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On a subspace of Cesaro Summable Difference Sequence Space $C_1(\Delta)$

Parth Sarthi

Assistant Professor Mathematics, Government College for Girls, Madlauda, Panipat (Haryana)

ABSTRACT

In this paper author prove that the subset $\{(x_k) \in C_1(\Delta) : ((x_1 - x_{k+1})/k)$ converges to zero$\}$ of the Cesaro summable difference sequence space $C_1(\Delta)$ is separable with respect to the norm $||.||_\Delta$ where $||x||_\Delta = |x_1| + \sup\{|(x_1 - x_{k+1})/k| : k \geq 1\}$. As a corollary of this, it is proved that the space $(l_\infty, ||.||_\Delta)$ where l_∞ the spaces of all bounded sequences x = (x_k) with complex terms is separable. Thus the norms $||.||_\Delta$ and $||.||_\infty$ where $||x||_\infty = \sup\{|x_k| : k \geq 1\}$ on the set l_∞ are not equivalent.
MSC: 46A45, 54D65

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1. INTRODUCTION

The normed linear space $(C_1(\Delta), ||.||_\Delta)$ introduced in [1] is proved to be inseparable. But we see that the proof is not sound (see Remark 3.2 below). We don't know whether the Cesaro summable difference sequence space $C_1(\Delta)$ is separable or not. But we find a subset of $C_1(\Delta)$ containing the set l_∞ which is separable with respect to the norm $||.||_\Delta$. It follows that $(l_\infty, ||.||_\Delta)$ is separable. This proves that the norms $||.||_\Delta$ and $||.||_\infty$ are not equivalent.

2. NOTATION AND DEFINITIONS

IN, Q and IR denote the set of all natural numbers, the set of all rational numbers and the set of all real numbers respectively. By *s* we shall denote the linear space of all complex sequences over the field of complex numbers. l_∞ denote the spaces of all bounded sequences x = (x_k) with complex terms, respectively, normed by $||x||_\infty = \sup\{|x_k| : k \geq 1\}$. Let $C_0(\Delta) = \{(x_k) \in C_1(\Delta) : ((x_1 - x_{k+1})/k)$ converges to zero$\}$. A sequence space X is called normal or solid if $y = (y_k) \in X$ whenever $|y_k| \leq (x_k)$, $k \geq 1$, for some $(x_k) \in X$.

3. SEPARABLE, NORMAL SPACE

Lemma. 3.1. Let (X,d) be a separable metric space. Let $\delta > 0$. Let A be a non-empty subset of X such that $d(a,b) > \delta$ for every two distinct points a, b of A. Then A is countable.

Proof. We prove the result by contradiction. Suppose not. Then A is uncountable. Let D be any dense set in (X,d). We define a function $f : A \to D$ as follows. Let $a \in A$. Since D is dense in (X,d), so there is some $a_D \in D$ such that $d(a,a_D) < \delta/2$. By axiom of choice, we can set $f(a) = a_D$. We prove that f is one-one. Let $a, b \in A$, $a \neq b$. Now by given condition, $d(a,b) > \delta$. Now $d(a,b) \leq d(a,b_D) + d(b_D,b) < d(a, b_D) + \delta/2$. This implies that $d(a, b_D) > \delta/2$. Therefore $a_D \neq b_D$. Thus f is one-one. This proves that D is uncountable as A is uncountable. Hence X is not separable which is a contradiction to the given condition. The proof is complete.

Remark. 3.2. Since the real line IR is separable, it follows from Lemma 3.1 that the set A taken in the proof of Theorem 3.7 of [1] is countable. So the proof is not sound to prove $C_1(\Delta)$ as an inseparable space.

Corollary. 3.3. If a metric space (X,d) has an uncountable set A such that for every two distinct points a, b of A, $d(a,b) > \delta$ for some $\delta > 0$, then X is inseparable.

We don't know whether the sequence space $(C_1(\Delta), ||.||_\Delta)$ is separable or not. But below, we prove that $C_0(\Delta)$ is separable with respect to the norm $||.||_\Delta$. As a corollary of this, we see that the space $(l_\infty, ||.||_\Delta)$ is also separable.

