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Classic and Modified Log-Barrier Methods Algorithms

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ABSTRACT

We study two alternative ways for solving nonlinear programming problem with inequality constraints, The first method is a classic barrier method .this method suffer from some computational disadvantages and are not entirely efficient -in which the hessian of the barrier function becomes increasingly ill conditioned for this reasons it may be desirable to improve the classic barrier method by alternative method, the second one is modified logarithmic barrier methods which are not only theoretically but also computationally superior to classic barrier methods when applied to nonlinear problems s., The solution to the problem is obtained by modified barrier function (MBF) at each step by using the Newton method and updating Lagrange multipliers. The Lagrange multipliers are updated by using the value of the constraints at the minimum of the MBF. We give the algorithms of these methods and some numerically result is reported.

Keywords: barrier method, classic barrier method, modified barrier method, Newton step, nonlinear programming, optimization.

1. INTRODUCTION

Penalty and barrier methods are procedures for approximating constrained optimization problems by unconstrained problems. The approximation is accomplished in the case of penalty methods by adding to the objective function a term that prescribes a high cost for violation of the constraints, and in the case of barrier methods by adding a term that favors points interior to the feasible region over those near the boundary. Basically, there are two alternative approaches. The first is called the penalty or the exterior penalty function method, in which a term is added to the objective function to penalize any violation of the constraints. This method generates a sequence of infeasible points whose limit is an optimal solution to the original problem.

The second method is called the barrier or interior penalty function method, in which a barrier penalty term that prevents the points generated from leaving the feasible region is added to the objective function. This method generates a sequence of feasible points whose limit is an optimal solution to the original problem. The earlies historically of these methods based on the idea of courtant [17] who transformed a constrained mininzation problem into an unconstrained, in the middle of 1950's frish[11] and at the outset of 1960 Carroll recommended the classic barrier function for solving optimization problems, later these functions were extensively studied by Fiacco and McCormick [3]. They are still among the most popular ones for some cases of problems, although there are some modifications that are more often used of penalty and barrier functions.

When a classical barrier method is applied to the solution of nonlinear problem with inequality constraints the Hessian of the barrier function becomes increasingly ill conditioned also it is not entirely efficient for these reasons it may be desirable to improve the classic barrier method by alternative one the modified barrier methods which avoids the inefficiency. It is the purpose of this work is to present two different algorithms for applying the methods to nonlinear problems, and indicate which problems arising from the original methods.

This paper is organized as follows. Section 2 describes the problem formula. Section 3 represents the algorithms of classic barrier method. Section 4, 5 drive the modified barrier method for general nonlinear programming and give the properties of the modified barrier function and give the algorithm Finally, in section 6 rate of convergent and computational results are presented and discussed and The basic conclusion which can be drawn from the computational tests and some concluding remarks are made.
We will focus on log barrier methods.

2. The statement of the problem

Consider the nonlinear programming problem

$$\min_{x\in R^n} f(x) \qquad (2.1a)$$

$$s.t \;\; c_i(x) \ge 0 \qquad i \in I \qquad (2,1b)$$

Where $x=(x_1,\ldots,x_n)^T$ and I is the index set for inequality . We assume that f and c_i, i=1,..,m, are twice continuously differentiable. Such problems arise in a variety of application in science, engineering, industry and management.The main idea of the penalty-barrier methods to find a solution for problem (2.1)
is to solve a sequence of unconstrained minimization sub-problems if the logarithmic barrier functions is used, then the unconstrained problems have the form

