Planar Annihilating-Ideal Graph of Commutative Rings

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ABSTRACT

For a commutative ring with identity , let AG(R) be the set of ideals of R with non-zero annihilators .The annihilating-ideal graph of R with vertex set AG(R)*= AG(R)-{0} and two distinct vertices I and J are adjacent if and only if I,J=0 .In this paper we investigate and find the graph AG(R) to be planar. Also we give some basic properties of AG(R), where R finite local rings. Finally we find planarity Zn.

Keyword: annihilating-ideal graph, planar graph, finite rings, local rings,

1-INTRODUCTION

Let R be a finite commutative ring with identity, and Z(R) (A(R)) the set of zero divisors( ideals with non-zero annihilator , respectively). We associate a simple graph Γ(R)[2]AG(R) respectively) with vertices Z(R)*=Z(R)-{0} ((A(R)*=A(R)-{0}, respectively) and two vertices x and y (I and J, respectively) are adjacent if and only if xy=0(IJ=(0), respectively).The first study of planar of zero divisor graph in 2001[3] when an interesting question was proposed by Anderson, Frazier, Lauve and Livingston: For which finite commutative rings R is Γ(R) planar? answer this question was given from by some authors see[1 ,3 , 6 ].Our goal in this paper is to investigate finite commutative rings whose Annihilating-ideal graph are planar. It is clear that if Γ(R) is not - planar then AG(R) is need not to be planar, for example, Γ(Z32) is shown in figure( 1-1), and figure (1-2) shows AG(Z32)

Fig. 1.1 and 1.2

For notation, we let Kn represents the complete graph on n vertices, if n=3 , then is called triangle and Km,n the complete bipartite graph with part sizes m and n. We will repeatedly use Kuratowski’s theorem, which states that a graph is planar if and only if it does not contain a subdivision ofK5 orK3,3 [7].

When working with polynomial rings, say K[X]/I, we will let X denote the cosetX+I. In particular Fn is denoted by a field of order n Σ_m is the set of coset representatives of F*=F –{0}.Σ_m* = Σ_m U {0}.The symble“—” is
denoted the edge between two vertices ,[9]. Zn denoted the ring of integers modulo n. Finally ann(X) denoted by annihilator of se X .

2. PLANARITY OF COMMUTATIVE LOCAL RINGS.

It well known that if (R,M) is a finite local ring with maximal ideal M, then |R|=pt, for a some positive prime number p and positive integer t.In this section we investigate planarity of local rings of order pt

Question:

Under what condition finite commutative ring R is AG(R) is planar?

First we prove some results in a finite local rings

Proposition 2.1:

Let R be a local ring , then every minimal ideal of R adjacent with every ideal vertices in annihilating ideal graph of R.

Proof:

Let K minimal ideal of R,if K is not adjacent with every ideal vertices then there exists an ideal vertex J of AG(R)such that KJ ≠0 ,so that J ∉ ann(K) but ann(K) maximal and R local ring which implies a contradicts therefor K,J=0 and hence K adjacent with every vertices in AG(R).

A converse of Proposition2.1 is not true in general as the following example shows:

Example 1:

Let R ≅ Z[x,y]/(x^4,xy, y^2) the ideals of R, I_1=(X) ,I_2=(Y) , I_3=(X,Y) , I_4=(X^2),I_5=(X^2,Y) , I_6=(X+Y), I_7=(X^2+Y), I_8=(X^3) , I_9=(X^3,Y), I_10=(X^3+Y) it's clearly I_9 adjacent with every other ideal vertices but not minimal ideal .

Theorem 2.2:

If R local ring with maximal ideal M and M^2=0 , then either R has exactly one ideal M or R contains at least two minimal ideals.

Proof:

If R contains at least two minimal ideals we are done . Suppose that R contains one minimal ideal, since M^2=0 , then M≅ ann(M) , but M maximal ideal , so that M=ann(M). On the other hand ann(M) is minimal ideal . Which leads M minimal and maximal ,since R local then for every ideal J of R,M⊆ J ⊆ M by chosen R contains one minimal ideal and hence J=M. Which implies that R contains one ideal.

Proposition 2.3:

Let R be a local ring with M^2=0, then AG(R) is complete graph.

Proof:

If R contained one ideal,then AG(R) = k_1,we are done .If not ,let I and J be any ideal vertices of AG(R). Since IJ⊆ MM=(0), then any ideal vertices adjacent in AG(R).Therefore AG(R) is complete graph.

