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Study of Unified Integrals Associated with H-Functions

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ABSTRACT

In this chapter our aim is presenting some generalized integral formulas involving Fox H-function and M-Series. Results derived in this paper are in terms of H-function and due to generous nature of H-function, several particular cases are considered in the form of corollaries.

Notations and Result Required

i. $(a)_n = a(a+1)(a+2)(a+3)\dots\dots(a+n-1)$

ii. $(a)_{mn} = m^{mn}\left(\frac{a}{m}\right)_n\left(\frac{a+1}{m}\right)_n \dots\dots \left(\frac{a+m-1}{m}\right)_n$

iii. $(a)_{2n} = 2^{2n}\left(\frac{a}{2}\right)_n\left(\frac{a+1}{2}\right)_n$

iv. $(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}$

v. $(a)_{n-m} = \frac{(-1)^m (a)_n}{(1-a-n)_m}$

vi. $Re(a) = \ Realpart of\ 'a'$

vii. $\prod_{i=1}^{p}(a_i)_n = (a_1)_n(a_2)_n(a_3)_n \dots\dots (a_p)_n$

viii. $B(\alpha,\beta) = \frac{\Gamma\alpha\Gamma\beta}{\Gamma(\alpha+\beta)}$

INTRODUCTION

The importance of the H-Function and M-Series are realized by scientists, engineers and statisticians (Caputo [7] , Glockle and Nonnenmacher [14], Mainardi et al. [17], Hilfer [15] etc.) due to vast potential of their applications in diversified fields of science and engineering such as fluid flow, rheology, diffusion in porous media, propagation of seismic waves, anomalous diffusion and turbulence etc.

In view of importance and popularity of the H-function a large number of integral formulas involving this function have been developed by many authors. For example, Garg and Mittal [13] obtained an interesting unified integral involving Fox H-function, Choi and Agarwal [8] derived unified integrals associated with Bessel functions. Further, Ali [5] gave three interesting unified integrals involving the hypergeometric function. Recently, many useful integral formulas associated with the Bessel functions of several kinds and Hypergeometric functions have been studied by Agarwal ([1]-[3]), Agarwal et al. [4], Choi and Agarwal [9] and Choi et al. [10].

Fox [12] introduced the H-function, via a Mellin-Barnes type integral for integers m, n, p, qsuch that $0 = m = q;\ 0 = n = p$;for $a_i,\ b_j \in C$and for $\alpha_i, \beta_j \in R_+ = (0,1),\ (i = 1,2,\dots,p;\ j = 1,2,\dots,q)$; as

$$H_{p,q}^{m,n}\left[z\middle|\begin{matrix}(a_i,\alpha_i)_1^p\\(b_j,\beta_j)_1^q\end{matrix}\right] = \frac{1}{2\pi i}\int_L H_{p,q}^{m,n}(s)z^{-s}ds \quad \text{where } H(s) \text{ is given by}$$

$$H_{p,q}^{m,n}(s) = \frac{\prod_{j=1}^{m}\Gamma(b_j+\beta_j s)\prod_{i=1}^{n}\Gamma(1-a_j-\alpha_i s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j-\beta_j s)\prod_{i=n+1}^{p}\Gamma(a_j+\alpha_i s)}$$

with all convergence conditions as given by Braaksma [6], Mathai [18], Kilbas and Saigo [16]. On putting $\alpha_j = \beta_j = 1$ in H-function, we obtain the Meijer's $G-$functions $G_{p;q}^{m;n}(z)$ (Fox [12]).

Oberhettinger [19] established the following interesting integral formula

$$\int_0^\infty x^{\mu-1}\left(x+a+\sqrt{x^2+2ax}\right)^{-\lambda}dx = 2\lambda a^{-\lambda}\left(\frac{a}{2}\right)^{\mu}\frac{\Gamma(2\mu)\Gamma(\lambda-\mu)}{\Gamma(1+\lambda+\mu)} \quad (1.1.1)$$

where $0 < Re(\mu) < Re(\lambda)$.

$$\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}[a+bx]^{-\alpha-\beta}dx = \frac{1}{(a+b)^{\alpha}a^{\beta}}B(\alpha,\beta) \quad 1.1.2$$

We also use the following integral relation of Beta function for the prove of our results.

$$\int_0^1 x^{\alpha-1}[a+bx]^{-\alpha-\beta}dx = \frac{1}{b^{\alpha}a^{\beta}}B(\alpha,\beta) \quad 1.1.3$$

One of the most interesting following relations of *Beta function* is as follows,
$B(\alpha,\beta) = B(\alpha+1,\beta) + B(\alpha,\beta+1)$ 1.1.4