$$P(x;\mu) = f(x) - \mu\sum_{i\in I}\log c_i(x) \qquad (2.2)$$

where $\mu > 0$ is referred to here as the barrier parameter. From now on, we refer to $P(x;\mu)$ itself as the "logarithmic barrier function .then under certain conditions, it can be shown that as the minimizer of $P(x;\mu)$, which we denote by $x(\mu)$, approaches a solution of (2.1) as $\mu \downarrow 0$ [3,7].For further discussion, see the recent survey by Wright [16]. It was originally envisaged that each of the sequence of barrier functions be minimized using standard methods for unconstrained minimization. However Lootsma [6] and Murray [22,23] painted a less optimistic picture by showing that, under most circumstances, the spectral condition number of the Hessian matrix of the barrier function increases without bound as μ shrinks. This has important repercussionsas it indicates that a simple-minded sequential minimization is likely to encounternumerical difficulties. Consequently, the initial enthusiasm for barrier function methods declined. Methods which alleviate these difficulties have been proposed(see, e.g., Murray [21], Wright [26], Murray and Wright [7], Gould [24], and Mc-Cormick [25]) that are immediately applicable to smaller dense problems. Nashand Sofer [20] have recently discussed an approach that is applicable to large-scale,nonlinear problems, although their experience is only with simple bounds.We will examine two approaches classic and modified barrier methods. Techniques for doing this as follows:

3. Classic Barrier methods

The classical logarithmic barrier method of Fiacco and McCormick [3] was designed to solve the problem (2.1) . problem (2.1) transform to

$$\min_x P(x;\mu) = f(x) - \mu\sum_{i\in I}\log c_i(x) \qquad (3.1)$$

For a sequence of positive barrier parameters $\mu\rightarrow 0$, let $x(\mu)$be the mininmizer of $p(x,\mu)$. Under a certain conditions it can be show that any limit point x^* of the sequence $x(\mu)$ is a solution of (2.1) furthermore, the optimal Lagrange multipliers $\lambda_i^*, i\in I$, can be estimated as follows

$$\lambda_i^* \approx \frac{\mu}{c_i(x)}$$

Then $x(\mu)\rightarrow x, \lambda(\mu)\rightarrow\lambda$,where $\lambda$is the Lagrange multipliers corresponding to x
The Newton direction for the barrier subproblem (3.1)at the point x which are

$$\nabla^2_{xx} p(x,\mu)p = -\nabla p(x,\mu)$$

we examine the structure of the
gradient and Hessian of $P(x;\mu)$.We have

$$\nabla_x p(x;\mu) = \nabla f - \sum_{i\in I}\frac{\mu}{c_i(x)}\nabla c_i(x) \qquad (3.2)$$

$$\nabla^2_{xx} p(x;\mu) = \nabla^2 f(x) - \sum_{i\in I}\frac{\mu}{c_i(x)}\nabla^2 c_i(x) + \sum_{i\in I}\frac{\mu}{ci(x)}\nabla c_i(x)\nabla c_i(x)^T \qquad (3.3)$$

Unfortunately, the minimizer $x(\mu)$ becomes more and more difficult to find as $\mu \downarrow 0$. The scaling of thefunction $P(x;\mu)$ becomes poorer and poorer, and the quadratic Taylor series approximation does not adequately capture the behavior of the true function $P(x;\mu)$,except in a small neighborhood of $x(\mu)$. (see[27]). Also an important problem with classic logarithmic barrier methods was the need to determine an initial feasible point, which can be as difficult as solving the actual problem.The classic barrier method avoids this conditions by using an approximation to the Newton direction for barrier function , this approximation is obtained by examing the range and null space components of the search direction . A different way to avoid the ill condition but that requires explicit matrix factorizations and to develop the formula for search direction see [3-7] finally based on [7-18] we give algorithm in next section

Algorithm of classic Log-Barrier Function ,

Based on above analyses, an algorithm can be written as follows.

Framework 1 (Log-Barrier).

1. Given $\mu_0 > 0$, tolerance $\tau_0 > 0$ Select an initial feasible point x_o^s;

for k = 0, 1, 2, . . .