Corollary 2.4:

Let R be a local ring with M^2=(0). Then AG(R) is planar if and only if s≤4, where s=|A(R)*|.

Proof:

Applying Proposition 2.3 AG(R) is complete graph so that AG(R) is planar if and only if s≤4.

Theorem 2.5:

Let R be a local ring .Then either M^3 = 0 or AG(R) has a triangle
Proof:

Since R is a local ring, there exists an integer \( n \geq 2 \) such that \( M^n = 0 \) and \( M^{n+1} \neq 0 \). Clearly \( M^{n+1} \) is adjacent to every non-zero ideal vertex I of AG(R). Now if \( M^2 = 0 \) we are done. If not, then \( M^2 \) and \( M^{n+2} \) will be adjacent, so that AG(R) has a triangular 5-connected subgraph. Therefore AG(R) is 5-regular, and thus AG(R) is planar. For all other local rings R of order \( p^4 \), AG(R) is planar.

Theorem 2.6:

Let R be a local ring with \(|R|=p^i\), where p is a positive prime number and \( t=2,3 \). Then R is planar.

Proof:

If \( t=2 \), then by \([8] \) \( R \cong Z_{p^2} \) or \( Z_p[X]/(X^2) \). So that AG(R) = \( K_4 \) which is planar. If \( t=3 \), then \( R \cong F_p[X]/(X^3) \), \( F[X,Y]/(X,Y^2) \), \( A[X]/(pX,X^2 – ap) \), or \( Z_p^2 \), where \( a \neq 0 \) and \( a \in \sum_0^2 \). So that AG(R) = \( K_4 \) where \( R \cong F_p[X]/(X^3) \), \( A[X]/(pX,X^2) \), \( Z_p^3 \) and \( AG(R) \cong K_4 \), where \( F[X,Y]/(X,Y^2) \), \( A[X]/(pX,X^2 – ap) \), and \( a \in \sum_0^2 \). For all cases R is planar.

Theorem 2.7:

If R is isomorphic to one of the following six rings of order \( p^4 \):

\[ F_p[X,Y]/(X,Y,P)^5 \], \( F_p[X,Y,Z]/(X,Y,Z)^3 \) or \( F_p[X]/(X^3,P) \), \( F_p[X,Y]/(X^4,XY,Y^2) \), \( Z_p^2 \) \( [X,Y]/(X^2 – p,XY,Y^2,pX) \), \( Z_p^2 [X,Y]/(X^2 – p,XY,Y^2 – p, pX) \), \( Z_p^3 [X]/(X^2 – P^2, pX) \), \( Z_p^2 [X]/(pX,3F_p[X]/(X^3,pX)) \), \( F_p[X,Y]/(X,Y,p)^3 \) and \( F_p[X,Y,Z]/(X,Y,Z)^2 \).

It is easy to check that if \( R \cong F_p[X]/(X^2) \) or \( Z_p^2 [X]/(X^2 + X + 1) \), then R has non-zero one ideal, so that AG(R) isomorphic to \( K_4 \) and hence R is planar in this case. Consider the rings \( R \cong F_p[X]/(X^4) \), \( Z_p^2 [X]/(X^2 – ap) \) where \( p \neq 2 \) and \( a \in \sum_0^2 \), \( Z_p^2 [X]/(X^2 – 2X – 2) \), \( Z_p^2 [X]/(X^3 – p,2X) \) or \( Z_p^2 \), then R have non-zero three ideals and \( AG(R) \cong K_{1,2} \) and hence R is planar in this case. Consider the rings \( R \cong F_p[X,Y]/(X^2,Y^2) \), \( Z_p^2 [X,Y]/(X^2,Y^2 – p,Y^2 – p,pX) \), \( Z_p^2 [X]/(X^2 – P^2, pX) \), \( Z_p^2 [X]/(pX,3)F_p[X]/(X^3,pX) \) and \( F_p[X,Y]/(X,Y,p)^3 \) and \( F_p[X,Y,Z]/(X,Y,Z)^2 \), then R have non-zero five ideals and \( AG(R) = K_{1,4} \), so that R is planar.

