importance in life testing experiments and is named after Rayleigh [1]. It also finds an application in communication theory, in physical sciences to model wave heights, wind speed etc., in medical imaging sciences. When the shape parameter of two parameter Weibull distribution equals to 2, the resultant distribution turns out to be the Rayleigh distribution. The origin and properties of Rayleigh distribution were studied by Siddiqui [2].Some other authors who contributed to this model are Merovci et al [3], Ahmad et al [4],Howlader and Hossian [5] and Abd Elfattah et al. [6].

There are various methods available in literature that could be used to make the distributions richer and flexible to model the real life data. One of the procedure available in literature is Alpha Power Transformation (APT) suggested by Mahdavi and Kundu [7]. Mahdavi and Kundu [7] proposed Alpha Power Exponential distribution based on APT. Recently Nassar et al. [8] proposed "Alpha Power Weibull distribution: Properties and Application". Let X be a continuous random variable with cumulative density function (CDF) $F(x)$, then the CDF of X under APT is given as:

$$F_{APT}(x) = \begin{cases} \dfrac{\alpha^{F(x)}-1}{\alpha-1} & ;\alpha>0\ ,\alpha\neq 1 \\ F(x) & ;\alpha=1 \end{cases} \tag{1}$$

And the corresponding probability density function (PDF) is given as:

$$f_{APT}(x) = \begin{cases} \dfrac{\log\alpha}{\alpha-1} f(x)\alpha^{F(x)} & ;\alpha>0\ ,\alpha\neq 1 \\ f(x) & ;\alpha=1 \end{cases} \tag{2}$$

The main aim of this paper is to put forward a new distribution based on APT called Alpha Power Rayleigh (APR) distribution so as to increase the flexibility of base model for modelling purposes. The rest of the paper is organized as follows: in section 2 the new distribution, namely, APR distribution is introduced and its reliability measures are discussed. Various mathematical properties such as moments, order statistics, entropy etc.is studied in section 3. In section 4 the parameters of the distribution are estimated using maximum likelihood estimation. Application of the proposed distribution is debated with the help of two real life data sets. Finally some discussion and conclusions are given in section 6 and 7 respectively.

II. APR DISTRIBUTION

The random variable X is said to follows two parameter APR distribution with the scale parameter $\theta > 0$ if the CDF of x>0 is given by:

$$F_{APR}(x) = \begin{cases} \dfrac{\alpha^{1-e^{-\frac{x^2}{2\theta^2}}} - 1}{\alpha - 1} & ;\alpha > 0 \ ,\alpha \neq 1 \\ 1 - e^{-\frac{x^2}{2\theta^2}} & ;\alpha = 1 \end{cases} \tag{3}$$

And the corresponding probability density function (PDF) of APR distribution is given by:

$$f_{APR}(x) = \begin{cases} \dfrac{\log\alpha}{\alpha - 1} \dfrac{x}{\theta^2} e^{-\frac{x^2}{2\theta^2}} \alpha^{1-e^{-\frac{x^2}{2\theta^2}}} & ;\alpha > 0 \ ,\alpha \neq 1, \theta > 0 \\ \dfrac{x}{\theta^2} e^{-\frac{x^2}{2\theta^2}} & ;\alpha = 1, \theta > 0 \end{cases} \tag{4}$$