2. solve the sub-problem (3.1) to find an approximate minimizer x_k
3. if μ is small enough , then stop .otherwise select $\mu_{k+1} < \mu_k$, set k=k+1 and go to step 2

Note that when $\mu_k \to 0$, the Hessian of P(x,μ)is ill conditioned. On the other hand, selection of a good value for μ_{k+1} is difficult .the minimizing condition for P(x,μ) is $\nabla p(x;\mu_k) = 0$ if μ→0,then μk/ci(x)→λi,which is the Lagrange multiplier associated to the ith inequality constraint. and the logarithmic barrier algorithm is to choose an initial feasible x_k^s obtained by extrapolating along the path defined by the previous approximate minimizers $x_{k-1}, x_{k-2}, \ldots$ the choice $x_k^s = x_{k-1} + (\mu_k - \mu_{k-1})x^{\bullet}$ resolve these problems.

If the problem (2.1) is well-conditioned it has numerical difficulties and this the major reason why classic barrier methods are not using to solve nonlinear problems. Polyak [19] proposed the modified logarithmic barrier function also which will be describe for the constrained problem in the next section.

4. The modified Barrier Method

An extensive major literature of modified barrier methods can be found in ([4-7-8]) , These methods are combinations of thebest properties of the Classic Lagrangian and the Classic Barrier functions, but are free from their most essential drawbacks. At each major iteration the modified barrier method an unconstrained problem is solved , Polyak [19] suggested the following modified log barrier function

$$M(x,\lambda,\mu) = f(x) - \mu \sum_{i=1}^{m} \lambda_i \log\left(\frac{ci(x)}{\mu} + 1\right) \qquad (4.1)$$

Where λ_i are nonnegative estimates of the Lagrange multipliers associated to the inequality constants and μ>0 is the barrier parameter. For this modified log-barrier function, Polyak [19] established the convergence properties similar to those given by Bertsekas [12] for the multiplier method.

We must satisfy the following condition

$$\nabla f(x(\mu)) - \sum_{i \in I} \lambda_i(\mu) \nabla c_i(x(\mu)) = 0 \qquad (4.2)$$

The variable vector x are update as

$$x+ = x + \alpha \Delta x \qquad (4.3)$$

Where α>0 is the step length

5. Primal Dual Logarithmic Modified Barrier Method

By modifying the log-barrier viewpoint slightly, we can derive a new class of algorithms known as primal–dual interior-point methods see [2-15-16-20], In primal–dual methods, the Lagrange multipliers are treated as independent variables in the computations, with equal status to the primal variables x. As in the classic log-barrier approach, however, we still seek the primal–dual pair (x(μ), λ (μ)) that satisfies the approximate conditions which are as follows

$$c_i(x) \geq 0 \qquad \text{for all } i \in I \qquad (5.1)$$

$$\lambda_i \geq 0 \qquad \text{for all } i \in I \qquad (5.2)$$

$$\lambda_i c_i(x) \qquad \text{for all } i \in I \qquad (5.3)$$

$$\lambda_i(\mu) = \mu / c_i(x(\mu)) \qquad \text{for all } i \in I \qquad (5.4)$$

the Primal-Dual logarithmic barrier method, it is necessary to transform all inequality constraints into equality constraints by adding non-negative slack vectors, si ≥ 0 , so the problem can be written

$$\min_{(x,s)} f(x) \ subject \ to \begin{cases} c_i(x) = 0 & i \in E \\ c_i(x) - s_i = 0 & i \in I \\ s_i \geq 0 & i \in I \end{cases} \qquad (5.5)$$

S is strict positive , Suppose we rewrite the system of approximate KKT conditions
for the problem (2.1) which is (5.1)- (5.4), reformulating slightly by introducing slack variables s_i , i ∈ I, to obtain

$$\min f(x)$$

$$\begin{cases} c(x) = 0 \\ c(x) + s1 = 0 \\ x + s2 = x\max \\ -x + s3 = -x\min \\ s1 \geq 0 \\ s2 \geq 0 \\ s3 \geq 0 \end{cases} \qquad (5.6)$$