Theorem 1.2.1: Show that

$$\int_0^\infty x^{\mu-1}\left(x+a+\sqrt{x^2+2ax}\right)^{-\lambda} H_{p,q}^{m,n}\left[\frac{y^k\left(x+a+\sqrt{x^2+2ax}\right)^k}{\omega^k}\Big| \begin{matrix}(a_i,\alpha_i)_1^p\\(b_j,\beta_j)_1^q\end{matrix}\right]dx =$$

$$2a^{-\lambda}\Gamma(2\mu)\left(\frac{a}{2}\right)^{\mu} H_{p+2,q+2}^{m,n+1}\left[\frac{y^k a^k}{\omega^k}\Big| \begin{matrix}(-\lambda,-k),(\lambda,k),(a_i,\alpha_i)_1^p\\(\lambda-\mu,k),(-\lambda-\mu,,-k)(b_j,\beta_j)_1^q\end{matrix}\right] \quad (1.2.1.1)$$

The convergence conditions for the validity of (1.2.1.1) are as follows,

i. $0 \le m \le q, 0 \le p \le n$ for $a_i, b_j, \lambda, \mu \in C$ and $\alpha_i, \beta_j \in R^+$ and $i \in \{1,2,\ldots\ldots p\}$, $j \in \{1,2,\ldots\ldots q\}$ with $0 < Re(\mu) < Re(\lambda - sk)$ and $k > 0$; $x > 0$.

ii. $L = L_{i\gamma\infty}$is a contour starting at the point $\gamma - i\infty$and going to $\gamma + i\infty$where $\gamma \in R(-\infty,+\infty)$ such that all the poles of $\Gamma(b_j+\beta_j s) j \in \{1,2,\ldots\ldots m\}$are separated from those of $\Gamma(1-a_i-\alpha_i s)$, $i \in \{1,2,\ldots\ldots n\}$, The integral converges if $\alpha > 0, |\arg z| < \frac{1}{2}\pi\alpha,\ \alpha \neq 0$. The integral also converges if $\sigma = 0, \gamma\mu + Re(\delta) < -1, \arg z = 0$ and $z \neq 0$ where $\sigma = \sum_{j=1}^{n}\alpha_j - \sum_{j=n+1}^{p}\alpha_j + \sum_{j=1}^{m}\beta_j - \sum_{j=m+1}^{q}\beta_j$.

iii. $Re(\mu) > 0, Re(\mu) + k\min_{(1\le j\le m)} Re\left[\frac{b_j}{\beta_j}\right] > \max\{0,(\lambda - sk)\}$.

Proof: We know that,

$$H_{p,q}^{m,n}\left[z\Big| \begin{matrix}(a_i,\alpha_i)_1^p\\(b_j,\beta_j)_1^q\end{matrix}\right] = \frac{1}{2\pi i}\int_L H_{p,q}^{m,n}(s)z^{-s}ds$$

where $H(s)$ is given by

$$H_{p,q}^{m,n}(s) = \frac{\prod_{j=1}^{m}\Gamma(b_j+\beta_j s)\prod_{i=1}^{n}\Gamma(1-a_j-\alpha_i s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j-\beta_j s)\prod_{i=n+1}^{p}\Gamma(a_j+\alpha_i s)}$$

On taking LHS of (1.2.1.1) we have

$$\int_0^\infty x^{\mu-1}\left(x+a+\sqrt{x^2+2ax}\right)^{-\lambda} H_{p,q}^{m,n}\left[\frac{y^k\left(x+a+\sqrt{x^2+2ax}\right)^k}{\omega^k}\Big| \begin{matrix}(a_i,\alpha_i)_1^p\\(b_j,\beta_j)_1^q\end{matrix}\right]dx$$

$$= \int_0^\infty x^{\mu-1}\left(x+a+\sqrt{x^2+2ax}\right)^{-\lambda}\left(\frac{1}{2\pi i}\int_L H_{p,q}^{m,n}(s)\left(\frac{y^k\left(x+a+\sqrt{x^2+2ax}\right)^k}{\omega^k}\right)^{-s}ds\right)dx$$

By change of order of integration, we have

$$\frac{1}{2\pi i}\int_L H_{p,q}^{m,n}(s)\frac{y^{-sk}}{\omega^{-sk}}\left(\int_0^\infty x^{\mu-1}\left(x+a+\sqrt{x^2+2ax}\right)^{-\lambda-ks}dx\right)ds$$

