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New type of hybrid conjugate gradient methods for optimization

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ABSTRACT

This paper given a new type hybrid conjugate gradient method for solving unconstrained optimization. The basic idea is to choose a new formula, and by searching a particular direction, the new method possess the descent property. The global convergence is established. The numerical results show that the given method is competitive to the other conjugate gradient methods for the test problems.

Keyword: Unconstrained optimization, Conjugate gradient method, Hybrid conjugate gradient, Global convergent property.

INTRODUCTION

Optimization problems can be classified as unconstrained optimization problems and constrained optimization problems. The mathematical description of unconstrained optimization problems is that :

$$\min f(x) \ , \ x \in R^n \qquad \text{..........} \quad (1)$$

where f is smooth and its gradient g is available. Conjugate gradient method is quite useful in finding an unconstrained minimum of a high-dimensional function f. The iterates of conjugate gradient methods are obtained by :

$$x_{k+1} = x_k + \alpha_k d_k \ , \quad k = 0,1,\ldots \qquad \text{..........} \quad (2)$$

where α_k is step size is determined a line search step satisfying the sufficient descent condition :

$$g_{k+1}^T d_{k+1} \le -c \, \|g_{k+1}\|^2 \qquad \text{..........} \quad (3)$$

in addition to the standard Wolfe conditions, that is, the step size satisfying :

$$f(x_k + \alpha_k d_k) \le f(x_k) + \delta_1 \alpha_k d_k^T g_k \qquad \text{..........} \quad (4)$$

$$g(x_k + \alpha_k d_k)^T d_k \ge \delta_2 d_k^T g_k \qquad \text{..........} \quad (5)$$

where $0 < \delta_1 \le \delta_2 < 1$ are constants with the search direction are computed as :

$$d_{k+1} = \begin{cases} -g_{k+1} & \text{if } k = 0 \\ -g_{k+1} + \beta_k d_k & \text{if } k > 0 \end{cases} \qquad \text{..........} \quad (6)$$

β_k is a suitable scale known as the conjugate gradient parameter.

The Fletcher-Reevers and Hestenes-Stiefel methods are two well-known conjugate gradient methods, they are specified by :

$$\beta_k^{FR} = \frac{\|g_{k+1}\|^2}{\|g_k\|^2} \ , \quad \beta_k^{HS} = \frac{g_{k+1}^T y_k}{d_k^T y_k} \qquad \text{..........} \quad (7)$$

where $y_k = g_{k+1} - g_k$ and $\|.\|$ stands for the Euclidean norm. For ease of presentation we call the methods corresponding to FR method [4] and HS method [5], respectively. Other conjugate gradient methods can be found [2,3,8,10] et al . . .

In the same context based on the quadratic model Hideaki and Yasushi [6] proposed the updating formulas as :

$$\beta_k^{HYG} = \frac{\|g_{k+1}\|^2}{2/\alpha_k (f_k - f_{k+1})}, \quad \beta_k^{HYY} = \frac{g_{k+1}^T y_k}{2/\alpha_k (f_k - f_{k+1})} \qquad \text{..........} \quad (8)$$

Important difference between **HYG** and **HYY** is that with **HYY** the has nice numerical results and on the other hand **HYG** have strong convergence properties. We combine the **HYY** method which has good numerical results with the HYG method which has strong convergence properties.

The paper is organized as follows. In Section 2 we construct a new hybrid method **BGY**, using the convex combination of parameters from the **HYG** method and from the **HYY** method. In this section we also find the formula for computing the parameter $\theta_k \in [0, 1]$, which is relevant for our method and we present the algorithm **BGY**. In Section 3 We prove that under some assumptions the search direction of our method satisfies the descent condition and global convergence is established. Section 4 contains some numerical experiments. Finally we present the conclusion in the last part.