The TheLagrangian function, L, associated with the problem (5.6) is given by:

$$L = f(x) - \mu\sum_{j=1}^{r}\log(s_{1j}) - \mu\sum_{i=1}^{n}\log(s_{2i}) - \mu\sum_{i=1}^{n}\log(s_{3i}) + \sum_{i=1}^{m} \lambda_i c_i(x) + \sum_{j=1}^{r}\pi 1_j (cj(x) + s_{1j})$$

$$-\sum_{l=1}^{n}[\pi_{2l}(x_l + s_{2l} - x_{\max}) + \pi_{3l}(x_l - s_{3l} - x_{\min})] \qquad (5.7)$$

Where $\lambda, \pi_1, \pi_2, \pi_3$ are Lagrange multiplier vectors, called dual variable the stationary condition of L, is given by

$$\nabla L(x, s1, s2, s3, \lambda, \pi1, \pi2, \pi3) = 0 \qquad (5.8)$$

Which is equivalent to

$$\nabla L = \begin{bmatrix} \nabla f(x) - A(x)^T \lambda - A_1(x)^T(\pi 1 + \pi 2 + \pi 3) \\ -\frac{\mu}{s1} + \pi 1 \\ -\frac{\mu}{s2} + \pi 2 \\ -\frac{\mu}{s3} + \pi 3 \\ c_i(x) = 0, \\ c(x) + s1 \\ x + s2 - x_{\max} \\ -x + s3 + x_{\min} \end{bmatrix} \qquad (5.9)$$

A(x) is the matrix of constraint gradients, that is,

$$A(x)^T = \nabla c_i(x) \qquad i \in I$$

The solution of equation (9) can be obtained by Newton's method and can be represented as:

$$W\Delta p = -\nabla L \qquad (5.10)$$

Where

$$W = \begin{bmatrix} \nabla^2_{xx}L & 0 & 0 & 0 & A(x)^T & A_1(x)^T & I & I \\ 0 & \pi_1 s_1 & 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & \pi_2 s_2 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & \pi_3 s_3 & 0 & 0 & 0 & -I \\ A(x) & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ A_1(x) & I & 0 & 0 & 0 & 0 & 0 & 0 \\ I & 0 & I & 0 & 0 & 0 & 0 & 0 \\ I & 0 & 0 & -I & 0 & 0 & 0 & 0 \end{bmatrix} \quad (5.11)$$

$$\nabla^2_{xx}L = \nabla^2_{xx} f(x) + \sum_{p=1}^{m} \lambda_p \nabla^2_{xx} c_p(x) + \sum_{j=1}^{r} \pi \nabla^2_{xx} c_j(x)$$

S is the diagonal matrices whose diagonal elements are $\frac{\mu}{(s_{1p})^2}, \frac{\mu}{(s_{2r})}, \frac{\mu}{(s_{3r})}$ for s_1, s_2, s_3 respectively ,The correction vectors are used to update x,s,λ and π , as follows:

$$\begin{aligned} x_{k+1} &= x_k + \alpha_p \Delta x_k & \lambda^{k+1} &= \lambda^k + \alpha d \Delta \lambda^k \\ s_1^{k+1} &= s_1^k + \alpha_p \Delta s_1^k & \pi_1^{k+1} &= \pi_1^k + \alpha_d \Delta \pi_1^k \\ s_2^{k+1} &= s_2^k + \alpha_p \Delta s_2^k & \pi_2^{k+1} &= \pi_2^k + \alpha_d \Delta \pi_2^k \\ s_3^{k+1} &= s_3^k + \alpha_p \Delta_3^k & \pi_3^{k+1} &= \pi_3^k + \alpha_d \Delta \pi_3^k \end{aligned} \quad (5.12)$$

where the step length $\alpha \in (0,1]$ is chosen to preserve the feasibility of all the variables. It is chosen as follows:

$$\alpha_p^{\max} = \min\left\{ \frac{-s}{\Delta s} : \Delta s < 0 \right\} \quad (5.13)$$

