# Toric Varieties and the Implementation of the Bèzout Resultant Block Matrix

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**Abstract: The construction of the Bézout matrix in the hybrid resultant formulation involves theories from algebraic geometry. The underlying theory on toric varieties has very nice properties such as the properties of fan (or cones), homogeneous coordinate ring, normality, and Zariski closure are related to the structure of the lattice polytopes in R. This paper presents the application of these properties in the construction and implementation of the Bézout resultant block matrix for unmixed bivariate polynomial systems. The construction reveals a complete combinatorial description for computing the entries of the matrix.**

**Keywords: Algebraic geometry, Bézout matrix, combinatorics, toric varieties.**

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## Introduction

Toric varieties are algebraic varieties defined by combinatorial data, and there is a wonderful interplay between algebra, combinatorics and geometry involved in their study. Many of the key concepts of abstract algebraic geometry have very concrete interpretations in the toric case [1], for example, constructing a variety by gluing affine pieces. Thus the theory on toric varieties is an important area of research in algebraic geometry which features in many applications. Basics about toric varieties can be found in [2, 3].

In realizing this wonderful theory, this paper presents its applications in the construction of the Bézout block of the hybrid Sylvester-Bézout resultant matrix which was introduced in [4]. Considering an unmixed polynomial system in two variables in the following form:

$$f_i := \sum_{\alpha \in A} C_{i\alpha} x^{\alpha} \ , \ i = 1, \ldots, n+1 \tag{1}$$

Let $f_1, \ldots, f_{n+1} \in \mathbf{C}[x_1, \ldots, x_n]$ be $n + 1$ polynomials in $n$ variables with the same Newton polytopes $Q \subset \mathbf{R}^n$ which is known as a convex hull of the supports, $Q = \text{conv}(A)$ and $A = Q \cap \mathbf{Z}^n$, $A = \{\alpha_1, \ldots, \alpha_N\}$ is a set of the supports of the polynomials affinely span $\mathbf{Z}^n$. Note that $\mathbf{R}$ is the set of real numbers, $\mathbf{Z}$ is the set of integer numbers and both sets are in the affine space $\mathbf{C}^n$.

This works continues from [4, 5] and intensively concentrates on the formulation of cubic polynomials in the coefficients of the input polynomials which then corresponds to the entries of the Bézout block. The resultant matrix approach provides a very practical technique for eliminating variables in the polynomial equations. Bézout matrix is not trivial because its entries are compact form and presented in brackets. The brackets correspond to the determinants of $(n+1) \times (n+1)$ matrices. The entries of Bézout block or submatrix are linear forms in the bracket variables, $[uvw]$ defined as follows:

$$[uvw] = \begin{vmatrix} C_{1u} & C_{1v} & C_{1w} \\ C_{2u} & C_{2v} & C_{2w} \\ C_{3u} & C_{3v} & C_{3w} \end{vmatrix} \tag{2}$$

Where $C_{iu}$, $C_{jv}$ and $C_{kw}$ are the coefficients of $x^{\alpha} \in f_i$. The toric variety, which is denoted by $\mathbf{X}_A$, is the dimension $n$ variety defined as the Zariski closure of the following set in $\mathbf{P}^{N-1}$:

$$\mathbf{X}_A^0 = \left\{ \left( x^{\alpha_1} : \cdots : x^{\alpha_N} \right) : x = (x_1, \ldots, x_n) \in \left( \mathbf{C}^* \right)^n \right\}$$

Therefore the system of polynomials represented in (1) is described based on the following function:

$$\Phi_A : \left( \mathbf{C}^* \right)^n \rightarrow \mathbf{P}^{N-1} \tag{3}$$

defined by $\Phi_A(x_1,\ldots,x_n)=\left(x^{\alpha_1},\ldots,x^{\alpha_N}\right)$, where $\left(x^{\alpha_1},\ldots,x^{\alpha_N}\right)$ is never zero because $x_i \in \mathbf{C}^*$ for all $i$. Thus $\Phi_A$ is defined on all of $\left(\mathbf{C}^*\right)^n$, that is $(x_1,\ldots,x_n)$ in $\left(\mathbf{C}^*\right)^n$. This operation gives an element $\Phi_A(x_1,\ldots,x_n)$ in $\mathbf{P}^{N-1}$. This function shows the generalization of the affine space and the projective space to toric space. The main fact is that the equations of polynomials extend naturally to $X_A \subset \mathbf{P}^{N-1}$ which satisfies (3). $X_A$ is the notation for the associated toric variety with respect to the supports determined by a fan $\sum$, in $N \cong \mathbf{Z}^n$ and cones $\sigma$, which forms the fan.