Hybridization of HYG and HYY

In this section, we completely describe a new hybrid method. Our new method is a convex combination of HYG and HYY method. Now we define the next conjugate gradient parameter as:

$$\beta_k^{HBA} = (1 - \vartheta_k)\beta_k^{HYY} + \vartheta_k \beta_k^{HYG} \qquad \text{..........} \quad (9)$$

Hence, the direction d_k is given by :

$$d_0^{HBA} = -g_0, \quad d_{k+1}^{HBA} = -g_{k+1} + \beta_k^{HBA} s_k. \qquad \text{..........} \quad (10)$$

The parameter ϑ_k is the scalar parameter to be determined later. Obviously, if $\vartheta_k = 1$, then $\beta_k^{HBA} = \beta_k^{HYG}$, and if $\vartheta_k = 0$, then $\beta_k^{HBA} = \beta_k^{HYY}$. On the other hand, if $0 < \vartheta_k < 1$, then, β_k^{HBA} is a convex combination of the parameters β_k^{HYY} and β_k^{HYG}.

Theorem 1.

If the relations (9) and (10) hold, then :

$$d_{k+1}^{HBA} = (1 - \vartheta_k) d_{k+1}^{HYY} + \vartheta_k d_{k+1}^{HYG} \qquad \text{..........} \quad (11)$$

Proof :

Having in view the relations β_k^{HYG} and β_k^{HYY}, the relation (9) becomes :

$$\beta_k^{HBA} = (1 - \vartheta_k) \frac{g_{k+1}^T y_k}{2/\alpha_k (f_k - f_{k+1})} + \vartheta_k \frac{g_{k+1}^T g_{k+1}}{2/\alpha_k (f_k - f_{k+1})} \qquad \text{..........} \quad (12)$$

so, the relation (10) becomes :

$$d_{k+1}^{HBA} = -g_{k+1} + (1 - \vartheta_k) \frac{g_{k+1}^T y_k}{2/\alpha_k (f_k - f_{k+1})} s_k + \vartheta_k \frac{g_{k+1}^T g_{k+1}}{2/\alpha_k (f_k - f_{k+1})} s_k \qquad \text{..........} \quad (13)$$

In further consideration of the relation (13), we can get :

$$d_{k+1}^{HBA} = -(\vartheta_k g_{k+1} + (1 - \vartheta_k) g_{k+1}) + \beta_k^{HBA} s_k, \qquad \text{..........} \quad (14)$$

$$d_{k+1}^{HBA} = -(\vartheta_k g_{k+1} + (1 - \vartheta_k) g_{k+1}) + ((1 - \vartheta_k)\beta_k^{HYY} + \vartheta_k \beta_k^{HYG}) s_k. \qquad \text{..........} \quad (15)$$

The last relation yields ;

$$d_{k+1}^{HBA} = \vartheta_k (-g_{k+1} + \beta_k^{HYG} s_k) + (1 - \vartheta_k)(-g_{k+1} + \beta_k^{HYY} s_k). \qquad \text{..........} \quad (16)$$

From (16) we finally conclude :

$$d_{k+1}^{HBA} = (1-\vartheta_k)d_{k+1}^{HYY} + \vartheta_k d_{k+1}^{HYG} . \qquad \text{..........} \quad (17)$$

We shall find the value of the parameter ϑ_k in such a way that the conjugacy condition :

$$y_k^T d_k^{HBA} = 0 . \qquad \text{..........} \quad (18)$$

holds.

Firstly, we multiply both sides of the relation (13) by y_k^T from the left :

$$y_k^T \left[-g_{k+1} + (1-\vartheta_k)\frac{g_{k+1}^T y_k}{2/\alpha_k(f_k - f_{k+1})} s_k + \vartheta_k \frac{g_{k+1}^T g_{k+1}}{2/\alpha_k(f_k - f_{k+1})} s_k \right] = 0 \qquad \text{..........} \quad (19)$$

$$-y_k^T g_{k+1} + (1-\vartheta_k)\frac{g_{k+1}^T y_k}{2/\alpha_k(f_k - f_{k+1})}(y_k^T s_k) + \vartheta_k \frac{g_{k+1}^T g_{k+1}}{2/\alpha_k(f_k - f_{k+1})}(y_k^T s_k) = 0, \qquad \text{..........} \quad (20)$$