$$\alpha_d^{\max} = \min\left\{ \frac{-\pi_1}{\Delta \pi_1} : \Delta \pi_1 > 0 , \frac{-\pi_2}{\Delta \pi_2} : \Delta \pi_2 > 0 , \frac{-\pi_3}{\Delta \pi_3} : \Delta \pi_3 > 0 \right\} \quad (5.14)$$

$$\alpha = \min\left\{ \tau \alpha_p^{\max}, \tau \alpha_d^{\max}, 1 \right\}$$

Where $\tau \in (0,1)$,Various heuristics have been proposed for the choice of the new barrier parameter μ at each iteration. the condition $\nabla_\delta L = 0$ where δ = (x,λ ,π ,s) .suggests that μ could be reduced on the basis of a predicted decrease of the complementarity gap [11]. A strictly feasible starting point is not necessary, but the conditions $s > 0, \pi_1 \le 0, \pi_2 \le 0$ and $\pi_3 \ge 0$ must be satisfied at every point. the correction vector is

$\Delta p^T = (\Delta x, \Delta s_1, \Delta s_2, \Delta s_3, \Delta\lambda, \Delta\pi_1, \Delta\pi_2, \Delta\pi_3)$, the estimators Lagrange multipliers λ suggests to update through the following rule

$$\lambda_{1j}^{k+1} = \frac{\lambda_1^k \mu^{k+1}}{s_{1j}^{k+1} + \mu^{k+1}} \quad (5.15)$$

The vectors of the variables x, s, λ and π are updated as in (5.12) and the vector of the variables λ is updated as in (5.15). The factors of barrier is reduced as $\mu_{k+1} = \mu_k * \beta$, β>1 is a parameter. Finally, in order to have the penalty-barrier algorithm, we mustspecify the optimization method used for solving (8.16) for each set of parameters ($\sigma_k, \lambda_k, s_k, \beta_k$). Since (8.16) is a simple bounded constrained optimization problem,we can apply any method for solving this type of problems [9-13-14], or truncatedNewton with simple bounds (Nash, [20]). In our implementation of the algorithm, we have used the truncated Newton with simple bounds ([1],[5], [20]). The approximate solution xk is used as the startingpoint for the next sub-problem. Now, the following penalty-barrier algorithm with quadratic extrapolation of the inequality constraints can be presented [1 -4-8-9-10]

Algorithm of modified penalty barrier :

i. Set k = 0 , Given initial guesses of p0= (x0 , s_1 , s_2 , s_3 ,λ_0 ,π_1 ,π_2 ,π_3)and μ0 make starting estimates for p k = (x k ,s k ,λk ,π k) , u1, u2, u3 and μ k that satisfy the proposal conditions.

ii. Calculate the Newton direction p.

iii. If the KKT conditions are satisfied, END. The problem is solved.

iv. Evaluate the barrier parameter μ.

v. Evaluate the matrix W as a function of p.

vi. Solve the system W∆p = −∇ L for ∆p.

vii. Update p by ∆p.

Set k = k +1 and return to step ii

The initial values allocated to the variables x should be in the feasible region of the problem. In the evaluate of W matrix in step v, the updatedvalue of μ must be used.

6. NUMARICAL RESULTS

In this section we present results obtained from the matlab package to test the problems as we discuss The approach in these methods is that to transform the constrained optimization problem into an equivalent unconstrained problem or into a problem with simple constraints, and solved using one (or some variant) of the algorithms for unconstrained optimization problems. Algorithms and MATLAB codes are developed using Newton method to find the search direction p .

$$\min \quad f(x) = (x_1 - 3)^2 + (x_2 - 4)^2$$
$$s.t \quad x_1^2 - x_2 = 3$$
$$e^{-x_1} - x_2 \le 0$$
$$-x_1 + 2x_2 - 2 \le 0$$

the resultsfor problems optimum solution is (1.332,1712) take starting point $[10,10]^T$ and tol$\in$=5*10^{-5},β=0.09

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