The main contribution of this paper is to provide a detail computation and computer implementation of the Bézout block construction in the Sylvester-Bézout resultant matrix formulation given in [4]. In this formulation, the bracket variables satisfy a certain set of rules, known as the 5-Rule. This formulation involves a combinatorial approach which is applied to the structure of the Newton polytope of the associated polynomial system (1). In this paper, the basis of the Bézout block matrix construction is illustrated. In the implementation, all supports must consecutively satisfy the inequalities given in this 5-Rule. In addition, two new conditions are introduced so that the bracket variables are well defined entries of the Bézout block which thus satisfy the appropriate dimension of the matrix specified by the appropriate linear map involved in the construction.

The first phase of the construction is to homogenize the supports of the polynomial equations systematically, from the affine space to the projective space by using homogeneous coordinate ring and the properties of the toric variety. The key ingredient will again be the structure of the Newton polytopes $Q = \text{conv}(A)$ associated with the equations. In particular, the facets and inward normals of $Q$ are used. Interested reader may refer to [1-3, and 6] for further details on these structures.

In the following section, some underlying theoretical concepts on toric varieties are described.

## Toric Varieties

A simple example of a variety is the **affine varieties** by considering $V = \mathbf{V}(f_1,\ldots,f_{n+1}) \subset \mathbf{C}^n$ which can be defined by the polynomial equations $f_1,\ldots,f_{n+1} = 0$. Another example is the **projective varieties**, $V = \mathbf{V}(F_1,\ldots,F_{n+1}) \subset \mathbf{P}^n$ defined by the homogeneous equations $F_1,\ldots,F_{n+1} = 0$. Thus toric varieties are generalizations of both the affine space $\mathbf{C}^n$ and the projective space $\mathbf{P}^n$. In particular, the resultant vanishes if and only if the polynomial equations have a solution in a toric variety; hence it is regarded as a very important set of solutions in algebraic geometry. Toric varieties can be viewed in the theory of algebraic groups and also in the theory of polytopes. This paper only discusses toric varieties in the polytope's setting. The action of gluing affine pieces leads to the concept of a fan.

A. Cones and Fan of a Polytope

Cones are geometric objects that are used to define the affine varieties. A fan is defined as a finite collection of cones in a Newton polytope that satisfies a natural notion of gluing affine varieties. Fans are traditionally used in the description of toric varieties, $\mathbf{X}_A$ with a given very ample line bundle determined by the Newton polytope $Q$. The fan in the toric variety serves as a tool in formulating inequalities in the 5-Rule. All explanations follow from [1, 6] . The fan $\sum$ has the following three properties:

- Each $\sigma \in \sum$ is a strongly convex rational polyhedral cone.
- If $\sigma \in \sum$ and $\tau$ is a face of $\sigma$, then $\tau \in \sum$
- If $\sigma, \tau \in \sum$ then $\sigma \cap \tau$ is a face of each.

One dimensional cones form the set of one dimensional faces of cones, $\tau \in \sum_{(1)}$ then $\sigma_{(1)} = \{ \tau \in \sum_{(1)} : \tau \subset \sigma \}$ and its cardinality is $s$. Each facet corresponds to an irreducible $T$-invariant Weil divisor $D_\tau$ in $\mathbf{X}$. The free abelian group of $T$-invariant Weil divisor on $\mathbf{X}$ will be denoted $\mathbf{Z}^s$. Thus an element $D_i$ in $\mathbf{Z}^s$ is a sum $D = \sum_{i=1}^{s} a_i D_i$. For a projective toric variety $\sum$ is a normal fan of a rational convex polytope in $\mathbf{R}^n$. In general, every normal toric variety is determined by a fan. Simple examples of fans when $n = 1$ and $n = 2$ can be viewed in [1].

When n = 1, the fan is just a straight line that has been glued by three cones to give $\mathbf{P}^1$: one generated by $e_1(\sigma_1 = [0, \infty))$, another generated by $e_2(\sigma_2 = (-\infty, 0])$ and $\{0\}$. The cones $\sigma_1$ and $\sigma_2$ give affine varieties $U_{\sigma_1}$ with coordinate ring $\mathbf{C}[\mathbf{X}]$ and $U_{\sigma_2}$ with coordinate ring $\mathbf{C}[\mathbf{X}^{-1}]$, they are isomorphic, while $\{0\}$ is given by both $\mathbf{X}$ and $\mathbf{X}^{-1}$.