So,

$$y_k^T g_{k+1} - \frac{g_{k+1}^T y_k}{2/\alpha_k(f_k - f_{k+1})}(y_k^T s_k) = \vartheta_k \left[\frac{g_{k+1}^T g_{k+1}}{2/\alpha_k(f_k - f_{k+1})}(y_k^T s_k) - \frac{g_{k+1}^T y_k}{2/\alpha_k(f_k - f_{k+1})}(y_k^T s_k) \right] \qquad \text{..........} \quad (21)$$

i.e.

$$\frac{(\xi_{k+1} - y_k^T s_k)}{2/\alpha_k(f_k - f_{k+1})}(y_k^T g_{k+1}) = \vartheta_k \frac{g_{k+1}^T g_k}{2/\alpha_k(f_k - f_{k+1})}(y_k^T s_k) \qquad \text{..........} \quad (22)$$

Finally,

$$\vartheta_k = \frac{(2/\alpha_k(f_k - f_{k+1}) - y_k^T s_k)(y_k^T g_{k+1})}{g_{k+1}^T g_k (y_k^T s_k)} . \qquad \text{..........} \quad (23)$$

It is possible that ϑ_k, calculated as in (23), has the values outside the interval [0, 1]. However. In order to have a real convex combination in (14) the following rule is used : if $\vartheta_k \le 0$, then set $\vartheta_k = 0$ in (13) i.e. $\beta_k^{HBSQ} = \beta_k^{INQ}$, if $\vartheta_k \ge 1$, then set $\vartheta_k = 1$ in (13) i.e. $\beta_k^{HBSQ} = \beta_k^{BSQ}$. Therefore, under this rule for ϑ_k selection, the direction d_{k+1} combines the properties of the HYY and the HYG algorithms in a convex way.

Now, we can outline our new algorithm as follows:

Outline of the new algorithms:

Step 1. Input parameters : $x_1 \in R^n$, $0 < \delta_1 < \delta_2$, $d_1 = -g_1$, $f(x_1)$ $\alpha_1 = 1/\|g_1\|$.

Step 2. If $\|g_{k+1}\| \le 10^{-6}$, then stop.

Step 3. Determine the biggest $\alpha_{k+1} > 0$ it holds :

$$f(x_k + \alpha_k d_k) \le f(x_k) + \delta_1 \alpha_k g_k^T d_k .$$

$$d_k^T g(x_k + \alpha_k d_k) \ge \delta_2 d_k^T g_k .$$

and update the variables $x_{k+1} = x_k + \alpha_k d_k$.

Step 4. If $g_{k+1}^T g_k (y_k^T s_k) = 0$, then $\vartheta_k = 0$, else find ϑ_k from (23) .

Step 5. Compute : $\beta_k^{HBA} = (1-\vartheta_k)\dfrac{g_{k+1}^T y_k}{2/\alpha_k(f_k - f_{k+1})} + \vartheta_k \dfrac{g_{k+1}^T g_{k+1}}{2/\alpha_k(f_k - f_{k+1})}$.

Step 6. If $|g_{k+1}^T g_k| \ge 0.2 \|g_{k+1}\|^2$, then $d_{k+1} = -g_{k+1}$, else $d_{k+1} = -g_{k+1} + \beta_k^{HBA} s_k$.

Step 7. $k = k+1$, go to step 1.

CONVERGENCE ANALYSIS

For further considerations we need the next assumptions :

i- The level set $L = \{x \in R^n \mid f(x) \le f(x_0)\}$ is bounded.

ii- In some neighborhood U and L, $f(x)$ is continuously differentiable and its gradient id Lipschitz continuous, namely, there exists a constant $L > 0$ such that :

$$\|g(x_{k+1}) - g(x_k)\| \le L\|x_{k+1} - x_k\|, \ \forall \ x_{k+1}, x_k \in U \quad . \qquad \text{..........} \quad (24)$$

Under these assumptions on f, there exists a constant then a constant $\Gamma > 0$ exists, such that :

$$\|g_{k+1}\| > \Gamma , \qquad \text{..........} \quad (25)$$

for all $x \in L$. More details can be found in [9].

