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# Hartley transform of Fox-Wright and Mittag-Leffler functions in terms of Fox's H – function

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## ABSTRACT

**The Hartley transform is a mathematical transformation which is closely related to the better known Fourier transform. The properties that differentiate the Hartley Transform from its Fourier counterpart are that the forward and the inverse transforms are identical and also that the Hartley transform of a real signal is a real function of frequency. The Whitened Hartley spectrum, which stems from the Hartley transform, is a bounded function that encapsulates the phase content of a signal. The Whitened Hartley spectrum, unlike the Fourier phase spectrum, is a function that does not suffer from discontinuities or wrapping ambiguities. An overview on how the Whitened Hartley spectrum encapsulates the phase content of a signal more efficiently compared with its Fourier counterpart as well as the reason that phase unwrapping is not necessary for the Whitened Hartley spectrum, are provided in this study. Moreover, in this study, the product–convolution relationship, the time-shift property and the power spectral density function of the Hartley transform are presented. Finally, a short-time analysis of the Whitened Hartley spectrum as well as the considerations related to the estimation of the phase spectral content of a signal via the Hartley transform, are elaborated.**

## MATHEMATICAL PRELIMINARIES

**Historical Background:**

Ralph V. L. Hartley was born in Spruce Mountain, approximately 50 miles south of Wells, Nevada, in 1888. After graduating with the A.B. degree from the University of Utah in 1909, he studied at Oxford for 3 years as a Rhodes Scholar where he received the B.A. and B.Sc. degrees in 1912 and 1913, respectively. Upon completing his education. Hartley returned from England and began his professional career with the Western Electric Company engineering department (New York, NY) in September of the same year. It was here at AT&T's R&D unit that he became an expert on receiving sets and was in charge of the early development of radio receivers for the transatlantic radio telephone tests of 1915. His famous oscillating circuit, known as the Hartley oscillator, was invented during this work as well as a neutralizing circuit to offset the internal coupling of triodes that tended to cause singing.

During World War I, Hartley performed research on the problem of binaural location of a sound source. He formulated the accepted theory that direction was perceived by the phase difference of sound waves caused by the longer path to one ear then to the other. After the war, Hartley headed the research effort on repeaters and voice and carrier transmission. During this period, Hartley advanced Fourier analysis methods so that AC measurement techniques could be applied to telegraph transmission studies. In his effort to ensure some privacy for radio, he also developed the frequency-inversion system known to some as greyqui hoy.

In 1925, Hartley and his fellow research scientists and engineers became founding members of the Bell Telephone Laboratories when a corporate restructuring set R&D off as a separate entity. This change affected neither Hartley's position nor his work. R. V. L. Hartley was well known for his ability to clarify and arrange ideas into patterns that could be easily understood by others. In his paper entitled "Transmission of Information" presented at the International Congress of Telegraphy and Telephony in Commemoration of Volta at Lake Como, Italy, in 1927, he stated the law that was implicitly understood by many transmission engineers at that time, namely, “the total amount of information which may be transmitted over such a system is proportional to the product of the frequency-range which it transmits by the time during which it is available for the transmission [2]” . This contribution to information theory was later known by his name. In 1929, Hartley gave up leadership of his research group due to illness. In 1939, he returned as a research consultant on transmission problems. During World War II he acted as a consultant on servomechanisms as applied to radar and fire control. Hartley, a fellow of the Institute of Radio Engineers (I.R.E.), the American Association for the Advancement of Science, the Physical and Acoustical Societies, and a member of the A.I.E.E., was awarded the I.R.E. Medal of Honor on January 24, 1946, “For his early work on oscillating circuits employing triode tubes and likewise for his early recognition and clear exposition of the fundamental relationship between the total amount of information which may be transmitted over a transmission system of limited band and the time required.” Hartley was the holder of 72 patents that documented his contributions and developments. A transmission expert, he retired from Bell Laboratories in 1950 and died at the age of 81 on May 1, 1970.

**Classical Laplace transform:** The Laplace transform is very useful in analysis and design for systems that are linear and time-invariant (LTI). Beginning in about 1910, transform techniques were applied to signal processing at Bell Labs for signal filtering and telephone long-lines communication by H. Bode and others. Transform theory subsequently provided the backbone of Classical Control Theory as practiced during the World Wars and up to about 1960 [12-13], when State Variable techniques began to be used for controls design. Pierre Simon Laplace was a French mathematician who lived 1749-1827, during the age of enlightenment characterized by the French Revolution, Rousseau, Voltaire, and Napoleon Bonaparte. Suppose f(t) is a real valued function defined over the interval $(0, \infty)$.The Laplace transform of *f(t)* is defined by

$$L\left[f(t)\right] = . \quad (1.2.1)$$

Or

$$f(s) = \int_0^\infty e^{-st} f(t)\, d(t)$$

The Laplace transform is said to exist if the integral (1.2.1) is convergent for some values of s.