Besides, when $n = 2$, the fan generates a toric variety which is isomorphic to $\mathbf{P}^2$. Let $(\mathbf{C}^*)^2 \subset \mathbf{P}^2$. Each cone $\sigma \in \sum$ gives an affine toric variety $U_\sigma \cong \mathbf{C}^2$ glued together in the usual way to give $\mathbf{P}^2$. If $\tau$ is a face of $\sigma$, then $U_\tau$ can be regarded as a Zariski open subset of $U_\sigma$. This leads to the following definition.

**Definition 1** [1] *Given a fan $\sum$ in $\mathbf{R}^n$, $X_{\sum}$ is the variety obtained from the affine varieties $U_\sigma$, $\sigma \in \sum$, by gluing together $U_\sigma$ and $U_\tau$ along their common open subset $U_{\sigma \cap \tau}$ for all $\sigma, \tau \in \sum$.*

To this end it can be realized that the fan $\sum$ has a close relation to the structure of the toric variety $\mathbf{X}_A$. In the next section the toric variety of the polytopes used in this paper shall be illustrated.

B.   Toric Variety of a Polytope

Let $Q = \text{conv}(A)$, a finite subset of $\mathbf{Z}^n$, be the lattice polytope of dimension $n$ and let $A = \{ \alpha_1,\ldots,\alpha_N \}$. The toric variety $\mathbf{X}_A$ = $\overline{\Phi_A\left(\left(\mathbf{C}^*\right)^n\right)} \subset \mathbf{P}^{N-1}$ is the dimension $n$ variety defined as the *Zariski closure* of the set $\left(x^{\alpha_1},\ldots,x^{\alpha_N}\right)$ that is the image of $\Phi_A$ in (3)  where $\sigma_i$ ranges over the elements of $A$ and $\boldsymbol{x} = ( x_1,\ldots,x_n) \in (\mathbf{C}^*)^n$.

Polytopes have special subsets called faces. For example a polytope in $\mathbf{R}^3$ has faces (polygons lying in planes), edges (line segments connecting certain pairs of vertices) and vertices (points) [2]. To define a face of an arbitrary Newton polytope, an affine hyperplane is needed. Let $\eta$ be a nonzero vector in $\mathbf{R}^n$. An affine hyperplane is defined by an equation of the form

$$m \cdot \eta = -a .$$

Therefore for every unmixed system of polynomial equations, there exists an associated Newton polytope $Q$ defined by

$$a_i = -\min_{m \in Q}\langle m \cdot \eta \rangle . \tag{4}$$

Thus, any Newton polytope $Q$ can be defined by its facet inequalities given by

$$Q = \left\{ m \in \mathbf{R}^n \mid \langle m,\eta_i \rangle \geq -a_i, i = 1,\ldots,s \right\} \tag{5}$$

for some integers $a_1,\ldots,a_s$ considered as the data of $Q$. The inner normals $\eta_1,\ldots,\eta_s$ are called rays. The set of data and inner normals are unique for the facet $\tau_i$.

Along this line, the formula for partitioning a fan into three cones is considered as follows, refer [4]:
$R_1 = \{ i : \eta_i = c_1\eta_1 + c_2\eta_2, \text{ with } c_1 \geq 0 \text{ and } c_2 \leq 0 \}$
$R_2 = \{ i : \eta_i = c_1\eta_1 + c_2\eta_2, \text{ with } c_1 \leq 0 \text{ and } c_2 \geq 0 \}$   (6)
$R_3 = \{ i : \eta_i = c_1\eta_1 + c_2\eta_2, \text{ with } c_1 < 0 \text{ and } c_2 < 0 \}$.
Precisely, these are three sets of specific partition of the edges of polytope. This partition is then applied in the formulation of the 5-Rule:
*5-Rule*

$$
\begin{aligned}
&\text{Rule1} : \ \forall\, k \in R_3,\ w_k > a_k \\
&\text{Rule2} : \ \exists\, j \in R_2,\ w_j \leq a_j \\
&\text{Rule3} : \ \forall\, j \in R_2,\ v_j + w_j > a_j \\
&\text{Rule4} : \ \exists\, i \in R_1,\ v_i + w_i \leq a_i \\
&\text{Rule5} : \ \forall\, i \in R_1,\ u_i + v_i + w_i > a_i
\end{aligned}
\tag{7}
$$

The formulation of the 5-Rule involves a combinatorial approach where all the nonzero coefficients in the polynomials are considered. The bracket variables $[uvw]$ in (2) the formulation of the Bèzout block entries must satisfy this 5-Rule and they are also known as the coefficients of the facet variables for the polynomial ring $S$. The brackets are called Plücker coordinates by [3, 5].