By using (1.1.1) we have

$$\frac{1}{2\pi i}\int_L H_{p,q}^{m,n}(s)\frac{y^{-sk}}{\omega^{-sk}}\left(\int_0^\infty x^{\mu-1}\left(x+a+\sqrt{x^2+2ax}\right)^{-(\lambda+ks)}dx\right)$$

$$\int_0^\infty x^{\mu-1}\left(x+a+\sqrt{x^2+2ax}\right)^{-(\lambda+ks)}dx = 2(\lambda+ks)a^{-\lambda}\left(\frac{a}{2}\right)^{\mu}\frac{\Gamma(2\mu)\Gamma(\lambda+ks-\mu)}{\Gamma(1+\lambda+ks+\mu)}$$

$$= 2a^{-(\lambda+ks)}\left(\frac{a}{2}\right)^{\mu}\frac{\Gamma(2\mu)\Gamma(\lambda+ks+1)\Gamma(\lambda+ks-\mu)}{\Gamma(\lambda+ks)\Gamma(1+\lambda+ks+\mu)}$$

$$= 2a^{-\lambda}\Gamma(2\mu)\left(\frac{a}{2}\right)^{\mu}\frac{1}{2\pi i}\int_L \frac{y^{-sk}a^{-sk}}{\omega^{-sk}}H_{p,q}^{m,n}(s)\left(\frac{\Gamma(\lambda+ks+1)\Gamma(\lambda+ks-\mu)}{\Gamma(\lambda+ks)\Gamma(1+\lambda+ks+\mu)}\right)ds$$

$$= 2a^{-\lambda}\Gamma(2\mu)\left(\frac{a}{2}\right)^{\mu}\frac{1}{2\pi i}\int_L \frac{y^{-sk}a^{-sk}}{\omega^{-sk}}H_{p,q}^{m,n}(s)\left(\frac{\Gamma(\lambda+ks+1)\Gamma(\lambda+ks-\mu)}{\Gamma(\lambda+ks)\Gamma(1+\lambda+ks+\mu)}\right)ds$$

$$= 2a^{-\lambda}\Gamma(2\mu)\left(\frac{a}{2}\right)^{\mu}H_{p+2,q+2}^{m,n+1}\left[\frac{y^k a^k}{\omega^k}\Big|\begin{matrix}(-\lambda,-k),(\lambda,k),(a_i,\alpha_i)_1^p\\(\lambda-\mu,k),(-\lambda-\mu,,-k)(b_j,\beta_j)_1^q\end{matrix}\right]$$

This complete the prove of the Theorem.

Theorem1.2.2: Show that

$$\int_0^\infty x^{\mu-1}\left(x+a+\sqrt{x^2+2ax}\right)^{-\lambda}H_{p,q}^{m,n}\left[\frac{y^k x^k\left(x+a+\sqrt{x^2+2ax}\right)^k}{\omega^k}\Big|\begin{matrix}(a_i,\alpha_i)_1^p\\(b_j,\beta_j)_1^q\end{matrix}\right]dx = \frac{2a^{-\lambda}}{\Gamma(1+\lambda+\mu)}\left(\frac{a}{2}\right)^{\mu}H_{p+2,q+3}^{m,n+1}\left[\frac{(2y)^k}{\omega^k}\left(\frac{a}{2}\right)^{2k}\Big|\begin{matrix}(-\lambda,-k),(\lambda,k),(a_i,\alpha_i)_1^p\\(-1,0),(2\mu,-2k),(\lambda-\mu,2k),(b_j,\beta_j)_1^q\end{matrix}\right] \quad (1.2.2.1)$$

The convergence conditions for the validity of (1.2.2.1) are as follows,

i. $0 \le m \le q, 0 \le p \le n$ for $a_i, b_j, \lambda, \mu \in C$ and $\alpha_i, \beta_j \in R^+$ and $i \in \{1,2,\ldots\ldots p\}$, $j \in \{1,2,\ldots\ldots q\}$ with $0 < Re(\mu) < Re(\lambda)$ and $k > 0;\ x > 0$.

ii. $L = L_{i\gamma\infty}$ is a contour starting at the point $\gamma - i\infty$ and going to $\gamma + i\infty$ where $\gamma \in R(-\infty,+\infty)$ such that all the poles of $\Gamma(b_j+\beta_j s) j \in \{1,2,\ldots\ldots m\}$ are separated from those of $\Gamma(1-a_i-\alpha_i s)$, $i \in \{1,2,\ldots\ldots n\}$, The integral converges if $\alpha > 0, |\arg z| < \frac{1}{2}\pi\alpha,\ \alpha \ne 0$. The integral also converges if $\sigma = 0, \gamma\mu + Re(\delta) < -1, \arg z = 0$ and $z \ne 0$ where $\sigma = \sum_{j=1}^{n}\alpha_j - \sum_{j=n+1}^{p}\alpha_j + \sum_{j=1}^{m}\beta_j - \sum_{j=m+1}^{q}\beta_j$.

iii. $Re(\mu) > 0, Re(\mu) + k\min_{(1\le j\le m)} Re\left[\frac{b_j}{\beta_j}\right] > \max\{0,(\lambda - sk)\}$.