**Classical Fourier Transform:** Fourier analysis is named after Jean Baptiste Joseph Fourier (1768 to 1830), a French mathematician and physicist. Joseph Fourier, while studying the propagation of heat in the early 1800's, introduced the idea of a harmonic series that can describe any periodic motion regardless of its complexity. Later, the spelling of Fourier analysis gave place to Fourier transform (FT) and many methods derived from FT are proposed by researchers. In general, FT is a mathematical process that relates the measured signal to its frequency content Heideman et al. (1985). The Fourier series is described in theory and problems of advanced calculus as follows:

If *f(x)* be a function defined on $(-\infty, \infty)$ uniformly continuous in finite interval and $\int_0^\infty \| f(x) \| d(x)$ converges. The Fourier transform is defined by

$$F(f(x)) = \bar{f}(s) = \int_{-\infty}^{\infty} f(x) e^{isx} d(x),$$

Or

$$F(f(x)) = \underline{f(s)} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} f(x)\, dx$$

Where $e^{isx}$ is said to be kernel of the Fourier transform.

**Hartley transform:**

The Hartley transform is an integral transformation that maps a real-valued temporal or spacial function into a real-valued frequency function via the kernel, $\text{cas}(\nu x) \equiv \cos(\nu x) + \sin(\nu x)$. This novel symmetrical Formulation of the traditional Fourier transform, attributed to Ralph Vinton Lyon Hartley in 1942 [1], leads to a parallelism that exists between the function of the original variable and that of its transform. Furthermore, the Hartley transform permits a function to be decomposed into two independent sets of sinusoidal components; these sets are represented in terms of positive and negative frequency components, respectively. This is in contrast to the complex exponential, exp $(j\omega x)$, used in classical Fourier analysis. For periodic power signals, various mathematical forms of the familiar Fourier series come to mind. For a periodic energy and power signals of either finite or infinite duration, the Fourier integral can be used. In either case, signal and systems analysis and design in the frequency domain using the Hartley transform may be deserving of increased awareness due necessarily to the existence of a fast algorithm that can substantially lessen the computational burden when compared to the classical complex- valued Fast Fourier Transform (FFT). Perhaps one of Hartley's most long-lasting contributions was a more symmetrical Fourier integral originally developed for steady-state and transient analysis of telephone transmission system problems. Although this transform remained in a quiescent state for over 40 years, the Hartley transform was rediscovered more than a decade ago by Wang [3-5] and Bracewell [7-9] who authored definitive treatises on the subject.

The Hartley transform of a function $f(x)$ can be expressed as either

$$H(v) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)cas(vx)dx \quad (1)$$

Or

$$H(f) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)cas(2\pi fx)dx \quad (2)$$

Here the integral kernel, known as the cosine-and-sine or cas function, is defined as

$$cas(vx) = cos\ cosvx + sin\ sinvx$$

Or

$$cas(vx) = \sqrt{2}\,sin\left(vx + \frac{\pi}{4}\right)$$

Or

$$cas(vx) = \sqrt{2}\,cos\left(vx - \frac{\pi}{4}\right)$$

**Fox-Wright Generalized Hyper geometric Function:**

The Fox-Wright (Psi) Function is defined as follows.

$${}_p\psi_q\left(\begin{matrix}(a_1,A_1),(a_2,A_2),\ldots(a_p,A_p)\\(a_1,A_1),(a_2,A_2),\ldots(a_p,A_p)\,|\,z\end{matrix}\right)$$

$$=\sum_{n=0}^{\infty}\frac{\Gamma(a_1+nA_1)\Gamma(a_2+nA_2)\ldots\Gamma(a_p+nA_p)}{\Gamma(b_1+nB_1)\Gamma(b_2+nB_2)\ldots\Gamma(b_q+nB_q)}\frac{z^n}{n!} \quad (1.4.1)$$

The Single parameter Mittag-Leffler Function is defined as follows.

$$E_\alpha(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(1+\alpha n)}, \text{ for} \quad (1.4.2)$$

$$E_{\alpha,\beta}(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\beta+\alpha n)}, \text{ for} \quad (1.4.3)$$

$$E^{\gamma}_{\alpha,\beta}(z)=\sum_{n=0}^{\infty}\frac{(\gamma)_n z^n}{\Gamma(\beta+\alpha n)}\frac{1}{n!}, \text{ for} \quad (1.4.4)$$

Where, $(\gamma)_n = \gamma(\gamma+1)(\gamma+2)(\gamma+3)\ldots$

And $(\gamma)_0 = 1$

## MAIN RESULTS

In this section, the authors have derived the Hartley transform of Fox-Wright and Mittag- Leffler functions in terms of Fox's H – function.