Each facet $F = \{\tau_1,\ldots,\tau_s\}$ of $Q$ is a polytope of dimension less than $\dim Q$. Vertices are faces of dimension 0 and edges are faces of dimension 1. So if $Q$ has dimension $n$, then facets are faces of dimension $n - 1$. For all facets $F$ containing $\tau_i$, the set of cones $\sigma_\tau$ are generated by inner normals. Then $\sum_Q = \{ \sigma_\tau \mid \tau$ is a face of $Q \}$ is the normal fan of $Q$. This gives a toric variety denoted as $\mathbf{X}_A$. Each vertex of the cones is spanned by the inner normal $\eta_i$ corresponding to facets which are incident to the vertex. The characterization of the normal fan is stated in the following theorem.

**Theorem 1** [1] *The normal toric variety of a fan $\sum \in \mathbf{R}^n$ is projective if and only if $\sum$ is the normal fan of an n-dimensional lattice polytope in $\mathbf{R}^n$.*

Besides a complete characterization of the polytope $Q$ in terms of the rays in its normal fan, Weil divisors are described in the following proposition.

**Proposition 1** [4, 5] *The $\eta_i$ are in one to one correspondence with the T-invariant prime Weil divisors on $\mathbf{X}_A$. Let $D_i$ denote the divisor corresponding to $\eta_i$.*

A divisor $D = \sum a_i D_i$ determines a convex polytope of (5). The divisor is best described as a line bundle and a global section of that line bundle. Any line bundle on the Grassmannian is a degree of the hyperplane bundle. Thus the divisor can be represented simply as a polynomial in the Plücker coordinates [3]. Precisely, the following proposition describes the space of global sections of a divisor.

**Proposition 2** [4] *The space of global sections of a divisor $D = \sum a_i D_i$ is the space spanned by the lattice points in the polytopes (5), that is $H^0(\mathbf{X}_A, O(D)) \cong \mathbf{C}\{Q \cap \mathbf{Z}^n\}$.*

Now the polynomials are in $n+1$ hyperplane sections. Thus the system defines a codimension $n+1$ plane and the set of all codimension $n+1$ planes meeting $\mathbf{X}_A$ defines a hypersurface in the Grasmannian $G(n+1, N)$. The equation of this

hypersurface is called the *Chow form* of **X** [8]. The equation is the determinant of the complexes as described in the next section.

On the other hand, a general notion of a toric variety to the following general class of algebraic varieties is defined as follows.

**Definition 2** [3] *A toric variety is an irreducible complex algebraic variety **X** equipped with an action of the algebraic torus* $(C^*)^n$ *having an open dense orbit.*

The variety **X** is the closure of the orbit of the point (1:1: … :1) obtained under the action of the torus $(C^*)^n$ on the projective space $P^{N-1}$, so it is a toric variety. The toric variety is equivariantly embedded, that is the action of the torus on $X_A$ extends to the whole $P^{N-1}$ [3]. An action of $(C^*)^n$ on $P^{N-1}$ by projective transformation can always be lifted to a linear action on $C^n$. We denote the point lying on the open orbit of the torus by $\omega = (\omega_1,\dots,\omega_N)$ and some of the $\omega_i$ may equal 0.

Now we are ready to present the homogeneous coordinate ring of the toric variety $X_A$ which was introduced by [6]. In particular, new variables called *facet variables*, $y_1,\dots,y_s$ are introduced. All explanation follows from [6].

### The Complexes and the Homogeneous Coordinate Ring

The complexes presented in [4] are examined for the construction of Bèzout matrix. Besides earlier concepts given in the previous section, the materials relevant in this section is also due to Cox [6].

**Definition 3** [4] *The homogeneous coordinate ring for $X = X_A$ is the polynomial ring $S_X = K[y_1,\dots,y_s]$ where the monomials are graded.*

Let $S$ be the polynomial ring, $S = C[y_1,\dots,y_s]$ with one variable for each ray in the fan $\sum_Q$ determining the toric variety $X_A$. Then by considering the complexes (8) as below, it can be shown that the action of $G$ induces a natural grading on $S$.

$$0 \longrightarrow Z^n \xrightarrow{\Phi} Z^s \xrightarrow{\pi} G \longrightarrow 0 \qquad (8)$$

where $\Phi(m) = (\langle m, \eta_1 \rangle,\dots,\langle m, \eta_s \rangle)$, and $G$ is the cokernel of $\Phi$, so that $G = Z^s / \mathrm{Im}(\Phi) = \{\mathrm{Im}(\Phi) + a : a \in Z^s\}$. $Z^s$ is a free abelian group of $T$-invariant Weil divisor on $X_A$. Thus an element $D \in Z^s$ is a sum $D = \sum a_i D_i$, $D_i$ denote the divisor corresponding to $\eta_i$. Furthermore, the map $Z^n \to Z^s$ is defined by $m \mapsto D_m = \sum_i \langle m, \eta_i \rangle D_i$ . The polynomial ring $S_X$ is graded by the elements of $G$. Suppose $g \in G$, then deg $y^\alpha = \pi(\alpha) = g$ and have bases corresponding to lattice points in the Newton polytopes $Q$. Here $y^\alpha \in S$ and $\pi(\alpha) \in G$. There exists a one to one mapping between the monomials in $S_X$ and the lattice points in $Q$. The following proposition is the characterization of the interior points of $Q$.

