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Equality of Connected Edge Domination and Total Edge Domination in Graphs

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Abstract: Let G be a (p,q) –graph with connected edge domination number γ'_c and total edge domination number γ'_t. In this paper we investigate the structure of graphs in which some of the edge domination parameters are equal. We characterize graphs for which $\gamma'_c = \gamma'_t$.

Key words: Edge domination number, Connected edge domination number and Total edge domination number.

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1. Introduction

By a graph G = (V,E) we mean a finite undirected graph without loops or multiple edges. Terms not defined here are used in the sense of Harary [1] The concept of edge domination was introduced by Mitchell and Hedetniemi. A subset X of E is called an edge dominating set of G if every edge not in X is adjacent to some edge in X. The edge domination number $\gamma'(G)$ (or γ' for short) of G is the minimum cardinality taken over all edge dominating sets of G. An edge dominating set X of G is called a total edge dominating set of G if the induced subgraph $\langle X \rangle$ has no isolated edges. The total edge domination number $\gamma'_t(G)$(or γ'_t for short) of G is the minimum cardinality taken over all total edge dominating sets of G. An edge dominating set X of is called a connected edge dominating set of G if the induced subgraph $\langle X \rangle$ is connected. The connected edge domination number $\gamma'_c(G)$(or γ'_c for short) of G is the minimum cardinality taken over all cfonnected edge dominating sets of G.

Allan and Laskar [2] proved that for any $K_{1,3} -$ free graph, the domination number and independent domination number are equal. Topp and Volkmann [3] generalized the result of Allan and Laskar and constructed new classes of graphs with equal domination and independent domination number. Harary and Livingston [4] characterized caterpillars with equal domination and independent domination number. In [5] they gave the characterization of trees with equal domination and independent domination number. Payan and Xuong [6] proved that for any graph G on 9 vertices, $\gamma = \overline{\gamma = 3}$ if and only if $G = K_3 \times K_3$. Arumugam and Paulraj Joseph [7] studied the class of graphs for which connected domination number and domination number are equal. In this paper we initiate a study of graphs in which some of the edge domination parameters are equal. We characterize graphs for which $\gamma'_c = \gamma'_t$. For this we need the following Theorem.

Theorem 1.1 [8] Let P denote the property that $\gamma' = \gamma'_c = n.$ A connected graph G is P – critical if and only if G is isomorphic to $S(K_{1,n})$

2. Main Results

We observe that for any connected graph G with $\gamma'_c = 2$ or 3, $\gamma'_c = \gamma'_t$. we now proceed to characterize connected graphs with $\gamma'_c = \gamma'_t = n \geq 4.$

Definition 2.1: Let $P^{(n)}$ denote the property that $\gamma'_t = \gamma'_c = n.$ A connected graph G is $P^{(n)} - critical$ if G satisfies $P^{(n)}$ but no proper connected sub graph H of G satisfies $P^{(n)}$.

Lemma 2.2 For $n \geq 4$, a connected subgraph G is $P^{(n)} - critical$ if and only if G satisfies the following conditions.

(i) G contains a subgraph H which is either P_5 or a tree with at least 4 edges, exactly one vertex u of degree at least 3 and all its pendent vertices are at distance 1 or 2 from u.

(ii) For each pendent edge x_i of H, there is exactly one edge y_i of G such that y_i is adjacent to x_i but not adjacent to any other edge of H.

(iii) Every vertex $v \in V(G) \setminus V(H)$ is incident with at most two y_i's.

(iv) For $n \geq 5$, if H has a vertex of degree at least 3 and there exists a vertex w in $V(G) \setminus V(H)$ incident with two y_i's, say $y_1 = u_1w$ and $y_2 = u_2w$, then u_1 and u_2 are adjacent to u.

(v) The subgraph induced by x_i's and y_i's is G.

Proof: Suppose G is $P^{(n)}-critical$. Then $\gamma'_t(G) = \gamma'_c(G) = n\,(\geq 4)$. Let $S = \{x_i \setminus 1 \leq i \leq n\}$ be any minimum connected edge dominating set of G. Then by Lemma 5.17, H = $\langle S \rangle$ satisfies the condition in (i). Also since G is $P^{(n)}-critical$, for each pendent edge x_i in H, there is exactly one edge $y_i \in E(G) \setminus E(H)$ such that y_i is adjacent to x_i but not adjacent to any other edge of H.

Suppose there exists a vertex $v \in V(G) \setminus V(H)$ such that v is incident with three y_i's, say y_1, y_2, y_3. Then $(S \setminus \{x_1, x_3\}) \cup \{y_2\}$ is a total edge dominating set of G with cardinality $\gamma'_c - 1$ which is a contradiction. Hence each vertex v in $V(G) \setminus V(H)$ is incident with at two y_i's. Thus (ii) holds.