In [7] it is proved that for any conjugate gradient method with the strong Wolfe line search, it holds :

Lemma 1.

Let assumptions (i) and (ii) holds. Consider the method (2) and (6), where d_{k+1} is a descent direction and α_k is received from the strong Wolfe line search. If the Lipschitz condition holds, then either :

$$\sum_{k=0}^{\infty} \frac{\|g_{k+1}\|^4}{\|d_{k+1}\|^2} < \infty , \qquad \text{..........} \quad (26)$$

then

$$\lim_{k \to \infty} \inf \|g_{k+1}\| = 0 \ . \qquad \text{..........} \quad (27)$$

We first prove the descent property in this subsection.

Theorem 2.

Let Assumptions (i) and (ii) hold and let Wolfe conditions $(4) - (5)$ hold. Also, let $\{\|s_k\|\}$ tend to zero, and let there exist some nonnegative constants η_1 , η_2 such that :

$$2/\alpha_k (f_k - f_{k+1}) \ge \eta_1 \|s_k\|^2 , \qquad \text{..........} \quad (28)$$

$$\|g_{k+1}\|^2 \le \eta_2 \|s_k\| \ . \qquad \text{..........} \quad (29)$$

then d_k^{HBA} satisfies the descent condition for all k .

Proof :

It holds $d_0 = -g_0$. So, for $k = 0$, it holds $g_0^T d_0 = -\|g_0\|^2 < 0$. Multiplying (17) by g_{k+1}^T from the left, we get :

$$g_{k+1}^T d_{k+1}^{HBA} = (1 - \vartheta_k) g_{k+1}^T d_{k+1}^{HYY} + \vartheta_k g_{k+1}^T d_{k+1}^{HYG} \ . \qquad \text{..........} \quad (30)$$

If $\vartheta_k = 0$, the relation (30) becomes :

$$g_{k+1}^T d_{k+1}^{HBA} = g_{k+1}^T d_{k+1}^{HYY} \ . \qquad \text{..........} \quad (31)$$

So, if $\vartheta_k = 0$, the sufficient descent holds for the hybrid method, if it holds for HYY method. We can prove the descent for HYY method under the conditions of **Theorem 2. .** It holds:

$$d_{k+1}^{HYY} = -g_{k+1} + \beta_k^{HYY} s_k \ . \qquad \text{..........} \quad (32)$$

Multiplying (32) by g_{k+1}^T from the left, we get :

$$g_{k+1}^T d_{k+1}^{HYY} = -g_{k+1}^T g_{k+1} + \beta_k^{HYY} g_{k+1}^T s_k \ . \qquad \text{..........} \quad (33)$$

Using β_k^{HYY} , we get:

$$g_{k+1}^{T} d_{k+1}^{HYY} = -g_{k+1}^{T} g_{k+1} + \frac{g_{k+1}^{T} y_k}{2/\alpha_k (f_k - f_{k+1})} g_{k+1}^{T} s_k . \qquad \text{.......... } (34)$$

From (33), we get :

$$g_{k+1}^{T} d_{k+1}^{HYY} \leq -\|g_{k+1}\|^2 + \frac{\|g_{k+1}\|^2 \|y_k\| \|s_k\|}{2/\alpha_k (f_k - f_{k+1})} . \qquad \text{.......... } (35)$$

From Lipschitz condition we have $\|y_k\| \leq L \|s_k\|$, so :

$$g_{k+1}^{T} d_{k+1}^{HYY} \leq -\|g_{k+1}\|^2 + \frac{\|g_{k+1}\|^2 L \|s_k\|^2}{2/\alpha_k (f_k - f_{k+1})} . \qquad \text{.......... } (36)$$