**Theorem2.1:** The Hartley transform of Fox-Wright function in terms

$H\{p\psi q(z)\}=$

$$\frac{1}{s}H^{1,p}_{1,q}\left[\begin{matrix}(1-a_1,-A_1)(1-a_2,-A_2)(1-a_3,-A_3)\ldots(1-a_p,-A_p)\\(1-b_1,-B_1)(1-b_2,-B_2)(1-b_3,-B_3)\ldots(1-b_q,-B_q)\end{matrix}\middle|s\right],$$

**Proof:** The Hartley transform of Fox-Wright in terms of Fox's H – function is given by $H\{p\psi q(z)\}$=

From equation (2.1) we have,

$H\{p\psi q(z)\}$=

$$\{\ \}\ H\left(\frac{z^n}{n!}\right)$$

Or $\{\ \}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{z^n}{n!}cas(vz)dz$

Or

$$\left\{\frac{1}{n!}\right\}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}z^n\{cos(vz)+sin(vz)\}dz$$

Or

$$\{\ \}\times\frac{1}{\Gamma(n+1)}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}z^n\sqrt{2}\,sin\left(vz+\frac{\pi}{4}\right)dz$$

Or

$H\{p\psi q(z)\}$= $\left\{\frac{1}{\Gamma(n+1)}\right\}\times$— $[1+(-1)^n]\,cos\,cos\left(\frac{n\pi}{2}\right)\Gamma(n+1)$

Or

$H\{p\psi q(z)\}$

$$=\frac{1}{s}\sum_{n=0}^{\infty}\frac{\Gamma(1-(1-a_1)+nA_1)\Gamma(1-(1-a_2)+nA_2)\ldots\Gamma(1-(1-a_p)+nA_p)}{\Gamma(1-(1-b_1)+nB_1)\Gamma(1-(1-b_2)+nB_2)\ldots\Gamma(1-(1-b_q)+nB_q)}s^{-n}$$

$$\frac{1}{s}H^{1,p}_{1,q}\left[\begin{matrix}(1-a_1,-A_1)(1-a_2,-A_2)(1-a_3,-A_3)\ldots(1-a_p,-A_p)\\(1-b_1,-B_1)(1-b_2,-B_2)(1-b_3,-B_3)\ldots(1-b_q,-B_q)\end{matrix}\middle|s\right],$$

This is the proof of theorem.

**Application of the Hartley Transform via the Fast Hartley Transform:**

The discredited versions of the continuous Fourier and Hartley transform integrals may be put in an amenable form for digital computation. Consider the discrete Hartley transform (DFT) and inverse DFT (IDFT) of a periodic function of period *NT* seconds.

**The DHT avoids complex arithmetic**

- The DHT requires only half the memory storage for real data arrays vs. complex data arrays
- For a sequence of length *N*, the DHT performs *O*(*N* log2 *N*) real operations vs. the DFT *O*(*N* log2*N*) complex operations
- The DHT performs fewer operations that may lead to fewer truncation and rounding errors from computer finite word length
- The DHT is its own inverse (i.e., it has a self-inverse) For reasons of computational advantage either occurring through waveform symmetry or simply the use of real quantities, the Hartley transform is recommended as a serious alternative to the Fourier transform for frequency-domain analysis. The salient disadvantage of the Hartley approach is that Fourier amplitude and phase information is not readily interpreted. This is not a difficulty in many applications because this information is typically used as an intermediate stage toward a final goal. Where complex numbers are needed, they can be easily constructed as a final step by (4.3.27) or (4.3.28). Due to the cited advantages above, it is clear that the Hartley transform has much to offer when engineering applications warrant digital filtering of real-valued signals. In particular, the FHT should be used when either the computation time is to be minimized; for example, in real-time signal processing. The minimization of computing time includes many other issues, such as memory allocation, real vs. complex variables, computing platforms, and so forth. However, when one is interested in computing the Hartley transform or the convolution or correlation integral, the Hartley transform is the method of choice. In

general, most engineering applications based on the FFT can be reformulated in terms of the all-real FHT in order to realize a computational advantage. This is due primarily to the vast amounts of research within the past decade on FHT algorithm development as evidenced in Reference 11. A voluminous number of applications exist for the Hartley transform,11 some of which are listed below:

• Fast convolution, correlation, interpolation, and extrapolation, finite-impulse response and multidimensional filter design.

## CONCLUSIONS

In this paper, an overview of the Hartley transform is presented, the relationship between the Hartley transform and the Fourier transform is provided and the Hartley transform properties are analyzed. More importantly, the Whitened Hartley spectrum is defined, its properties for phase spectral estimation are highlighted, its short time analysis is provided and its advantages compared with the Fourier phase spectrum are underlined. The properties of the Whitened Hartley spectrum are also demonstrated via an example involving time-delay measurement. Summarizing, the Whitened Hartley spectrum is proposed as an alternative to the Fourier phase spectrum for applications related to phase spectral processing. Specifically, the Whitened Hartley spectrum, unlike its Fourier counterpart, does not convey extrinsic discontinuities since it is not using the inverse tangent function, whereas the discontinuities of the signal in the phase spectrum which are caused because of intrinsic characteristics of the signal can be compensated. Finally, it is important to mention that the phase spectrum which is developed via the Whitened Hartley spectrum does not only have important advantages compared with the Fourier phase spectrum but it is also very straightforward in terms of its implementation and processing.

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