The graph of PDF and CDF are given below

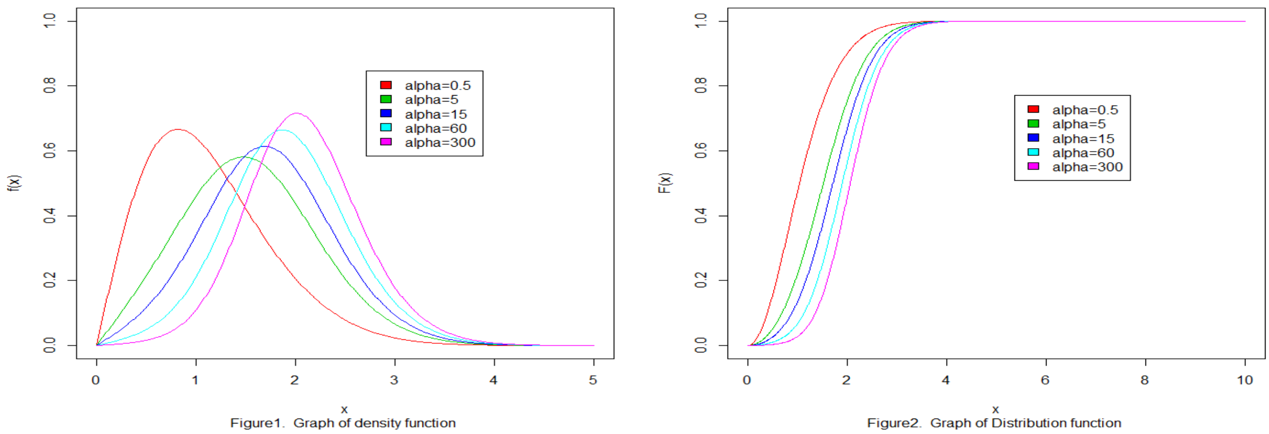

Figure 1: plots of APR distribution for $\theta = 1$ and different values of α

From Fig. 1 it can be seen that for α <1, $f(x)$ tends to be right skewed and unimodal and for α >1, $f(x)$ is unimodal and tends to become symmetrical as the value of α is increased.

2.1. Reliability analysis

2.1.1. Reliability function

The reliability function of APR distribution is given as:

$$R_{APR}(x) = \begin{cases} \dfrac{\alpha(1 - \alpha^{-e^{-\frac{x^2}{2\theta^2}}})}{\alpha - 1} & ;\alpha > 0 \ ,\alpha \neq 1 \\ e^{-\frac{x^2}{2\theta^2}} & ;\alpha = 1 \end{cases}$$

2.1.2. Hazard function

The hazard function of APR distribution is given as:

$$h_{APR}(x) = \begin{cases} \dfrac{\log\alpha}{\alpha(1 - \alpha^{-e^{-\frac{x^2}{2\theta^2}}})} \dfrac{x}{\theta^2} e^{-\frac{x^2}{2\theta^2}} \alpha^{1-e^{-\frac{x^2}{2\theta^2}}} & ;\alpha > 0 \ ,\alpha \neq 1 \\ \dfrac{x}{\theta^2} & ;\alpha = 1 \end{cases}$$

