Equality of Connected Edge Domination and Total Edge Domination in Graphs

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Abstract: Let $G$ be a $(p,q)$-graph with connected edge domination number $\gamma'_c$ and total edge domination number $\gamma'_t$. In this paper we investigate the structure of graphs in which some of the edge domination parameters are equal. We characterize graphs for which $\gamma'_c = \gamma'_t$.

Key words: Edge domination number, Connected edge domination number and Total edge domination number.

1. Introduction

By a graph $G = (V,E)$ we mean a finite undirected graph without loops or multiple edges. Terms not defined here are used in the sense of Harary [1]. The concept of edge domination was introduced by Mitchell and Hedetniemi. A subset $X$ of $E$ is called an edge dominating set of $G$ if every edge not in $X$ is adjacent to some edge in $X$. The edge domination number $\gamma'(G)$ (or $\gamma'$ for short) of $G$ is the minimum cardinality taken over all edge dominating sets of $G$. An edge dominating set $X$ of $G$ is called a total edge dominating set of $G$ if the induced subgraph $X$ has no isolated edges. The total edge domination number $\gamma_t'(G)$ (or $\gamma'_t$ for short) of $G$ is the minimum cardinality taken over all total edge dominating sets of $G$. An edge dominating set $X$ of $G$ is called a connected edge dominating set of $G$ if the induced subgraph $X$ is connected. The connected edge domination number $\gamma_c'(G)$ (or $\gamma'_c$ for short) of $G$ is the minimum cardinality taken over all connected edge dominating sets of $G$.

Allan and Laskar [2] proved that for any $K_{1,3} -$ free graph, the domination number and independent domination number are equal. Topp and Volkmann [3] generalized the result of Allan and Laskar and constructed new classes of graphs with equal domination and independent domination number. Harary and Livingston [4] characterized caterpillars with equal domination and independent domination number. In [5] they gave the characterization of trees with equal domination and independent domination number. Payan and Xuong [6] proved that for any graph $G$ on 9 vertices, $\gamma = \gamma = 3$ if and only if $G = K_3 \times K_3$. Arumugam and Paulraj Joseph [7] studied the class of graphs for which connected domination number and domination number are equal. In this paper we initiate a study of graphs in which some of the edge domination parameters are equal. We characterize graphs for which $\gamma'_c = \gamma'_t$. For this we need the following Theorem.

Theorem 1.1 [8] Let $P$ denote the property that $\gamma' = \gamma'_c = n$. A connected graph $G$ is $P - critical$ if and only if $G$ is isomorphic to $S(K_{1,n})$.

2. Main Results

We observe that for any connected graph $G$ with $\gamma'_c = 2$ or $3$, $\gamma'_c = \gamma'_t$. We now proceed to characterize connected graphs with $\gamma'_c = \gamma'_t = n \geq 4$.

Definition 2.1: Let $P^{(n)}$ denote the property that $\gamma'_t = \gamma'_c = n$. A connected graph $G$ is $P^{(n)} - critical$ if $G$ satisfies $P^{(n)}$ but no proper connected sub graph $H$ of $G$ satisfies $P^{(n)}$.

Lemma 2.2 For $n \geq 4$, a connected subgraph $G$ is $P^{(n)} - critical$ if and only if $G$ satisfies the following conditions.

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(i) G contains a subgraph H which is either P₃ or a tree with at least 4 edges, exactly one vertex u of degree at least 3 and all its pendant vertices are at distance 1 or 2 from u.

(ii) For each pendant edge xᵢ of H, there is exactly one edge yᵢ of G such that yᵢ is adjacent to xᵢ but not adjacent to any other edge of H.

(iii) Every vertex v ∈ V(G) \ V(H) is incident with at most two yᵢ’s.

(iv) For n ≥ 5, if H has a vertex of degree at least 3 and there exists a vertex w in V(G) \ V(H) incident with two yᵢ’s, say y₁ = u₁w and y₂ = u₂w, then u₁ and u₂ are adjacent to u.

(v) The subgraph induced by xᵢ’s and yᵢ’s is G.