But, using $(28) - (29)$, we get :

$$g_{k+1}^{T} d_{k+1}^{HYY} \leq -\|g_{k+1}\|^2 + \frac{\eta_2 L \|s_k\|}{\eta_1} . \qquad \text{.......... } (37)$$

But, because of the assumption $\|s_k\| \Rightarrow 0$, the second summand in (37) tends to zero, so there exists a number $0 < \delta \leq 1$, such that :

$$\frac{1}{\eta_1} \eta_2 L \|s_k\| \leq \delta \|g_{k+1}\|^2 . \qquad \text{.......... } (38)$$

Now, (37) becomes :

$$g_{k+1}^{T} d_{k+1}^{HYY} \leq -\|g_{k+1}\|^2 + \delta \|g_{k+1}\|^2 , \qquad \text{.......... } (39)$$

i.e.

$$g_{k+1}^{T} d_{k+1}^{HYY} \leq -(1-\delta)\|g_{k+1}\|^2 < 0 . \qquad \text{.......... } (40)$$

On the other hand, for $\vartheta_k = 1$, the relation (30) becomes :

$$g_{k+1}^{T} d_{k+1}^{HBA} = g_{k+1}^{T} d_{k+1}^{HYG} . \qquad \text{.......... } (41)$$

But, under the Wolfe line search, HYG method satisfies the descent condition [6].

Now, let $0 < \vartheta_k < 1$ and from (30), we get :

$$g_{k+1}^{T} d_{k+1}^{HBA} \leq (1-\vartheta_k) g_{k+1}^{T} d_{k+1}^{HYY} + \vartheta_k g_{k+1}^{T} d_{k+1}^{HYG} . \qquad \text{.......... } (42)$$

We obviously can conclude now :

$$g_{k+1}^{T} d_{k+1}^{HBA} \leq 0 . \qquad \text{.......... } (43)$$

Next, we show the convergence of the HBA method.

Theorem 3.

Consider the iterative method, defined by algorithm **HBA**. Let all conditions of Theorem 2. hold. Then either $g_{k+1} = 0$, for some $k+1$, or

$$\lim_{k \to \infty} \inf \|g_{k+1}\|^2 = 0 . \qquad \text{.......... } (44)$$

Proof:

Let $g_{k+1} \neq 0$ for all k. Then, we are going to prove (44). Suppose, on the contrary, that (44) doesn't hold. Then there exists a constant $c > 0$, such that :

$$g_{k+1} \geq c , \quad \forall \ k . \qquad \text{.......... } (45)$$

From (17), we get :

$$\|d_{k+1}^{HBA}\| \leq \|d_{k+1}^{HYY}\| + \|d_{k+1}^{HYG}\| . \qquad \text{.......... } (46)$$

Next, it holds :

$$\left\| d_{k+1}^{HYG} \right\| \le \left\| g_{k+1} \right\| + \left| \beta_k^{HYG} \right| \left\| s_k \right\|. \qquad \text{..........} \quad (47)$$

Further,

$$\left\| d_{k+1}^{HYG} \right\| \le \Gamma + \frac{\eta_2}{\eta_1}. \qquad \text{..........} \quad (48)$$

Also,

$$\left\| d_{k+1}^{HYY} \right\| \le \left\| g_{k+1} \right\| + \left| \beta_k^{HYY} \right| \left\| s_k \right\|. \qquad \text{..........} \quad (49)$$

It holds :

$$\left\| d_{k+1}^{HYY} \right\| \le \Gamma + \frac{\Gamma L}{\eta_1}. \qquad \text{..........} \quad (50)$$

So, using (48), (50) and (46) we get :

$$\left\| d_{k+1}^{HBA} \right\| \le 2\Gamma + \frac{\Gamma L}{\eta_1} + \frac{\eta_2}{\eta_1}. \qquad \text{..........} \quad (51)$$