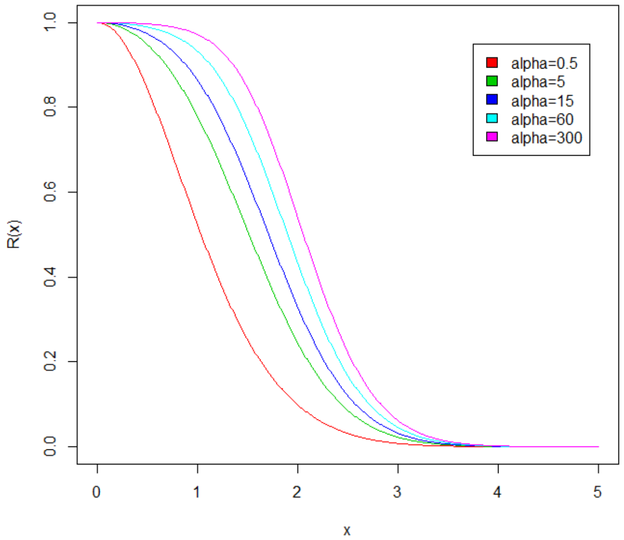

Figure3. Graph of Reliability function

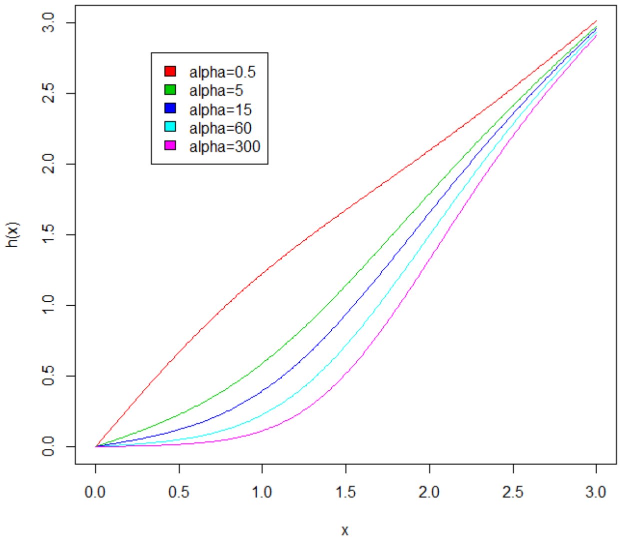

Figure4. Graph of hazard function

2.1.3. Stress strength Reliability

Let X_1 and X_2 be two independent random variables where $X_1 \sim APR(\alpha_1, \theta_1)$ and $X_2 \sim APR(\alpha_2, \theta_2)$, then the stress strength parameter denoted by R is given as:

$$R = \int_{-\infty}^{\infty} f_1(x) F_2(x) dx$$

$$R = \int_0^{\infty} \frac{\alpha_1 \log \alpha_1}{(\alpha_1 - 1)} \frac{x}{\theta_1^2} e^{-\frac{x^2}{2\theta_1^2}} \alpha_1^{-e^{-\frac{x^2}{2\theta_1^2}}} \frac{(\alpha_2^{-e^{-\frac{x^2}{2\theta_2^2}}} - 1)}{(\alpha_2 - 1)} dx$$

Using (5) and simplifying we get

$$R = \frac{\alpha_1 \log \alpha_1}{(\alpha_1 - 1)(\alpha_2 - 1)\theta_1^2} \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{(-\log \alpha_1)^k (-\log \alpha_1)^j}{k!\, j!} \int_0^{\infty} x e^{-\frac{x^2}{2\theta_1^2}} e^{-\frac{kx^2}{2\theta_1^2}} e^{-\frac{jx^2}{2\theta_2^2}} dx - \frac{1}{(\alpha_2 - 1)}$$

$$R = \frac{\alpha_1 \log \alpha_1}{(\alpha_1 - 1)(\alpha_2 - 1)\theta_1^2} \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{(-\log \alpha_1)^k (-\log \alpha_1)^j}{k!\, j! \left(\frac{1}{2\theta_1^2} + \frac{k}{2\theta_1^2} + \frac{j}{2\theta_2^2} \right)} - \frac{1}{(\alpha_2 - 1)}$$

III. MATHEMATICAL PROPERTIES

In this section various mathematical properties of APR distribution are discussed. Note that for the rest of the paper we assume that $\alpha \neq 1$.

3.1. Quantiles and simulation

The APR distribution can be easily simulated by inverting Eq. (3) as follows: if p follows uniform distribution $U(0,1)$, then

$$x = \left\{ -2\theta^2 \log \left[1 - \frac{\log(1 - (\alpha - 1)p)}{\log \alpha} \right] \right\}^{1/2}$$

Quantile function is given by:

$$x_q = \left\{ -2\theta^2 \log \left[1 - \frac{\log(1 - (\alpha - 1)q)}{\log \alpha} \right] \right\}^{1/2}$$

Median is obtained by putting q=0.5 in above equation. The resultant equation is given below:

$$x_{0.5}=\left\{-2\theta^2\log\left[1-\frac{\log(1-(\alpha-1)(0.5))}{\log\alpha}\right]\right\}^{1/2}$$

3.2. Moments and moment generating function

The r^{th} moment about origin is given as:

$$\mu_r{}'=E[X]^r=\int_0^\infty x^r\frac{\log\alpha}{\alpha-1}\frac{x}{\theta^2}e^{-\frac{x^2}{2\theta^2}}\alpha^{1-e^{-\frac{x^2}{2\theta^2}}}dx$$

.Using the power series expansion

$$a^{-u}=\sum_{i=0}^{\infty}\frac{(-\log\alpha)^i u^i}{i!} \qquad (5)$$

We get

$$\mu_r{}'=(2\theta)^{r/2}\left(r/2\right)!\frac{\alpha}{(1-\alpha)}\sum_{i=0}^{\infty}\frac{(-\log\alpha)^{i+1}}{(i+1)^{r/2+1}i!} \qquad (6)$$

Also, the moment generating function is given as:

$$M_X(t)=(2\theta)^{r/2}\left(r/2\right)!\frac{\alpha}{(1-\alpha)}\sum_{i=0}^{\infty}\frac{(-\log\alpha)^{i+1}}{(i+1)^{r/2+1}i!} \qquad (7)$$

3.3. Information measures

3.3.1. Renyi entropy

The Renyi entropy denoted by $I_R(\rho)$ is defined as:

$$I_R(\rho)=\frac{1}{1-\rho}\log\left\{\int_{-\infty}^{\infty}f(x)^\rho dx\right\}.$$

Where $\rho>0$ *and* $\rho\neq1$.