**Proof:** Suppose G is \( P(n) - critical \). Then \( \gamma_e' (G) = \gamma_c' (G) = n \geq 4 \). Let \( S = \{ x_i | 1 \leq i \leq n \} \) be any minimum connected edge dominating set of G. Then by Lemma 5.17, H = \( \langle S \rangle \) satisfies the condition in (i). Also since G is \( P(n) - critical \), for each pendant edge xᵢ in H, there is exactly one edge yᵢ ∈ E(G) \ E(H) such that yᵢ is adjacent to xᵢ but not adjacent to any other edge of H.

Suppose there exists a vertex v ∈ V(G) \ V(H) such that v is incident with three yᵢ’s, say y₁, y₂, y₃. Then \( \langle S \setminus \{ x_1, x_3 \} \rangle \cup \{ y_2 \} \) is a total edge dominating set of G with cardinality \( \gamma_c' - 1 \) which is a contradiction. Hence each vertex v in V(G) \ V(H) is incident with at most two yᵢ’s. Thus (ii) holds.

Now let n ≥ 5. Suppose H has a vertex u of degree at least 3 and there exists a vertex w in V(G) \ V(H) incident with two yᵢ’s, say y₁, y₂. If y₁ = u₁w and d(u₁, u₂) = 2 then \( \langle S \setminus \{ x_2, x_3 \} \rangle \cup \{ y_1 \} \) where x₃ is an edge of H adjacent to x₁, is a total edge dominating set of G of cardinality \( \gamma_c' - 1 \) which is a contradiction. Hence (iv) holds.

Now let H₁ be the subgraph induced by xᵢ’s and yᵢ’s. Since \( \gamma_e' (H_1) = \gamma_c' (H_1) = n \) and G is \( P(n) - critical \), it follows that G = H₁.

Conversely, suppose G satisfies the conditions (i), (ii), (iii) and (iv) of the hypothesis. Then E(H) is a minimum connected edge dominating set as well as minimum total edge dominating set of G so that \( \gamma_c' (G) = \gamma_e' (G) \).

**Theorem 2.3** Let G be a connected graph. Then \( \gamma_e' = \gamma_c' = n \geq 4 \) if and only if it satisfies the following conditions.

(i) G contains a \( P(n) - critical \) subgraph H.

(ii) Every edge e ∈ E(G) \ E(H) has at least one end in V(H₁) where H₁ is the subgraph of H induced by a minimum connected edge dominating set of H.

(iii) Suppose H₁, P₃ = (u₁, u₂, u₃, u₄, u₅). If uᵢuⱼ ∈ E(G) where i + 1 < j ≤ 5 and 1 ≤ i ≤ 3, then for each k with i < k < j, there exists an edge of E(G) \ E(H) incident with uᵢ.

(iv) Suppose H₁ = K₁,₄. If the subgraph of G induced by the pendant vertices of H₁ contains a P₄, then there exists an edge of E(G) \ E(H) incident with the center of H₁.

(v) Suppose H₁ is

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    v₁ ---- v₂
   /    /  \
v₄----v₃  v₅
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If exactly one of \( v₁, v₄, v₁v₄ \) ∈ E(G), then there exists an edge of E(G) \ E(H) incident with \( v₂ \) and if both \( v₁v₄, v₁v₅ \) ∈ E(G), then there exists an edge \( e₂, e₃ \) in E(G) \ E(H) incident with \( v₂, v₃ \) respectively. If \( v₂v₄, v₂v₅ \) ∈ E(G) or \( v₂v₅, v₄v₅ \) ∈ E(G) then there exists an edge of E(G) \ E(H) incident with \( v₃ \).
Suppose $n \geq 5$ and let $u$ be the unique vertex of degree $\geq 3$ in $H_1$. If there exists an edge in $E(G) \setminus E(H)$ joining two vertices of degree 2 in $H_1$, then $\deg_{H_1} u = 3$. If two pendent vertices of $H_1$ are joined by an edge in $E(G) \setminus E(H)$, then they are adjacent to $u$ and all such edges are independent. Also there exists no edge in $E(G) \setminus E(H)$ joining a pendent vertex of $H_1$ and a vertex of degree 2 in $H_1$.