Proof: We know that,

$$H_{p,q}^{m,n}\left[z\Big|\begin{matrix}(a_i,\alpha_i)_1^p\\(b_j,\beta_j)_1^q\end{matrix}\right] = \frac{1}{2\pi i}\int_L H_{p,q}^{m,n}(s)z^{-s}ds$$

where $H(s)$ is given by

$$H_{p,q}^{m,n}(s) = \frac{\prod_{j=1}^{m}\Gamma(b_j+\beta_j s)\prod_{i=1}^{n}\Gamma(1-a_j-\alpha_i s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j-\beta_j s)\prod_{i=n+1}^{p}\Gamma(a_j+\alpha_i s)}$$

On taking LHS of (1.2.2.1) we have

$$\int_0^\infty x^{\mu-1}\left(x+a+\sqrt{x^2+2ax}\right)^{-\lambda}H_{p,q}^{m,n}\left[\frac{y^k x^k\left(x+a+\sqrt{x^2+2ax}\right)^k}{\omega^k}\Big|\begin{matrix}(a_i,\alpha_i)_1^p\\(b_j,\beta_j)_1^q\end{matrix}\right]dx =$$

$$\int_0^\infty x^{\mu-1}\left(x+a+\sqrt{x^2+2ax}\right)^{-\lambda}\left(\frac{1}{2\pi i}\int_L H_{p,q}^{m,n}(s)\left(\frac{y^k x^k\left(x+a+\sqrt{x^2+2ax}\right)^k}{\omega^k}\right)^{-s}ds\right)dx$$

By change of order of integration, we have

$$\frac{1}{2\pi i}\int_L H_{p,q}^{m,n}(s)\frac{y^{-sk}}{\omega^{-sk}}\left(\int_0^\infty x^{\mu-1}\left(x+a+\sqrt{x^2+2ax}\right)^{-\lambda-ks}dx\right)ds$$

By using (1.1.1) we have

$$\frac{1}{2\pi i}\int_L H_{p,q}^{m,n}(s)\frac{y^{-sk}}{\omega^{-sk}}\left(\int_0^\infty x^{\mu-ks-1}\left(x+a+\sqrt{x^2+2ax}\right)^{-(\lambda+ks)}dx\right)$$

$$\int_0^\infty x^{\mu-ks-1}\left(x+a+\sqrt{x^2+2ax}\right)^{-(\lambda+ks)}dx = 2(\lambda+ks)a^{-\lambda}\left(\frac{a}{2}\right)^{\mu-ks}\frac{\Gamma\big(2(\mu-ks)\big)\Gamma(\lambda+ks-\mu+ks)}{\Gamma(1+\lambda+ks+\mu-ks)}$$

$$= 2a^{-(\lambda+ks)}\left(\frac{a}{2}\right)^{\mu-ks}\frac{\Gamma(2\mu-2ks)\Gamma(\lambda+ks+1)\Gamma(\lambda+ks-\mu+ks)}{\Gamma(\lambda+ks)\Gamma(1+\lambda+ks+\mu)}$$

$$= 2a^{-\lambda}\left(\frac{a}{2}\right)^{\mu}\frac{1}{2\pi i}\int_L\frac{(2y)^{-sk}}{\omega^{-sk}}\left(\frac{a}{2}\right)^{-2ks}H_{p,q}^{m,n}(s)\left(\frac{\Gamma(2\mu-2ks)\Gamma(\lambda+ks+1)\Gamma(\lambda+ks-\mu+ks)}{\Gamma(\lambda+ks)}\right)ds$$

$$= 2a^{-\lambda}\left(\frac{a}{2}\right)^{\mu}\frac{1}{2\pi i}\int_L\frac{(2y)^{-sk}}{\omega^{-sk}}\left(\frac{a}{2}\right)^{-2ks}H_{p,q}^{m,n}(s)\left(\frac{\Gamma(2\mu-2ks)\Gamma(\lambda+ks+1)\Gamma(\lambda+ks-\mu+ks)}{\Gamma(\lambda+ks)}\right)ds$$

$$= \frac{2a^{-\lambda}}{\Gamma(1+\lambda+\mu)}\left(\frac{a}{2}\right)^{\mu}H_{p+2,q+3}^{m,n+1}\left[\frac{(2y)^k}{\omega^k}\left(\frac{a}{2}\right)^{2k}\Big|\begin{matrix}(-\lambda,-k),(\lambda,k),(a_i,\alpha_i)_1^p\\(-1,0),(2\mu,-2k),(\lambda-\mu,2k),(b_j,\beta_j)_1^q\end{matrix}\right]$$

This complete the prove of the Theorem.