The Renyi entropy of APR distribution can be obtained as

$$I_R(\rho)=\frac{1}{1-\rho}\log\left[\int_0^\infty\left(\frac{\log\alpha}{\alpha-1}\frac{x}{\theta^2}e^{-\frac{x^2}{2\theta^2}}\alpha^{1-e^{-\frac{x^2}{2\theta^2}}}\right)^\rho dx\right]$$

$$I_R(\rho)=\frac{1}{1-\rho}\log\left[\left(\frac{\alpha\log\alpha}{(\alpha-1)\theta^2}\right)^\rho\int_0^\infty\left(xe^{-\frac{x^2}{2\theta^2}}\alpha^{-e^{-\frac{x^2}{2\theta^2}}}\right)^\rho dx\right]$$

Using (5) the Renyi entropy of APR distribution is given as:

$$I_R(\rho)=\frac{1}{1-\rho}\log\left[\left(\frac{\alpha\log\alpha}{(\alpha-1)\theta^2}\right)^\rho(2\theta^2)^{(\rho-1)/2}\sum_{i=0}^{\infty}\rho^i\frac{(-\log\alpha)^i\Gamma\left(\frac{\rho+1}{2}\right)}{i!\,(\rho+k)^{\frac{\rho+1}{2}}}\right]$$

3.3.2. Shannon entropy

The Shannon entropy defined as:

$$\eta_X=E[-\log f(x)]$$

The Shannon entropy for APR distribution is given as:

$$\eta_X=E\left[-\log\left[\frac{\alpha\log\alpha}{(\alpha-1)\theta^2}xe^{-\frac{x^2}{2\theta^2}}\alpha^{-e^{-\frac{x^2}{2\theta^2}}}\right]\right] \qquad (8)$$

Using $\int_0^\infty e^{-\nu x}\log x\,dx=\frac{1}{\nu}(c+\log\nu)$

where C is the Euler constant. Hence Eq. (8) becomes

$$\eta_X = -\log\left[\frac{\alpha\log\alpha}{(\alpha-1)\theta^2}\right] + \frac{\alpha}{(1-\alpha)}\sum_{k=0}^{\infty}\frac{(-\log\alpha)^{k+1}}{k!}\left\{\frac{2(k+2)+2\log\alpha(k+1)^2-(k+1)(k+2)(\log(2\theta^2)-C-\log(k+1))}{(k+1)^2 2(k+2)}\right\}$$

3.4. Order statistics

Let $x_1, x_2, \ldots, x_n$ be an ordered sample of size n from APR distribution. Then the PDF of X_r is given as:

$$f_{r:n}(x) = \frac{n!}{(n-r)!(r-1)!}\left[1-F(x)\right]^{n-r}\left[F(x)\right]^{r-1} f(x)$$

$$= \frac{(-1)^{r-1}\alpha^{n-r}}{B(r,n-r+1)(\alpha-1)}\left[1-\alpha^{-e^{-\left(x^2/2\theta^2\right)}}\right]^{n-r}\left[1-\alpha^{-e^{-\left(x^2/2\theta^2\right)}}\right]^{r-1} f(x)$$

where $B(a,b)$ is beta function. Using binomial expansion we get

$$f_{r:n}(x) = \frac{(\log\alpha)}{B(r,n-r+1)\theta^2(\alpha-1)^n}\sum_{j=0}^{n-r}\sum_{k=0}^{r-1}(-1)^{r+j+k-1}\alpha^{n-r-k-+1}\alpha^{-(j+k+1)e^{-\left(x^2/2\theta^2\right)}}x e^{-\left(x^2/2\theta^2\right)}$$