**Proof:** Suppose $\gamma'_c = n (\geq 4)$. Let $S = \{x_1, x_2, \ldots, x_n\}$ be any minimum connected edge dominating set of $G$. By Lemma 5.17, $H_1 = (S) = P_5 = \{u_1, u_2, u_3, u_4, u_5\}$ or a tree with exactly one vertex $v$ of degree $\geq 3$ and the distance of every pendent vertex of $H_1$ from $u$ is either 1 or 2. Now as in Lemma 5.19, for each pendent edge $x_i$ of $H_1$, we choose an edge $y_i$ such that $y_i$ is adjacent to $x_i$ but not to any other edge of $S$ and the subgraph $H$ induced by $x_i$’s and $y_i$’s is $P^{(n)}(n - \text{critical})$. Hence (i) holds.

Now since $S$ is a minimum connected edge dominating set of $G$, (ii) holds. We consider the following cases.

**Case (i) $n = 4$**

In this case $H_1$ is either $P_5$, $K_{1,4}$ or $K_2 + K_3$ where $u$ is a pendent vertex.

Suppose $H_1 = P_5$. If $i, j, k$ are positive integers with $1 \leq i < k < j \leq 5$, $u_iu_j \in E(G)$ and no edge of $E(G) \setminus E(H)$ is incident with $u_k$, then $\left( E(H_1) \cup \{u_i, u_j\} \right) \setminus \{u_k, u_{k+1}\}$ is a total edge dominating set of $G$ with cardinality 3, which is a contradiction. Hence $G$ satisfies (iii).

Suppose $H_1 = K_{1,4}$ and the subgraph of $G$ induced by the pendent vertices of $H_1$ contains a $P_4 = (u_1, u_2, u_3, u_4)$. If no edge of $E(G) \setminus E(H)$ is incident with the center of $H_1$, then $\{u_1u_2, u_2u_3, u_3u_4\}$ is a total edge dominating set of $G$ with cardinality 3, which is a contradiction. Thus $G$ satisfies (iv).

**Case (ii) $n \geq 5$.**

Let $u$ be the unique vertex of degree $\geq 3$ in $H_1$. Suppose there exists an edge $u_1u_2$ in $E(G) \setminus E(H)$ where $u_1$, $u_2$ are vertices of degree 2 in $H_1$. If $\deg_{H_1} u \geq 4$, then $\left( S \setminus \{u_1, u_2\} \right) \cup \{u_1, u_2\}$ is a total edge dominating set of $G$ so that $\gamma'_c \leq n - 1 < n$, which is a contradiction. Hence $\deg_{H_1} u = 3$.

If there exists an edge $e = vw$ of $E(G) \setminus E(H)$ joining a pendent vertex $v$ of $H_1$ and a vertex $w$ of degree 2 in $H_1$, then $\left( S \setminus \{v, w\} \right) \cup \{v, w\}$ is a total edge dominating set of $G$ of cardinality $n-1$, which is a contradiction.

Now let $v$ and $w$ be pendent vertices of $H_1$ and $vw \in E(G) \setminus E(H)$. $\deg_{H_1} (v, w) = \deg_{H_1} (w, u) = 2$, then $\left( S \setminus \{v, w\} \right) \cup \{v, w\}$ is a total edge dominating set of $G$ of cardinality $n-1$ where $v$, $w$ are the vertices of $H_1$ adjacent to $v$, $w$ respectively. If
\( \deg_{H_1}(u, v) = 2 \) and \( \deg_{H_1}(u, w) = 1 \) then \( (S \setminus \{ u_1, u_2 \}) \cup \{ v, w \} \) is a total edge dominating set of \( G \) of cardinality \( n - 1 \). Hence \( v \) and \( w \) are adjacent to \( u \) in \( H_1 \).

Now if \( u_1, u_2, u_3 \) are pendent vertices of \( H_1 \) and \( u_1u_2, u_2u_3 \in E(G) \setminus E(H) \), then 
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(S \setminus \{ uu_1, uu_2, uu_3 \}) \cup \{ u_1u_2, u_2u_3 \}
\]
is a total edge dominating set of \( G \), which is a contradiction. Thus the set of all edges of \( E(G) \setminus E(H) \) joining pendent vertices of \( H_1 \) is an edge independent set of \( G \). Hence \( G \) satisfies the condition (vi).

Conversely suppose that \( G \) satisfies the six conditions of the hypothesis. Then \( E(H_1) \) is a minimum connected edge dominating set of \( G \) and also a minimum total edge dominating set of \( G \) so that \( \gamma'_c = n (\geq 4) \).

References