Theorem. 3.4. The sequence space $(C_0(\Delta), ||.||_\Delta)$ is separable.
Proof. Let $A = \{y = (y_k) : y_k \in Q$ for $1 \leq k \leq n$ and $y_k = 0$ for $k > n, n \in IN\}$. Then A is countable. Also A is a subset of $C_0(\Delta)$. We prove that A is dense in $(C_0(\Delta), ||.||_\Delta)$. Let $\in > 0$. Let $x = (x_k) \in C_0(\Delta)$. Then $((x_1 - x_{k+1})/k)$ converges to zero. So there is some natural number p such that $|(x_1 - x_{k+1})/k| < \in/6$ for all $k \geq p$. Let m be a natural number such that $|x_1|/m < \in/6$. Let $t = \max\{p,m\}$. Then $|(x_1 - x_{k+1})/k| < \in/6$ for all $k \geq t$ and $|x_1|/t < \in/6$. Since Q is dense in the real line, so there is rational numbers y_k, $1 \leq k \leq t$ such that $|x_k - y_k| < \in/6$. We set $y_k = 0$ for $k > t$. Let $y = (y_k)$. Then $y \in A$. Now $||x - y||_\Delta = |x_1 - y_1| + \sup\{|((x_1 - y_1) - (x_{k+1} - y_{k+1}))/k : k \geq 1\}$. Now for $1 \leq k \leq t - 1$, $|((x_1 - y_1) - (x_{k+1} - y_{k+1}))/k| \leq |x_1 - y_1|/k + |x_{k+1} - y_{k+1}|/k < \in/6k + \in/6k = \in/3k < \in/3$. For $k \geq t$, $|((x_1 - y_1) - (x_{k+1} - y_{k+1}))/k| = |((x_1 - y_1) - (x_{k+1}))/k| \leq |x_1 - y_1| + |x_{k+1}/k| < \in/6 + |(x_{k+1} - x_1)/k| + |x_1/k| < \in/6 + \in/6 + \in/6 = \in/2$. It follows that $||x - y||_\Delta < \in/6 + \in/2 < \in$. This proves that A is dense in $(C_0(\Delta), ||.||_\Delta)$.

Lemma. 3.5. $l_\infty \subset C_0(\Delta)$.
Proof. Let $x = (x_k) \in l_\infty$. Then there exists $M > 0$ such that $|x_1 - x_{k+1}| \leq M$ for all $k \geq 1$, and so $((x_1 - x_{k+1})/k) \to 0$ as $k \to \infty$.

Theorem. 3.6. The sequence space $(l_\infty, ||.||_\Delta)$ is separable.
Proof. Since subspace of a separable metric space is separable, so the theorem follows by Lemma 3.5 and Theorem 3.4.

Remark. 3.7. By Theorem 3.6, we see that the norms $||.||_\Delta$ and $||.||_\infty$ where $||x||_\infty = \sup\{|x_k| : k \geq 1\}$ on the set l_∞ are not equivalent.

Theorem. 3.8. $C_0(\Delta)$ is normal or solid.
Proof. Let (y_k) be such that $|y_k| \leq |x_k|$ for some $(x_k) \in C_0(\Delta)$. We prove that $(y_k) \in C_0(\Delta)$. Let $\in > 0$. Then there is some natural number p such that $|(x_1 - x_{k+1})/k| < \in/6$ for all $k \geq p$. Let m be a natural number such that $|x_1|/m < \in/6$. Let $t = \max\{p,m\}$. Then $|(x_1 - x_{k+1})/k| < \in/6$ for all $k \geq t$ and $|x_1|/t < \in/6$. Now all $k \geq t$, $|(y_1 - y_{k+1})/k| \leq |y_1/k| + |y_{k+1}/k| \leq |x_1/k| + |x_{k+1}/k| \leq \in/6 + |(x_{k+1} - x_1)/k| + |x_1/k| < \in/6 + \in/6 + \in/6 < \in$. This proves that $(y_k) \in C_0(\Delta)$.Thus $C_0(\Delta)$ is normal.

REFERENCES

[1] V.K. Bhardwaj and S. Gupta, Journal of Inequalities and Applications 2013, 2013:315.