Now let $n \geq 5$. Suppose H has a vertex u of degree at least 3 and there exists a vertex w in $V(G) \setminus V(H)$ incident with two y_i's, say y_1, y_2. If $y_1 = u_1w$ and $d(u_1, u) = 2$ then $(S \setminus \{x_2, x_3\}) \cup \{y_1\}$ where x_3 is an edge of H adjacent to x_1, is a total edge dominating set of G of cardinality $\gamma'_c - 1$ which is a contradiction. Hence (iv) holds.

Now let H_1 be the subgraph induced by x_i's and y_i's. Since $\gamma'_t(H_1) = \gamma'_c(H_1) = n$ and G is $P^{(n)}-critical$, it follows that $G = H_1$.

Conversely, suppose G satisfies the conditions (i), (ii), (iii) and (iv) of the hypothesis. Then E(H) is a minimum connected edge dominating set as well as minimum total edge dominating set of G so that $\gamma'_c(G) = \gamma'_t(G)$.

Theorem 2.3 Let G be a connected graph. Then $\gamma'_t = \gamma'_c = n\,(\geq 4)$ if and only if it satisfies the following conditions.

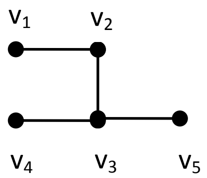
(i) G contains a $P^{(n)}-critical$ subgraph H.

(ii) Every edge $e \in E(G) \setminus E(H)$ has at least one end in $V(H_1)$ where H_1 is the subgraph of H induced by a minimum connected edge dominating set of H.

(iii) Suppose $H_1 = P_5 = (u_1, u_2, u_3, u_4, u_5)$. If $u_iu_j \in E(G)$ where $i + 1 < j \leq 5$ and $1 \leq i \leq 3$, then for each k with $i < k < j$, there exists an edge of $E(G) \setminus E(H)$ incident with u_k.

(iv) Suppose $H_1 = K_{1,4}$. If the subgraph of G induced by the pendent vertices of H_1 contains a P_4, then there exists an edge of $E(G) \setminus E(H)$ incident with the center of H_1.

(v) Suppose H_1 is



If exactly one of v_1v_4, $v_1v_5 \in E(G)$, then there exists an edge of $E(G) \setminus E(H)$ incident with v_2 and if both v_1v_4, $v_1v_5 \in E(G)$, then there exists an edges e_2, e_3 in $E(G) \setminus E(H)$ incident with v_2, v_3 respectively. If v_2v_4, $v_4v_5 \in E(G)$ or v_2v_5, $v_4v_5 \in E(G)$ then there exists an edge of $E(G) \setminus E(H)$ incident with v_3.

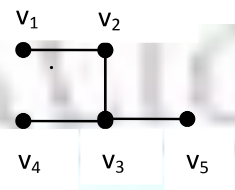
(vi) Suppose $n \geq 5$ and let u be the unique vertex of degree ≥ 3 in H_1. If there exists an edge in $E(G) \setminus E(H)$ joining two vertices of degree 2 in H_1, then $\deg_{H_1} u = 3$. If two pendent vertices of H_1 are joined by an edge in $E(G) \setminus E(H)$, then they are adjacent to u and all such edges are independent. Also there exists no edge in $E(G) \setminus E(H)$ joining a pendent vertex of H_1 and a vertex of degree 2 in H_1.

Proof: Suppose $\gamma'_t = \gamma'_c = n\,(\geq 4)$. Let $S = \{x_1, x_2, ..., x_n\}$ be any minimum connected edge dominating set of G. By Lemma 5.17, $H_1 = \langle S \rangle = P_5 = \{u_1, u_2, u_3, u_4, u_5\}$ or a tree with exactly one vertex u of degree ≥ 3 and the distance of every pendent vertex of H_1 from u is either 1 or 2. Now as in Lemma 5.19, for each pendent edge x_i of H_1, we can choose an edge y_i such that y_i is adjacent to x_i but not to any other edge of S and the subgraph H induced by x_i's and y_i's is $P^{(n)} - critical$. Hence (i) holds.

Now since S is a minimum connected edge dominating set of G, (ii) holds. We consider the following cases.

Case (i) n = 4

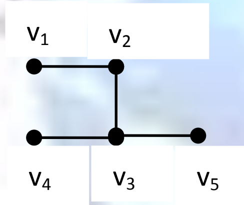
In this case H_1 is either P_5, $K_{1,4}$ or



Suppose $H_1 = P_5$. If i, j, k are positive integers with $1 \leq i < k < j \leq 5$, $u_iu_j \in E(G)$ and no edge of $E(G) \setminus E(H)$ is incident with u_k, then $(E(H_1) \cup \{u_iu_j\}) \setminus \{u_{k-1}u_k, u_ku_{k+1}\}$ is a total edge dominating set of G with cardinality 3, which is a contradiction. Hence G satisfies (iii).