Our next result is depending on formula 1.1.2, which is as follows,

Theorem1.2.3: Show

that $\int_0^\infty x^{\alpha-1}(1-x)^{\beta-1}[a+bx]^{-\alpha-\beta}H_{p,q}^{m,n}\left[\frac{y^k(1-x)^k}{\omega^k[a+bx]^k}\Big|\begin{matrix}(a_i,\alpha_i)_1^p\\(b_j,\beta_j)_1^q\end{matrix}\right]dx =$ $\frac{1}{(a+b)^\alpha a^\beta}H_{p+2,q+2}^{m,n+2}\left[\frac{y^k}{a^k\omega^k}\Big|\begin{matrix}(1-\alpha,0),(1,0),(a_i,\alpha_i)_1^p\\(\beta,-k),(-1-\alpha-\beta,k)(b_j,\beta_j)_1^q\end{matrix}\right]$ (1.2.3.1)

The convergence conditions for the validity of (1.2.3.1) are as follows,

i. $0\le m\le q, 0\le p\le n$ for $a_i, b_j, \lambda, \mu\in C$ and $\alpha_i,\beta_j\in R^+$ and $i\in\{1,2,\ldots\ldots p\}$, $j\in\{1,2,\ldots\ldots q\}$ with $0<Re(\mu)<Re(\lambda-sk)$ and $k>0;\ x>0$.

ii. $L = L_{i\gamma\infty}$ is a contour starting at the point $\gamma-i\infty$ and going to $\gamma+i\infty$ where $\gamma\in R(-\infty,+\infty)$ such that all the poles of $\Gamma\big(b_j+\beta_j s\big) j\in\{1,2,\ldots\ldots m\}$ are separated from those of $\Gamma(1-a_i-\alpha_i s)$, $i\in\{1,2,\ldots\ldots n\}$, The integral converges if $\alpha>0, |\arg z|<\frac{1}{2}\pi\alpha,\ \alpha\neq 0$. The integral also converges if $\sigma=0,\gamma\mu+Re(\delta)<-1,\arg z=0$ and $z\neq 0$ where $\sigma=\sum_{j=1}^n\alpha_j-\sum_{j=n+1}^p\alpha_j+\sum_{j=1}^m\beta_j-\sum_{j=m+1}^q\beta_j$.

iii. $Re(\mu)>0, Re(\mu)+k\min_{(1\le j\le m)}Re\left[\frac{b_j}{\beta_j}\right]>\max\{0,(\lambda-sk)\}$.

Proof: We know that,

$$H_{p,q}^{m,n}\left[z\Big|\begin{matrix}(a_i,\alpha_i)_1^p\\(b_j,\beta_j)_1^q\end{matrix}\right]=\frac{1}{2\pi i}\int_L H_{p,q}^{m,n}(s)z^{-s}ds$$

where $H(s)$ is given by

$$H_{p,q}^{m,n}(s)=\frac{\prod_{j=1}^m\Gamma\big(b_j+\beta_j s\big)\prod_{i=1}^n\Gamma\big(1-a_j-\alpha_i s\big)}{\prod_{j=m+1}^q\Gamma\big(1-b_j-\beta_j s\big)\prod_{i=n+1}^p\Gamma\big(a_j+\alpha_i s\big)}$$

On taking LHS of (1.2.3.1) we have

$$\int_0^\infty x^{\alpha-1}(1-x)^{\beta-1}[a+bx]^{-\alpha-\beta}H_{p,q}^{m,n}\left[\frac{y^k(1-x)^k}{\omega^k[a+bx]^k}\Big|\begin{matrix}(a_i,\alpha_i)_1^p\\(b_j,\beta_j)_1^q\end{matrix}\right]dx =$$

$$\int_0^\infty x^{\alpha-1}(1-x)^{\beta-1}[a+bx]^{-\alpha-\beta}\left(\frac{1}{2\pi i}\int_L H_{p,q}^{m,n}(s)\left(\frac{y^k(1-x)^k}{\omega^k[a+bx]^k}\right)^{-s}ds\right)dx$$