IV. PARAMETER ESTIMATION

Let $X_1, X_2, \ldots X_n$ be a random sample of size n from APR distribution then the likelihood function is given by:

$$L = \prod_{i=1}^{n}\left\{\frac{\log\alpha}{(\alpha-1)}\frac{x_i}{\theta^2}e^{-\left(\frac{x_i^2}{2\theta^2}\right)}\alpha^{1-e^{-\left(\frac{x_i^2}{2\theta^2}\right)}}\right\} \qquad (9)$$

Applying log on both sides of Eq. (8) we get

$$\log L = n\log(\log\alpha) - n\log(\alpha-1) + \sum_{i=1}^{n}\log x_i - 2n\log\theta - \sum_{i=1}^{n}\frac{x_i^2}{2\theta^2} + n\log\alpha - \sum_{i=1}^{n}e^{-\left(\frac{x_i^2}{2\theta^2}\right)}\log\alpha \qquad (10)$$

Differentiating Eq. (10) w.r.t α and θ respectively and equating to zero we get

$$\frac{\partial}{\partial\alpha}\log L = \frac{n}{\alpha\log\alpha} - \frac{1}{\alpha(\alpha-1)} - \frac{\sum_{i=1}^{n}e^{-\left(\frac{x_i^2}{2\theta^2}\right)}}{\alpha} = 0$$

$$\frac{\partial}{\partial\theta}\log L = -\frac{2n}{\theta} + \sum_{i=1}^{n}\frac{x_i^2}{\theta^3} - \sum_{i=1}^{n}e^{-\left(\frac{x_i^2}{2\theta^2}\right)}\left(\frac{x_i^2}{\theta^3}\right)\log\alpha = 0$$

Solving above two equations simultaneously we get the maximum likelihood estimates (MLE) of α and θ. To solve these nonlinear equations, methods such as Newton-Raphson method can be used. As $n\to\infty$, the asymptotic distribution of MLE can be treated as bivariate normal with mean zero and variance covariance matrix I^{-1} where I is the Fisher information matrix i.e.

$$\sqrt{n}\left(\hat{\alpha}-\alpha, \hat{\theta}-\theta\right) \to N(0, I^{-1})$$

$$\text{Where } I^{-1} = \begin{bmatrix} I_{\alpha\alpha} & I_{\alpha\theta} \\ I_{\theta\alpha} & I_{\theta\theta} \end{bmatrix}^{-1}$$

and $$I_{\alpha\alpha} = -E\left[\frac{\partial^2}{\partial\alpha^2}LogL\right] = \frac{n(\log\alpha+1)}{(\alpha\log\alpha)^2} - \frac{(2\alpha-1)}{(\alpha(\alpha-1))^2} - \frac{n}{\alpha(1-\alpha)}\sum_{k=0}^{\infty}\frac{(-\log\alpha)^{k+1}}{k!(k+2)}$$

$$I_{\theta\theta} = -E\left[\frac{\partial^2}{\partial\theta^2}LogL\right] = -\frac{2n}{\theta^2} + \frac{6n\alpha}{\theta^2(1-\alpha)}\sum_{k=0}^{\infty}\frac{(-\log\alpha)^{k+1}}{k!(k+1)^2} + \frac{6n\alpha}{\theta^2(1-\alpha)}\sum_{k=0}^{\infty}\frac{(-\log\alpha)^{k+2}(k+3)}{k!(k+2)^3}$$

$$I_{\theta\alpha} = I_{\alpha\theta} = -E\left[\frac{\partial}{\partial\theta\partial\alpha} LogL\right] = \frac{2n}{\theta(1-\alpha)}\sum_{k=0}^{\infty}\frac{(-\log\alpha)^{k+1}}{k!(k+2)^2}$$

Approximate $100(1-\gamma)\%$ two sided confidence interval for α and θ is respectively given as

$$\hat{\alpha} \pm z_{\gamma/2}\sqrt{I_{\alpha\alpha}^{-1}} \text{ And } \hat{\theta} \pm z_{\gamma/2}\sqrt{I_{\theta\theta}^{-1}}$$

where z_γ is the upper $\gamma - th$ percentile of the standard normal distribution.