**Proposition 3** [4] *The interior lattice points of the Newton polytope $Q$ correspond precisely to the monomials of $S_Q$ divisible by $y_1,\dots,y_s$. This corresponds to the graded piece defined by $g - \Phi_i(1,1,\dots,1)$ . This piece is denoted by $S_{int\,Q}$.*

By sequence (8), it follows that two monomials $\prod_i y_i^{\alpha_i}$ and $\prod_i y_i^{a_i}$ in $S$ have the same degree if and only if there exists $\alpha \in Z^n$ such that $\alpha_i = \langle \alpha, \eta_i \rangle + a_i$ for $i = 1,\dots,s$. As noted earlier, the monomials in $\Phi(a)$, $a = (a_1,\dots,a_s)$ graded piece of S are in one to one correspondence with the lattice points in $Q$ denoted $S_Q$. Therefore by Proposition 2, $H^0(X_A, O(D)) \cong S_Q$, where $O(D)$ is the coherent sheaf on **X** determined by the Weil divisor $D$.

There is a similar characterization of the interior lattice points of $Q$. Let $\omega_0 = (1,1,\dots, 1) \in Z^s$ the monomials in the $\pi(a - \omega_0)$ graded piece of $S$ are in one to one correspondence with the interior lattice points of $Q$ denoted $S_{int(Q)}$. Therefore, by Proposition 2, there exists the isomorphism $H^0(X_A, O(D - \sum_{i=1}^{s} D_i )) \cong S_{int(Q)}$.

The complexes (8) also provide a formulation for the homogenization of $\alpha$ in $Q$ written in the form of $s$ tuples $(\alpha_1, \dots , \alpha_s)$. The formulation is defined in Definition 4. The homogenization of $\alpha$ is to generate the degree for each facet variable in $S$ respectively.

**Definition 4** [4] *The $Q$-homogenization map $\Phi_Q : Z^n \to Z^s$ is defined by $\Phi_Q(\alpha)_i = \langle \alpha, \eta_i \rangle + a_i$ for $i=1,\dots,s$.*

Another approach for homogenization is by using a unique homogenizing variable $x_{i,\,l+1}$ to the polynomial equations with multihomogeneous representation as shown in [9,5, 10]. However in the formulation of the Bézout matrix in this work, the technique of homogeneous coordinate ring is used. Note that, the rays $\eta_i$ in Definition 4 can have negative coordinates, so

logically negative exponents can appear in the image of $\Phi_Q$. Nevertheless, Lemma 3.10 in [8] proved that $y^{\alpha(m)}$ has no negative exponents, so that it is in the homogeneous coordinate ring.

This homogenization of $x^\alpha$ determines the divisor, $D = \sum a_i D_i$. The description of (5) shows that the exponents of $y^\alpha$ are all nonnegatives ($\geq 0$), so that $y^\alpha$ is in the homogeneous coordinate ring. The theory of toric varieties in polytope setting is then applied to the construction of the Bézout matrix given in the next section.

### Algorithm for the Construction of the Bézout Matrix

A. **Framework of the Algorithm**

An algorithm for the computation of the entries of Bézout block of the hybrid Sylvester-Bézout matrix is based on Theorem 2 and is divided into three phases: Phase 1:To compute the data for $Q$ and the homogeneous coordinates for every element in the polynomial support. Phase 2: To compute the degree of facet variables and all the possible combinations ($u$, $v$, $w$) that satisfies the 5-Rule. Phase 3: To construct the Bézout Matrix.

**Theorem 2** [4] *The Bézout matrix is the matrix of the linear map* $\Delta_Q : (S_Q)^* \to S_{\text{int}(2Q)}$ *defined as follows:*

$$\Delta_Q\left((y^\alpha)^*\right) = \sum_{(u,v,w)\in F_\alpha \subset A^3} [uvw] y^{u+v+w-\alpha-\omega_0}$$

The procedures involved in Phase 1 and Phase 2 are in the following Listing 1 and Listing 2 respectively. All the procedures are based on the theoretical concepts of toric varieties discussed in the previous sections. Listing 1 gives the functions that compute $a = (a_1,\ldots , a_s)$ and $\alpha = (\alpha_1,\ldots,\alpha_N)$.