By change of order of integration, we have

$$\frac{1}{2\pi i}\int_L H_{p,q}^{m,n}(s)\frac{y^{-sk}}{\omega^{-sk}}\left(\int_0^\infty x^{\alpha-1}(1-x)^{\beta-ks-1}[a+bx]^{-\alpha-(\beta-ks)}dx\right)ds$$

By using (1.1.2) we have

$$\frac{1}{2\pi i}\int_L H_{p,q}^{m,n}(s)\frac{y^{-sk}}{\omega^{-sk}}\left(\frac{1}{(a+b)^\alpha a^{\beta-ks}}B(\alpha,\beta-ks)\right)ds$$

$$\frac{1}{(a+b)^\alpha a^\beta}\frac{1}{2\pi i}\int_L H_{p,q}^{m,n}(s)\frac{y^{-sk}}{(a\omega)^{-sk}}\frac{\Gamma\alpha\Gamma(\beta-ks)}{\Gamma(\alpha+\beta-ks)}ds$$

$$=\frac{1}{(a+b)^\alpha a^\beta}H_{p+2,q+2}^{m,n+2}\left[\frac{y^k}{a^k\omega^k}\Big|\begin{matrix}(1-\alpha,0),(1,0),(a_i,\alpha_i)_1^p\\(\beta,-k),(-1-\alpha-\beta,k)(b_j,\beta_j)_1^q\end{matrix}\right]$$

This complete the prove of the Theorem.

Theorem1.2.4: Show that

$$\int_0^\infty x^{\alpha-1}[a+bx]^{-\alpha-\beta}H_{p,q}^{m,n}\left[\frac{y^kx^k}{\omega^k[a+bx]^k}\Big|\begin{matrix}(a_i,\alpha_i)_1^p\\(b_j,\beta_j)_1^q\end{matrix}\right]dx=\frac{1}{(a+b)^\alpha a^\beta}H_{p+2,q+2}^{m,n+2}\left[\frac{y^k}{b^k\omega^k}\Big|\begin{matrix}(1-\alpha,0),(1,0),(a_i,\alpha_i)_1^p\\(\beta,0),(1-\alpha-\beta,k)(b_j,\beta_j)_1^q\end{matrix}\right] \quad (1.2.4.1)$$

The convergence conditions for the validity of (1.2.4.1) are as follows,

i. $0 \le m \le q, 0 \le p \le n$ for $a_i, b_j, \lambda, \mu \in C$ and $\alpha_i, \beta_j \in R^+$ and $i \in \{1,2,\dots .p\}$, $j \in \{1,2,\dots .q\}$ with $0 < Re(\mu) < Re(\lambda - sk)$ and $k > 0;\ x > 0$.

ii. $L = L_{i\gamma\infty}$is a contour starting at the point $\gamma - i\infty$and going to $\gamma + i\infty$where $\gamma \in R(-\infty, +\infty)$ such that all the poles of $\Gamma(b_j + \beta_j s) j \in \{1,2,\dots .m\}$are separated from those of $\Gamma(1 - a_i - \alpha_i s)$, $i \in \{1,2,\dots .n\}$, The integral converges if $\alpha > 0, |\arg z| < \frac{1}{2}\pi\alpha,\ \alpha \ne 0$. The integral also converges if $\sigma = 0, \gamma\mu + Re(\delta) < -1, \arg z = 0$ and $z \ne 0$ where $\sigma = \Sigma_{j=1}^n \alpha_j - \Sigma_{j=n+1}^p \alpha_j + \Sigma_{j=1}^m \beta_j - \Sigma_{j=m+1}^q \beta_j$.

iii. $Re(\mu) > 0, Re(\mu) + k \min_{(1\le j \le m)} Re\left[\frac{b_j}{\beta_j}\right] > \max\{0, (\lambda - sk)\}$.