Fig (2-1)
\[ R \cong F[X,Y]/(X^2,Y), Z_\sim(p^2 )[X,Y]/(X^2,XY- p,Y^2), Z_\sim(p^2 )[X]/(X^2- pX), or Z_\sim(p^2 )[X]/(X^2) \]

**Theorem 2.8:**

Let \( R = \mathbb{Z}_{p^m} \) be a ring of integer module \( p^m \) where \( p \) is prime and \( m \) positive number, then \( R \) is planar iff \( m \leq 8 \)

**Proof:**

Clearly \( \mathbb{Z}_{p^m} \) has \( (m-1) \) ideals, therefore \( (AG(R)) \leq 4 \) if \( m \leq 5 \) implies \( AG(R) \) is planar, if \( m=6,7 \) or \( 8 \) then \( AG(R) \) is planar see figures (2-2, 2-3 and 2-4).

If \( m \geq 9 \) then the vertices ideals \( (p^m-1), (p^{m-2}), (p^{m-3}), (p^{m-4}) \) adjacent so that \( \mathbb{Z}_{p^m} \) has \( K_5 \) as a sub graph, there for \( \mathbb{Z}_{p^m} \) is not planar.

![Figure 2-2 AG(ZP6)](image1)
![Figure 2-3 AG(ZP7)](image2)
![Figure 2-4 AG(ZP8)](image3)

**3. PLANARITY OF COMMUTATIVE NON-LOCAL RINGS**

In this section we investigate planarity of non-local rings. It well known that a finite ring \( R \), being Artinian, is isomorphic to a finite product of Artinian local rings. Thus if \( R \) is a finite ring, then \( R \cong R_1 \times R_2 \times \ldots \times R_n \) for some \( n \geq 1 \) and each \( R_i \) is an Artinian local ring.

**Theorem 3.1**

Let \( R \cong R_1 \times R_2 \times \ldots \times R_n \) for some \( n \geq 3 \) and each \( R_i \) is a local ring, then \( R \) is planar if and only if \( R \cong F \times F \times F \) or \( A \times F \times A \) where \( F \), \( F \times F \) and \( F \times A \) are fields and \( A \) is a local ring contains one ideal.

**Proof:**

If \( n \geq 4 \), then \( AG(R) \) has \( K_{3,3} \) as a sub-graph by \((0,0,\ldots,R_{n-1},0),(0,0,\ldots,0,R_n),(0,0,\ldots,R_{n-1},R_n)\) are all adjacent to \((R_1,0,0,\ldots),(R_1,R_2,0,\ldots),(R_2,0,\ldots)\). Then \( AG(R) \) is not planar.

If \( n = 3 \), then there exists three cases:
**Case 1:** if $R_1$ and $R_2$ not field, then there exists ideals $I_1 \subseteq R_1$ and $I_2 \subseteq R_2$ such that $I_i^2 = 0$, $i = 1,2$. Therefore $AG(R)$ is not planar by $(R_1,0,0)$, $(I_1,0,0)$ and $(R_1,I_2,0)$ are all adjacent to $(0,I_1,0)$, $(0,I_2,R_3)$ and $(0,0,R_3)$ is $K_{3,3}$ a sub-graph of $AG(R)$.

**Case 2:** If one of the $R_i$, $i=1,...,3$ without loss generality say $R_1$ not field, then there exists two sub-cases

Sub-cases a: If $R_1$ has at least two ideals, say $I_1$ and $I_2$ therefore the ideal vertices $(I_1,0,0)$, $(I_2,0,0)$ and $(R_1,0,0)$ are all adjacent to $(0,R_2,0)$, $(0,R_2,R_3)$, $(0,0,R_3)$ a $K_{3,3}$ sub-graph of $AG(R)$. Therefore $AG(R)$ not planar.

Sub-cases b: If $R_1$ has exactly one ideal say $M_1$ then by theorem 2.2 $M_1^2 = 0$. Since $R_2$ and $R_3$ field, then $R$ has ideals $I_1 = (R_1,R_2,0)$, $I_2 = (R_1,0,R_3)$, $I_3 = (R_1,0,0)$, $J_1 = (0,R_2,0)$, $J_2 = (0,0,R_3)$, $J_3 = (M_1,R_2,0)$, $J_4 = (M_1,0,R_3)$, $J_5 = (M_1,0,0)$, then $AG(R)$ is planar see figure (3-1).

**Case 3:**

If $R_1,R_2,R_3$ are field, then $R$ have ideals $J_1 = (R_1,0,0), J_2 = (R_1,R_2,0), J_3 = (R_1,0,R_3), J_4 = (0,R_2,0), J_5 = (0,0,R_3)$, whence $AG(R)$ planar see figure (3-2).