**Listing 1**: Computing data of polytope
```
BEGIN
    Initialize the value s
    FOR each m[n] is initialize, where n is less than s
            m[n][0] represent the x-axis
            m[n][1] represent the y-axis
     END FOR
     FOR each vector[n] is initialize, where n is less than s
            vector[n][0] represent the x-axis
            vector[n][1] represent the y-axis   }
      END FOR
    Display the value of m and vector
     FOR each a[n] is calculate, where n is less than s
             a[n]= -((m[n][0]*vector[n][0])+(m[n][1]*vector[n][1]));
     END FOR
     Display a
END
//Homogeneous coordinate ring
BEGIN
    Initialize the value maxN
    FOR each alpha[n] is initialize, where n is less than maxN
            alpha[n][0] represent the x-axis
            alpha[n][1] represent the y-axis
     END FOR
    Display alpha
     FOR each CumSum[n] and sum_alpha[n] is set to 0, where n is less than maxN
             FOR each sum_alpha[n][i] is set to 0, where i is less than s
            END FOR
     END FOR
     FOR each  CumSum[n] and sum_alpha[n] is calculate, where n is less than maxN
            FOR each sum_alpha[n][i] is calculate, where i is less than s
                  sum_alpha[n][i]=(((alpha[n][0]*vector[i][0])+ (alpha[n][1]*vector[i][1]))+a[i])
                  CumSum[n]=CumSum[n] + sum_alpha[n][i]
            END FOR
     END FOR
    Display sum_alpha
```

```
        FOR every CumSum[n], where n is less than (maxN -1)
                IF CumSum[n] is equal to CumSum[n+1]
                        hmd++
                END IF
        END FOR
        IF hmd is equal to (maxN-1)
                hcoord = CumSum[0]
                Display hcoord
        END IF
END
```

**Listing 2** : Computing the inequalities in the 5-Rule.
```
BEGIN
    Begin Rule 1
            FOR all k in R3, which k is less than maxN
                    IF w[k] greater than alpha[k]
                      Set w[k]
                    END IF
                    ELSE
                            Display rule 1 fail for k
                    END ELSE
            END FOR
    End Rule 1
    Begin Rule 2
            FOR some j in R2, which j is less than maxN
                    IF w[j] is smaller than or equal to alpha[j]
                     Set w[j]
                    END IF
                    ELSE
                            Delete w[j]
                            Display rule 2 fail for j
                    END ELSE
            END FOR
    End Rule 2
    Begin Rule 3
            FOR some j in R2, which j is less than maxN
                    IF (v[j]+w[j]) greater than alpha[j]
                            Set v[j]
                    END IF
                    ELSE
                            Display rule3 fail for j
                    END ELSE
            END FOR
    End Rule 3
    Begin Rule 4
            FOR some i in R1, which i is less than maxN
                    IF (v[i]+w[i]) less than or equal to alpha[i]
                            Set v[i]
                    END IF
                    ELSE
                            Delete w[i]
                            Display rule4 fail for i
                    END ELSE
            END FOR
    End Rule 4
    Begin Rule 5
            FOR some i in R1, which i is less than maxN
                    IF (u[i]+v[i]+w[i]) is greater than alpha[i]
                            Set u[i]
                    END IF
```

```
                    ELSE
                              Display rule 5 fail for i
                    END ELSE
            END FOR
      End Rule 5
END
```

Finally, the Bézout matrix is constructed using the output produced in Phase 1 and Phase 2. The procedures involved in Phase 3 are simplified in the following Listing 3.

**Listing 3**: Computing Bézout matrix.
```
BEGIN
      Get array UVW combination
      Set i =1 and count =0
      Loop:
                    IF uvw[i] is greater than uvw[i+1]
                            Bubble sort uvw to ascending order
                            count++
                    END IF
                    ELSE
                            IF i less than maxN
                                  IF count\%2 equals to 0
                                          Negative[i] ==false
                                  END IF
                                  Increase i by 1
                                  Set count to 0
                                  Repeat the loop
                            END IF
                            ELSE
                                  Exit the loop
                            END ELSE
                    END ELSE
                    Display [uvw] up to sign
                    Arrange [uvw] in Bezout matrix
                    Display Bezout matrix
END
```

**B.    Complexity of Algorithms**

Phase 1 consists of three procedures or subroutines: computation of the data in $Q$ which is defined by the equation of the corresponding hyperplane (4), degree of the homogeneous coordinates and test for homogeneous. Two sets of input are required: vertices $m_1,\dots,m_s$ and inner normal $\eta_1,\dots,\eta_s$. The output for the data in $Q$ is in the form of $s$-tuples represented by $a_1,\dots,a_s$.