V. APPLICATION

In this section, we use two real life data sets to demonstrate the flexibility of APR distribution. The analysis is performed by using R Software. The distribution that are being used for comparison purpose with the proposed model are

1. Rayleigh distribution(RD) with PDF

$$f(x,\theta) = \frac{x}{\theta^2} e^{-\frac{x^2}{2\theta^2}} \quad ;\theta > 0, x > 0$$

2. Alpha Power Exponential distribution (APED)with PDF

$$f(x,\theta) = \frac{\log\alpha}{\alpha - 1}\frac{1}{\theta} e^{-\frac{x}{\theta}} \alpha^{1-e^{-x/\theta}} \quad ;\alpha,\theta > 0, \alpha \neq 1, x > 0$$

3. Exponential distribution(ED) with PDF

$$f(x,\theta) = \frac{1}{\theta} e^{-\frac{x}{\theta}} \quad ;\theta > 0, x > 0$$

4. Weibull distribution(WD) with PDF

$$f(x,\theta) = \frac{\alpha}{\theta}\left(\frac{x}{\theta}\right)^{\alpha-1} e^{-\left(\frac{x}{\theta}\right)^{\alpha}} \quad ;\theta, \alpha > 0, x > 0$$

Data set 1

The given data set is on the breaking stress of carbon fibres of 50 mm length (GPA). The data has been previously used by Cordeiro and Lemonte [9] and Al-Aqtash et al. [10]. The data is as follows:

0.39, 0.85, 1.08, 1.25, 1.47, 1.57, 1.61, 1.61, 1.69 ,1.80, 1.84,1.87 ,1.89 ,2.03 ,2.03 ,2.05 ,2.12, 2.35 ,2.41, 2.43 ,2.48 ,2.50,2.53 ,2.55 ,2.55 ,2.56 ,2.59 ,2.67, 2.73 ,2.74, 2.79 ,2.81 ,2.82,2.85 ,2.87 ,2.88 ,2.93 ,2.95 ,2.96, 2.97 ,3.09 ,3.11 ,3.11 ,3.15,3.15 ,3.19 ,3.22 ,3.22 ,3.27 ,3.28, 3.31 ,3.31, 3.33 ,3.39 ,3.39,3.56 ,3.60 ,3.65 ,3.68 ,3.70 ,3.75, 4.20 ,4.38 ,4.42 ,4.70 ,4.90.

The summary of the data set 1 is given in table 1 and the maximum likelihood estimates and different information measures for models is given in table 2. Figure 5(a) shows the plots of the estimated pdfs of APR distribution and other competitive models.

Table 1: data summary of Data set 1

n	First Quartile	Third Quartile	Median	Skewness	Max.	Min.	Mean	Variance	Kurtosis
124	2.178	3.278	2.835	-0.1284	4.900	0.390	2.760	0.7946	0.12612

Table 2: Estimates and performance of distribution

Distribution	**MLE**		**-Log L**	**AIC**	**BIC**	**AICC**
	$\hat{\alpha}$	$\hat{\theta}$				
APRD	65.8132 (58.4115)	1.4398 (0.0789)	85.5182	175.0365	179.4158	175.1357
RD	-	2.0491 (0.1261)	98.2084	198.4168	200.6065	198.516
ED	-	2.7595457 (0.33967)	132.9944	267.9887	270.1784	268.0879
APED	3.008336e+05	9.114967e-01	92.3963	188.7927	193.172	193.172

	(1.678449e+04)	(4.536035e-02)				
WD	3.4411 (0.3309)	3.0622 (0.1149)	86.0675	176.1352	180.5145	176.2343

Data set 2

This data set is discussed in Smith and Naylor [11].The data are about the strengths of 1.5 cm glass fibres, measured at the National Physical Laboratory, England. The observed data are as follows:

12, 15, 22, 24, 24, 32, 32, 33, 34, 38, 38, 43, 44, 48, 52, 53, 54, 54, 55, 56,57, 58, 58, 59, 60, 60, 60, 60, 61, 62, 63, 65, 65, 67, 68, 70, 70, 72, 73, 75,76, 76, 81, 83, 84, 85, 87, 91, 95, 96, 98, 99, 109, 110, 121, 127, 129, 131,143, 146, 146, 175, 175, 211, 233, 258, 258, 263, 297, 341, 341, 376.