Suppose $H_1 = K_{1,4}$ and the subgraph of G induced by the pendent vertices of H_1 contains a $P_4 = (u_1, u_2, u_3, u_4)$. If no edge of $E(G) \setminus E(H)$ is incident with the center of H_1, then $\{u_1u_2, u_2u_3, u_3u_4\}$ is a total edge dominating set of G with cardinality 3, which is a contradiction. Thus G satisfies (iv).

Suppose H_1 =



Let us assume that exactly one of v_1v_4, v_1v_5, say v_1v_4 is in E(G). If no edge of $E(G) \setminus E(H)$ is incident with v_2, then $\{v_1v_4, v_3v_5, v_3v_4\}$ is a total edge dominating set of G. If both v_1v_4, $v_1v_5 \in E(G)$ and no edge of $E(G) \setminus E(H)$ is incident with at least one of v_2, v_3, say v_3, then $\{v_1v_2, v_1v_4, v_1v_5\}$ is a total edge dominating set of G. If v_2v_4, v_4v_5 (v_2v_5, v_4v_5) $\in E(G)$ and no edge of $E(G) \setminus E(H)$ is incident with v_3, then $\{v_2v_4, v_4v_5, v_2v_1\}$ $(\{v_2v_5, v_5v_4, v_2v_1\})$ is a total edge dominating set of G. Hence it follows that G satisfies (v).

Case (ii) $n \geq 5$.

Let u be the unique vertex of degree ≥ 3 in H_1. Suppose there exists an edge u_1u_2 in $E(G) \setminus E(H)$ where u_1, u_2 are vertices of degree 2 in H_1. If $\deg_{H_1} u \geq 4$, then $(S \setminus \{uu_1, uu_2\}) \cup \{u_1u_2\}$ is a total edge dominating set of G so that $\gamma'_t \leq n - 1 < n$, which is a contradiction. Hence $\deg_{H_1} u = 3$.

If there exists an edge e = vw of $E(G) \setminus E(H)$ joining a pendent vertex v of H_1 and a vertex w of degree 2 in H_1, then $(S \setminus \{uv, uw\}) \cup \{uw\}$ is a total edge dominating set of G of cardinality n-1, which is a contradiction.

Now let v and w be pendent vertices of H_1 and $vw \in E(G) \setminus E(H)$. $\deg_{H_1}(v, w) = \deg_{H_1}(w, u) = 2$, then $(S \setminus \{v_1v, uw_1\}) \cup \{vw\}$ is a total edge dominating set of G of cardinality n-1 where v_1, w_1 are the vertices of H_1 adjacent to v, w respectively. If

$\deg_{H_1}(u,v)=2$ and $\deg_{H_1}(u,w)=1$ then $(S\setminus\{uv_1,uw\})\cup\{vw\}$ is a total edge dominating set of G of cardinality n-1. Hence v and w are adjacent to u in H_1.

Now if u_1, u_2, u_3 are pendent vertices of H_1 and u_1u_2, $u_2u_3 \in$ E(G) \ E(H), then $(S\setminus\{uu_1,uu_2,uu_3\})\cup\{u_1u_2,u_2u_3\}$ is a total edge dominating set of G, which is a contradiction. Thus the set of all edges of E(G) \ E(H) joining pendent vertices of H_1 is an edge independent set of G. Hence G satisfies the condition (vi).

Conversely suppose that G satisfies the six conditions of the hypothesis. Then E(H_1) is a minimum connected edge dominating set of G and also a minimum total edge dominating set of G so that $\gamma'_t=\gamma'_c=n\,(\geq 4)$.

References

[1]. F. Harary, Graph Theory, Addison Wesley Reading Mass. (1972).
[2]. R.B. Allan and R.Laskar, On domination and independent domination of a graph, Discrete Math. 23(1978), 73 – 76.
[3]. J. Topp and L. Volkman, On graphs with equal domination and independent domination numbers, Discrete Math. 96 (1991) 75 – 80.
[4]. F. Harary and M. Livingston, Caterpillars with equal domination and independent domination numbers , Recent studies in Graph Theory, ed. V.R.Kulli, Vishwa International Publications (1989) 149 – 154.
[5]. F. Harary and M. Livingston, Characterization of trees with equal domination and independent domination numbers, Congr. Numer. 55 (1986), 121 – 150.
[6]. C.Payan and N.H. Xuong, Domination balanced graphs, Journal of Graph Theory 6 (1) (1982) 23 – 32.
[7]. S.Arumugam and J. Paulraj Joseph, On graphs with equal domination and connected domination numbers, Discrete Math. (To appear).
[8]. S.Velammal and S.Arumugam, Equality of edge domination and connected edge domination in graphs, International Journal of Advanced and innovative Research, Vol.2, Issue 2 (2013) 218-222.