Proof: We know that,

$$H_{p,q}^{m,n}\left[z\Big|\begin{matrix}(a_i,\alpha_i)_1^p\\(b_j,\beta_j)_1^q\end{matrix}\right]=\frac{1}{2\pi i}\int_L H_{p,q}^{m,n}(s)z^{-s}ds$$

where $H(s)$ is given by

$$H_{p,q}^{m,n}(s)=\frac{\prod_{j=1}^m\Gamma(b_j+\beta_js)\prod_{i=1}^n\Gamma(1-a_j-\alpha_is)}{\prod_{j=m+1}^q\Gamma(1-b_j-\beta_js)\prod_{i=n+1}^p\Gamma(a_j+\alpha_is)}$$

On taking LHS of (1.2.4.1) we have

$$\int_0^\infty x^{\alpha-1}[a+bx]^{-\alpha-\beta}H_{p,q}^{m,n}\left[\frac{y^kx^k}{\omega^k[a+bx]^k}\Big|\begin{matrix}(a_i,\alpha_i)_1^p\\(b_j,\beta_j)_1^q\end{matrix}\right]dx=$$

$$\int_0^\infty x^{\alpha-1}[a+bx]^{-\alpha-\beta}\left(\frac{1}{2\pi i}\int_L H_{p,q}^{m,n}(s)\left(\frac{y^kx^k}{\omega^k[a+bx]^k}\right)^{-s}ds\right)dx$$

By change of order of integration, we have

$$\frac{1}{2\pi i}\int_L H_{p,q}^{m,n}(s)\frac{y^{-sk}}{\omega^{-sk}}\left(\int_0^\infty x^{\alpha-ks-1}[a+bx]^{-(\alpha-ks)-\beta}dx\right)ds$$

By using (1.1.3) we have

$$\frac{1}{2\pi i}\int_L H_{p,q}^{m,n}(s)\frac{y^{-sk}}{\omega^{-sk}}\left(\frac{1}{a^\beta b^{\alpha-ks}}B(\alpha-ks,\beta)\right)ds$$

$$\frac{1}{b^\alpha a^\beta}\frac{1}{2\pi i}\int_L H_{p,q}^{m,n}(s)\frac{y^{-sk}}{(b\omega)^{-sk}}\frac{\Gamma(\alpha-ks)\Gamma(\beta)}{\Gamma(\alpha+\beta-ks)}ds$$

$$=\frac{1}{(a+b)^\alpha a^\beta}H_{p+2,q+2}^{m,n+2}\left[\frac{y^k}{b^k\omega^k}\Big|\begin{matrix}(1-\alpha,0),(1,0),(a_i,\alpha_i)_1^p\\(\beta,0),(1-\alpha-\beta,k)(b_j,\beta_j)_1^q\end{matrix}\right]$$

This complete the prove of the Theorem.

By using 1.1.4, we prove our next result, which is as follows,

Theorem1.2.5: Show that

$$\int_0^\infty x^{\alpha-1}(1-x)^{\beta-1}[a+bx]^{-\alpha-\beta}H_{p,q}^{m,n}\left[\frac{y^k(1-x)^k}{\omega^k[a+bx]^k}\Big|\begin{matrix}(a_i,\alpha_i)_1^p\\(b_j,\beta_j)_1^q\end{matrix}\right]dx=$$

$$\frac{1}{(a+b)^\alpha a^\beta}\left\{\begin{matrix}H_{p+2,q+2}^{m,n+2}\left[\frac{y^k}{a^k\omega^k}\Big|\begin{matrix}(-\alpha,0),(1,0),(a_i,\alpha_i)_1^p\\(\beta,-k),(-\alpha-\beta,k)(b_j,\beta_j)_1^q\end{matrix}\right]\\+\ H_{p+2,q+2}^{m,n+2}\left[\frac{y^k}{a^k\omega^k}\Big|\begin{matrix}(-\beta,k),(1,0),(a_i,\alpha_i)_1^p\\(\alpha,0),(-\alpha-\beta,k)(b_j,\beta_j)_1^q\end{matrix}\right]\end{matrix}\right\}\quad(1.2\ .1.1)$$

The convergence conditions for the validity of (1.2.1.1) are as follows,

iv. $0\le m\le q, 0\le p\le n$ for $a_i,b_j,\lambda,\mu\in C$ and $\alpha_i,\beta_j\in R^+$ and $i\in\{1,2,\ldots\ldots.p\}$, $j\in\{1,2,\ldots\ldots.q\}$ with $0<Re(\mu)<Re(\lambda-sk)$ and $k>0;\ x>0$.

v. $L=L_{i\gamma\infty}$is a contour starting at the point $\gamma-i\infty$and going to $\gamma+i\infty$where $\gamma\in R(-\infty,+\infty)$ such that all the poles of $\Gamma(b_j+\beta_j s)j\in\{1,2,\ldots\ldots.m\}$are separated from those of $\Gamma(1-a_i-\alpha_i s)$, $i\in\{1,2,\ldots\ldots.n\}$, The integral converges if $\alpha>0, |\arg z|<\frac{1}{2}\pi\alpha,\ \alpha\neq 0$. The integral also converges if $\sigma=0,\gamma\mu+Re(\delta)<-1,\arg z=0$ and $z\neq 0$ where $\sigma=\sum_{j=1}^n\alpha_j-\sum_{j=n+1}^p\alpha_j+\sum_{j=1}^m\beta_j-\sum_{j=m+1}^q\beta_j$.

vi. $Re(\mu)>\ 0, Re(\mu)+k\min_{(1\le j\le m)}Re\left[\frac{b_j}{\beta_j}\right]>\max\{0,(\lambda-sk)\}$.