Suppose there exists $N$ exponent vectors $\alpha$ of the unmixed support of the polynomial equations. Then the computation of the degree vectors for the homogeneous coordinates requires $N$ support lattice points and the $s$-tuples data in $Q$ as input. Computation in this procedure is based on Definition 3. The degree vector of the homogeneous coordinates for each support lattice point can be displayed in matrix form and is numbered $1,2,\dots,N$, which correspond to the rows against the $s$-tuples $\alpha_{i1},\dots,\alpha_{is}$ numbered $1,2,\dots,s$ for every $i \in \{1,2,\dots,N\}$ which correspond to the columns. The following example shows the output for the degree vectors of the homogeneous coordinates.

**Example 1**
Let $A = \{(0,0),(0,1),(1,0),(1,1),(2,1),(1,2)\}$, the degree vector for the homogeneous coordinates are displayed in matrix form as follows,
$\alpha_1=(0,0) : 0\ 0\ 1\ 3\ 1$
$\alpha_2=(1,0) : 1\ 0\ 0\ 2\ 2$
$\alpha_3=(0,1) : 0\ 1\ 2\ 2\ 0$
$\alpha_4=(1,1) : 1\ 1\ 1\ 1\ 1$
$\alpha_5=(2,1) : 2\ 1\ 0\ 0\ 2$
$\alpha_6=(1,2) : 1\ 2\ 2\ 0\ 0$

In Example 1, for every support lattice point $\alpha_i \in \mathbf{Z}^2$, the total degree of the corresponding exponent vector of the homogeneous coordinates is 5, which implies that the polynomial equations represented by the support have solutions in projective space $\mathbf{P}^{N-1}$.

Choosing the correct combination of $(u,v,w)$ for the brackets is based on the 5-Rule (7) where only three elements representing $u$, $v$, and $w$ are selected from each of the $s$-tuple of the homogeneous coordinates. The rules are dependent in ascending order, that is the computation must go through Rule 1 before Rule 2 and so on until Rule 5. For instance, Rule 2 verifies the elements in the set of $w$ that had been derived from Rule 1.

In order to realize these combinations, the Newton polytope is partitioned into three cones using formula (6). Each cone has two rays. In this study, a distinguished cone is picked out by choosing a vertex $p$ having a normal cone spanned by two rays $\{\eta_1, \eta_2\}$ which is incident to $p$. The total number of rays depends on the number of edges the Newton polytope.

Table 1: The Complexity of Procedures in Bézout Matrix Computation

| Procedures | Time Complexity | Space Complexity |
|---|---|---|
| Data of Q | $O(n)$ | $O(n)$ |
| Homogeneous Coordinate | $O(n^2)$ | $O(1)$ |
| Rule 1 | $O(n^2)$ | $O(1)$ |
| Rule 2 | $O(n^2)$ | $O(n)$ |
| Rule 3 | $O(n^3)$ | $O(n)$ |
| Rule 4 | $O(n^3)$ | $O(n)$ |
| Rule 5 | $O(n^4)$ | $O(n)$ |
| Test uvw | $O(n^3)$ | $O(1)$ |
| Compute degree of facet variables | $O(n^2)$ | $O(n)$ |
| Bézout matrix | $O(n^2)$ | $O(1)$ |

The $u_i$, $v_i$, $w_i$ is the $i$th component of these $s$-tuple respectively, so $i$ can take values from 1 to $s$ and corresponds to the elements in $R_1$. Similarly for $j$ and $k$ correspond to the elements in $R_2$ and $R_3$ respectively. For the purpose of effectiveness and efficiency, eliminating duplicate elements in the implementation of the algorithm, two new conditions are introduced:
The bracket variables $(u,v,w)$ is rejected if :
(1) (u = v) or (v = w) or (u = w) and
(2) (u, v) in Rule 5 $\neq$ (u, v) in Rule 4.
By introducing these conditions, the row elements are uniquely defined and the algorithm terminates with the correct matrix dimension. The complexity for each function used in the algorithm is presented in Table 1

**Implementation**

In this section, the computer program is implemented on unmixed bivariate polynomial systems of degree two and three where the results are known.

**Example 2** [10]
Unmixed bivariate polynomial equations of degree two. $f_i = C_{i1} + C_{i2}x + C_{i3}y + C_{i4}xy + C_{i5}x^2 + C_{i6}y^2, \quad i = 1, 2, 3.$

**Example 3** [4]
Unmixed bivariate polynomial equations of degree three. $f_i = C_{i1} + C_{i2}x + C_{i3}y + C_{i4}xy + C_{i5}x^2y + C_{i6}xy^2, \quad i = 1, 2, 3.$

The polynomials share the same Newton polytope of triangle and pentagon for Example 2 and Example 3 respectively. The vertices are sorted in counterclockwise. Table 2 and Table 3 show the input needed for computing the Bézout matrix.