The summary of the data set 2 is given in table 3 and the maximum likelihood estimates and different information measures for models is given in table 4.Figure 5(b) shows the plots of the estimated pdfs of APR distribution and other competitive models.

Table 3: Data summary of Data set 2

n	First Quartile	Third Quartile	Median	Skewness	Max.	Min.	Mean	Variance	Kurtosis
124	54.75	112.80	70.00	1.7589	376.00	12.00	99.82	6580.122	2.4595

Table 4: Estimates and performance of distribution

Distribution	**MLE**		**-Log L**	**AIC**	**BIC**	**AICC**
	$\hat{\alpha}$	$\hat{\theta}$				
APRD	0.02966 (0.0271)	130.5583 (16.3525)	395.0171	794.0341	798.5874	794.1333
RD	-	90.6990 (5.3444)	408.296	818.5921	820.8688	818.6913
ED		99.8194 (11.763)	808.9834	808.8843	811.1609	808.9834
APED	27.8632 (24.7065)	52.0209 (7.0089)	396.5697	797.1393	801.6926	797.2385
WD	1.39318 (0.11844)	110.555 (9.9344)	397.1477	798.2953	802.8487	798.3945

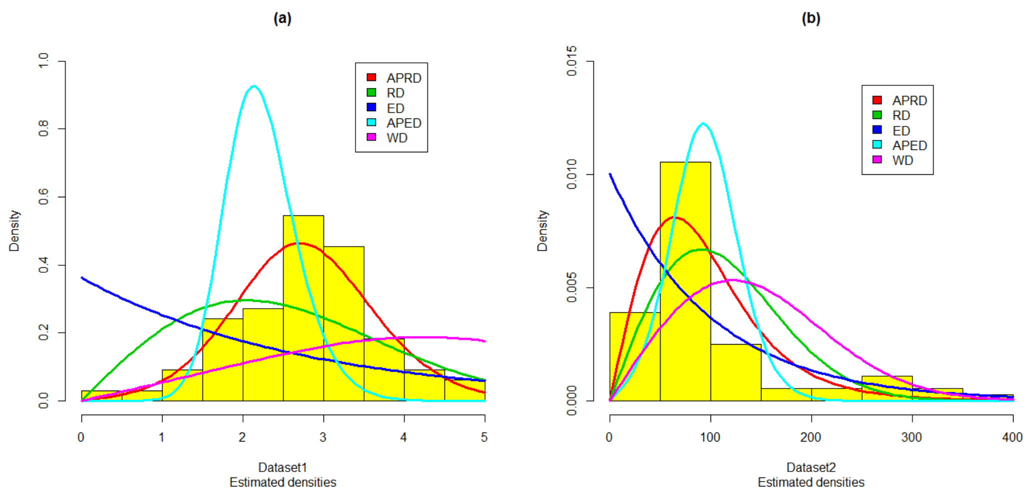

Figure 5. (a) Plots of the estimated PDF of APR distribution and other competitive models for Data set 1. (b) Plots of the estimated PDF of APR distribution and other competitive models for Data set 2.

VI. DISCUSSION

In order to compare the proposed model with other well-known models, the criteria such as -2log$l$, AIC (Akaike information criteria), BIC (Bayesian information criteria) and AICC (Corrected Akaike information criteria) are considered. The model with least value of -2log$l$, AIC, AICC and BIC is considered best fit for the given data. From table 2 and table 4 it is evident that APR distribution has the least value of AIC, AICC and BIC. Hence our proposed model i.e., APR distribution is best fit for the given data sets.

VII. Conclusion

In this paper a new two parameter Rayleigh distribution is proposed called APR distribution based on Alpha Power Transformation. The main aim behind generalization is to bring more flexibility to the distribution so that more data can be analysed using the distribution. Various mathematical properties such as moments, moment generating function, Quantile function etc. are discussed. The model parameters are estimated using maximum likelihood estimation. The usefulness of the proposed model is illustrated by means of two real life data sets whereby it is shown that APR distribution fits better than other competitive models to the given data sets. We hope that the new model will attract wider application in several areas.

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