Proof: We know that,

$H_{p,q}^{m,n}\left[z\Big|\begin{matrix}(a_i,\alpha_i)_1^p\\(b_j,\beta_j)_1^q\end{matrix}\right]=\frac{1}{2\pi i}\int_L H_{p,q}^{m,n}(s)z^{-s}ds$ where $H(s)$ is given by

$$H_{p,q}^{m,n}(s)=\frac{\prod_{j=1}^m\Gamma(b_j+\beta_j s)\prod_{i=1}^n\Gamma(1-a_j-\alpha_i s)}{\prod_{j=m+1}^q\Gamma(1-b_j-\beta_j s)\prod_{i=n+1}^p\Gamma(a_j+\alpha_i s)}$$

On taking LHS of (1.2.1.1) we have

$$\int_0^\infty x^{\alpha-1}(1-x)^{\beta-1}[a+bx]^{-\alpha-\beta}H_{p,q}^{m,n}\left[\frac{y^k(1-x)^k}{\omega^k[a+bx]^k}\Big|\begin{matrix}(a_i,\alpha_i)_1^p\\(b_j,\beta_j)_1^q\end{matrix}\right]dx=$$

$$\int_0^\infty x^{\alpha-1}(1-x)^{\beta-1}[a+bx]^{-\alpha-\beta}\left(\frac{1}{2\pi i}\int_L H_{p,q}^{m,n}(s)\left(\frac{y^k(1-x)^k}{\omega^k[a+bx]^k}\right)^{-s}ds\right)dx$$

By change of order of integration, we have

$$\frac{1}{2\pi i}\int_L H_{p,q}^{m,n}(s)\frac{y^{-sk}}{\omega^{-sk}}\left(\int_0^\infty x^{\alpha-1}(1-x)^{\beta-ks-1}[a+bx]^{-\alpha-(\beta-ks)}dx\right)ds$$

By using (1.1.3) we have

$$\frac{1}{2\pi i}\int_L H_{p,q}^{m,n}(s)\frac{y^{-sk}}{\omega^{-sk}}\left(\frac{1}{(a+b)^{\alpha}a^{\beta-ks}}B(\alpha,\beta-ks)\right)ds$$

By using (1.1.4) we have

$$\frac{1}{2\pi i}\int_L H_{p,q}^{m,n}(s)\frac{y^{-sk}}{\omega^{-sk}}\left(\frac{1}{(a+b)^{\alpha}a^{\beta-ks}}\left(B(\alpha+1,\beta-ks)+B(\alpha,1+\beta-ks)\right)\right)ds$$

$$\frac{1}{(a+b)^{\alpha}a^{\beta}}\left\{\begin{array}{l}\frac{1}{2\pi i}\int_L H_{p,q}^{m,n}(s)\frac{y^{-sk}}{(a\omega)^{-sk}}\frac{\Gamma(\alpha+1)\Gamma(\beta-ks)}{\Gamma(1+\alpha+\beta-ks)}ds\\ +\frac{1}{2\pi i}\int_L H_{p,q}^{m,n}(s)\frac{y^{-sk}}{(a\omega)^{-sk}}\frac{\Gamma\alpha\Gamma(1+\beta-ks)}{\Gamma(1+\alpha+\beta-ks)}ds\end{array}\right\}$$

$$=\frac{1}{(a+b)^{\alpha}a^{\beta}}\left\{\begin{array}{l}H_{p+2,q+2}^{m,n+2}\left[\frac{y^k}{a^k\omega^k}\Big|\begin{array}{c}(-\alpha,0),(1,0),(a_i,\alpha_i)_1^p\\(\beta,-k),(-\alpha-\beta,k)(b_j,\beta_j)_1^q\end{array}\right]\\ +\ H_{p+2,q+2}^{m,n+2}\left[\frac{y^k}{a^k\omega^k}\Big|\begin{array}{c}(-\beta,k),(1,0),(a_i,\alpha_i)_1^p\\(\alpha,0),(-\alpha-\beta,k)(b_j,\beta_j)_1^q\end{array}\right]\end{array}\right\}$$

This complete the prove of the Theorem.

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