**Theorem 3.2:**

Let $R \cong R_1 \times R_2$ where $R_1$, $R_2$ local ring with $M_1$, $M_2 \neq 0$, $M_1^2 = M_2^2 = 0$, then $R$ is planar iff $R \cong A \times B$, where $A$ and $B$ local ring with one ideal.

**Proof:**

Since $M_1,M_2 \neq 0$ then $R_1$ and $R_2$ not field, then by theorem 2.2 $R_1$ and $R_2$ either contains one ideal or contains at least two minimal ideals.

If $R_1$, $R_2$ contains one ideal, then $AG(R)$ is planar. If $R_1$, $R_2$ contains two minimal ideals say $I_1, I_2$ be minimal ideals in $R_1$ and $J_1, J_2$ minimal ideals in $R_2$, then the ideal vertices $(R_1,0),(I_1,0)$ and $(I_2,0)$ are all adjacent to $(0,R_2),(0,J_1)$ and $(0,J_2)$ a $K_{3,3}$ sub-graph. Therefore $AG(R)$ not planar.
If $R_1$ contains one ideal and $R_2$ contains at least two minimal ideals, let $I_1, I_2$ be minimal ideals in $R_2$ and $M_0, M_3$ maximal ideal in $R_3$, since $J_1J_2 \triangleq M_1$ and $I_1J_2 \neq M_2$, we get ideals $(R_1, 0, (R_1, J_1)$ and $(R_1, J_2)$ are all adjacent to $(0, M_2), (0, J_1), (0, J_2)$ a $K_{3,3}$ sub-graph in $AG(R)$ and hence $R$ is not planar.

**Theorem 3.3:**

Let $R$ be a finite ring such that $R \cong R_1 \times R_2$ where $R_1$ and $R_2$ are local rings with $M^2 \neq 0$, then $AG(R)$ is not planar.

**Proof:**

Since $R_2$ finite local ring, then exists an integer $n \geq 1$ such that $M_1^n = (0)$ and $M_2^{n-1} \neq 0$, but $M_2 \neq 0$, then we have $n \geq 5$. So that by proof of theorem 2.5 $R_2$ contains a triangle $M_1^2 \implies M_2^{n-1} \implies M_1^2$ we note that $(M_1^2)^2 = M_1^2.M_2^{n-1} = M_2^n.M_1^2 = 0$, where $s_i = n-2 > 0$, similarly $(M_1^2)^2 = M_1^2.M_2^{n-1} = 0$, where $s_i = n-4 > 0$. Therefore the ideal vertices $(R_1, 0)$, $(R_1, M_1^{s_1})$ and $(R_1, M_1^{s_2})$ are all adjacent to $(0, M_2^{s_1})$, $(0, M_2^{s_2})$, $(0, M_2^5)$ a $K_{3,3}$ sub-graph in $AG(R)$ and hence $R$ is not planar.

**Theorem 3.4:**

Let $R \cong R_1 \times R_2$ where $R_1$ and $R_2$ are local rings then $AG(R)$ is planar if and only if $R \cong A_1 \times A_2$ or $F \times B$, where $F$ is a field, $A_1, A_2$ are field or local rings with one ideal and $B$ local ring contains two or three ideals with maximal ideal $M$ satisfies $M^2 \neq 0$ and $M^3 = 0$.

**Proof:**

It clear that, if $R_1$ and $R_2$ fields or contains one ideal, then $R$ is planar and $R_i$ for some $i=1$ or $2$, contains triangular, then by Theorem 3.3 $R$ not planar. Also if $R_1$ and $R_2$ contains at least two ideals, then the ideal vertices ideals $(R_1, 0)$, $(I_1, 0)$ and $(I_2, 0)$ are all adjacent to $(0, R_2), (0, I_1)$ and $(0, I_2)$ in $AG(R)$. Therefore $K_{3,3}$ is a sub-graph of $AG(R)$ and therefore $AG(R)$ not planar. So we enough investigate two cases:

**Case 1:** If $R_1$ is a field and $R_2$ local ring contains at least four ideals say $I_1, I_2, I_3$ and $I_4$ without loss generality $I_1$ minimal ideal. Since $R_2$ local, then by Proposition 2.1 $I_1$ adjacent with every other ideal vertices $I_1, I_2, I_3$ and $I_4$. So that the ideal vertices $(R_1, 0)$, $(R_1, I_1)$ and $(I_2, 0)$ are all adjacent to $(0, I_1)$, $(0, I_2)$ and $(0, I_3)$ in $AG(R)$. Therefore $K_{3,3}$ is a sub-graph of $AG(R)$ and so $AG(R)$ not planar. Also if $R_2$ less than or equal three ideals, but not contains triangular say $I_1, I_2, I_3$, since $R_2$ not triangular $I_2^2 = 0$, $i=1,2$ and $I_1, I_2 = 0, I_1, I_2 = 0, I_2, J_1 = 0$ then the ideal in $R_1 \times R_2$ are $J_1 = (R_1, I_1), J_2 = (R_1, I_2), J_3 = (R_1, I_3), J_4 = (R_1, 0), J_5 = (0, R_2), J_6 = (0, I_1), J_7 = (0, I_2), J_8 = (0, I_3)$, then $AG(R)$ is planar see figure (3-3).

**Fig. (3-3)**

**Case 2:** If $R_1$ contains one ideal say $M_1$, then $M_1^2 = 0$. Now if $R_2$ contains at least three ideals $I_1, I_2$ and $I_3$ with minimal ideal $I_3$, then the ideal vertices $(R_1, 0)$, $(M_1, 0)$ and $(M_1, I_1)$ are all adjacent to $(0, I_1)(0, I_2)(0, I_3)$ in $AG(R)$ a $K_{3,3}$. Therefore $K_{3,3}$ is a sub-graph of $AG(R)$ and so $AG(R)$ not planar. If $R_2$ contains two ideals $I_1$ and $I_2$ with $I_1^2 = 0$ and $I_2^2 = 0$, clearly $I_1, I_2 = 0$ then $R$ have ideals $J_1 = (R_1, 0), J_2 = (R_1, I_1), J_3 = (R_1, I_2), J_4 = (M_1, 0), J_5 = (0, I_1), J_6 = (0, I_2), J_7 = (0, I_3)$ is planar see figure (3-4).
Theorem 3.5:

\[ \text{let } R \cong \mathbb{Z}_{p_1^{\alpha_1}} \times \mathbb{Z}_{p_2^{\alpha_2}} \times \mathbb{Z}_{p_3^{\alpha_3}} \times \ldots \times \mathbb{Z}_{p_m^{\alpha_m}}, \text{ where } p_i \text{ distinct primes and } \alpha_1, \alpha_2, \ldots, \alpha_m \text{ positive number } m \geq 2 \]

then \( R \) is planar iff \( R \cong \mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \mathbb{Z}_{p_3} \times \ldots \times \mathbb{Z}_{p_m} \)

Proof:

Since \( \mathbb{Z}_{p_1^{\alpha_1}} \times \mathbb{Z}_{p_2^{\alpha_2}} \times \mathbb{Z}_{p_3^{\alpha_3}} \times \ldots \times \mathbb{Z}_{p_m^{\alpha_m}} \)

If \( m \geq 3 \), then by Theorem 3.1 \( R \) is planar iff \( R \cong \mathbb{Z}_{p_1^{\alpha_1}} \times \mathbb{Z}_{p_2^{\alpha_2}} \times \mathbb{Z}_{p_3^{\alpha_3}} \) or \( \mathbb{Z}_{p_1^{\alpha_1}} \times \mathbb{Z}_{p_2^{\alpha_2}} \times \mathbb{Z}_{p_3^{\alpha_3}} \times \ldots \times \mathbb{Z}_{p_m^{\alpha_m}} \)

If \( m = 2 \), then by Theorem 3.4 \( R \) is planar iff \( R \cong \mathbb{Z}_{p_1^{\alpha_1}} \times \mathbb{Z}_{p_2^{\alpha_2}}, \text{ where } \alpha_1, \alpha_2 = 1,2 \)

Or \( R \cong \mathbb{Z}_{p_1^{\alpha_1}} \times \mathbb{Z}_{p_2^{\alpha_2}}, \text{ where } \alpha_2 = 3,4 \)

Example 2:

Let \( R = \mathbb{Z}_{p_1^{\alpha_1}} \times \mathbb{Z}_{p_2^{\alpha_2}} \) where \( p_1 = 3, p_2 = 2 \). Then \( R \cong \mathbb{Z}_{48} \) then \( R \) has ideals \( \{(2),(3),(4),(6),(8),(12),(16),(24)\} \). Hence \( \text{AG}(R) \) is planar.

FIG. (3-4)
REFERENCES