**Table 2.** Input for Example 2

| Vertices, $m$ | inner normals, $\eta$ | partition of fan | $\omega_0$ |
|---|---|---|---|
| (0,0), (1,0), (2,0), (1,1), (0,2),(0,1) | (1,0), (0,1), (0,1), (-1,-1), (-1,-1), (1,0) | $R_1 = \{1,6\}$, $R_2 = \{2,3\}$, $R_3 = \{4,5\}$ | (1,1,1,0,0,1) |

The final result for Example 2 is $6 \times 6$ matrix,

$$\begin{pmatrix} 0 & [124]-[135] & [126] & [156]-[234] & -[145]-[235] & -[236] \\ 0 & -[145] & [156] & [256]+[345] & -[245] & [356] \\ 0 & [156] & [146] & [246]+[356] & [256] & [346] \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

**Table 3.** Input for Example 3

| Vertices, $m$ | inner normals, $\eta$ | partition of fan | $\omega_0$ |
|---|---|---|---|
| (0,0), (1,0), (2,1), (1,1), (1,2),(0,1) | (1,0), (0,1), (-1,1), (-1,-1), (1,-1) | $R_1$= {1,5}, $R_2$ = {2,3}, $R_3$ = {4} | (1,1,1,1,1) |

The final result for Example 3 is also $6 \times 6$ matrix,

$$\begin{pmatrix} 0 & [124] & 0 & [126]-[234] & [235] & -[236] \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & [126]-[135] & 0 & [146]-[236] & [156]+[246]+[345] & [346] \\ 0 & -[145] & 0 & [156]-[345] & [256] & [356] \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & [156] & 0 & [356] & [456] & 0 \end{pmatrix}$$

The homogenization monomials which index the rows of the Bézout matrix in $S_{int(2Q)}$. The resultant for Example 2 and Example 3 can be respectively written as a homogeneous polynomial of degree 4 and degree 5 in the brackets [$uvw$], hence the resultants are polynomials of degree 12 and degree 15 respectively in 18 unknowns. However, the determinants of both the Bézout matrices in this construction vanish identically, due to the sparsity present to the supports. Thus they neither express the multihomogeneous resultant, nor provide any information on the roots of the system. Our previous work in [9] had shown that the unmixed bivariate polynomial systems can give Sylvester matrix by using a Maple package such as multires. Unfortunately, in terms of resultant, it is undefined because the determinant of the denominator submatrix is zero. To overcome this problem, a hybrid matrix is sought that is the matrix with the combination of Bézout and Sylvester submatrices. We are working on this hybrid matrix and almost complete.

### Acknowledgment

### Conclusion

The underlying theoretical basis for the resultant hybrid matrix construction reveals that toric varieties have very nice properties that can be explored and utilized. The algebraic geometry properties of fan (collection of cones), homogeneous coordinate ring, normality, and Zariski closure have relation with the structure of the Newton polytopes in $\mathbf{R}^n$. In fact, every Newton polytopes in $\mathbf{R}^n$ determines a projective toric variety of dimension $n$. Those properties are very important for the computation of the entries of B\'ezout matrix. The divisors and homogeneous coordinate ring are defined for the normal toric variety $\mathbf{X}_Q$ obtained from the normal fan of $Q$. This variety is the normalization of $\mathbf{X}_A$.

This paper provides a computer program for constructing the Bézout submatrix in the hybrid Sylvester-Bézout resultant formulation. The construction realizes the theoretical concepts of toric varieties. The essential ingredients for the construction are the determination and computations of the corresponding homogeneous coordinates and Plücker coordinates. The construction of the Plücker coordinates involves combinatorial selection of the exponent vectors from the polynomial's support. The fan in a projective toric variety is a normal fan of a rational convex polytope in $\mathbf{R}^n$. In general, every normal toric variety is determined by a fan.

The computer program is implemented on two examples of unmixed bivariate polynomial systems, taken from the examples given in [10, 4] respectively. Both examples can be viewed as multihomogeneous polynomials of type (1,1;2,2),

in which cases both are multigraded. However the polynomials are sparse and not fully multigraded since their supports do not contain all the vertices of a fully multigraded system of type (1,1;2,2). Thus, by the application of combinatorics on the support of the polynomial systems and its respective homogenous coordinates, satisfying the 5-rule, the sparseness conditions of sparse systems have been considered.

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