But, now we can get :

$$\frac{\left\| g_{k+1} \right\|^4}{\left\| d_{k+1} \right\|^2} \ge \frac{c^4}{\left[2\Gamma + \frac{\Gamma L}{\eta_1} + \frac{\eta_2}{\eta_1} \right]^2}. \qquad \text{..........} \quad (52)$$

Wherefrom

$$\sum_{k \ge 1}^{\infty} \frac{\left\| g_{k+1} \right\|^4}{\left\| d_{k+1} \right\|^2} = \infty. \qquad \text{..........} \quad (53)$$

Using **Lemma 1**, we conclude that this is a contradiction. So, we finish the proof.

NUMERICAL RESULTS

In this section we present the computational performance of a Fortran implementation of the CG algorithm on a set of **15** unconstrained optimization test problems. The test problems are the unconstrained problems in the CUTE library, along with other large-scale optimization problems presented in [1].

All algorithms implement the Wolfe line search conditions (4)-(5) with $\delta_1 = 0.001$ and $\delta_2 = 0.9$, and the same stopping criterion $\left\| g_{k+1} \right\| \le 10^{-6}$.

Tables 1 list numerical results. The meaning of each column is as follows : NI : number of iterations. NF : number of function evaluations.

So, the limited numerical experiments (Table 1) indicate that the algorithm **HBA** is potentially efficient.

CONCLUSION

In this paper, we propose a new hybrid conjugate gradient method known as **HBA**. This method possessed good performance when compared to other classical CG. Based on the theoretical proof and the numerical result in table 1, it is shown that this **HBA** converges globally.

Table 1: Comparison of different CG-algorithms with different test functions and different dimensions

		FR algorithm		HBA algorithm		HBA with $u = 0.5$	
P. No.	**n**	**NI**	**NF**	**NI**	**NF**	**NI**	**NF**
1	**100**	**19**	**35**	**19**	**35**	**18**	**34**
	1000	**38**	**65**	**36**	**66**	**38**	**65**
2	**100**	**43**	**88**	**40**	**84**	**41**	**87**
	1000	**46**	**92**	**40**	**92**	**40**	**89**
3	**100**	**32**	**64**	**13**	**25**	**13**	**25**
	1000	**77**	**129**	**16**	**30**	**14**	**28**
4	**100**	**180**	**313**	**72**	**137**	**69**	**133**

	1000	**F**	**F**	**81**	**154**	**65**	**124**
5	**100**	**124**	**231**	**51**	**90**	**78**	**146**
	1000	**445**	**711**	**183**	**317**	**172**	**309**
6	**100**	**71**	**110**	**29**	**56**	**31**	**58**
	1000	**47**	**84**	**31**	**62**	**34**	**65**
7	**100**	**101**	**217**	**85**	**206**	**87**	**210**
	1000	**101**	**214**	**87**	**207**	**85**	**203**
8	**100**	**32**	**65**	**26**	**57**	**28**	**60**
	1000	**53**	**116**	**37**	**88**	**34**	**84**
9	**100**	**40**	**65**	**34**	**53**	**43**	**67**
	1000	**43**	**68**	**41**	**67**	**F**	**F**
10	**100**	**398**	**605**	**414**	**657**	**351**	**555**
	1000	**F**	**F**	**F**	**F**	**F**	**F**
11	**100**	**121**	**218**	**90**	**142**	**77**	**125**
	1000	**345**	**634**	**266**	**423**	**245**	**383**
12	**100**	**74**	**123**	**84**	**135**	**80**	**128**
	1000	**370**	**616**	**259**	**430**	**239**	**401**
13	**100**	**9**	**18**	**8**	**16**	**8**	**16**
	1000	**12**	**82**	**9**	**20**	**7**	**15**
14	**100**	**23**	**45**	**18**	**33**	**20**	**36**
	1000	**27**	**55**	**19**	**44**	**21**	**46**
15	**100**	**25**	**43**	**23**	**46**	**22**	**44**
	1000	**46**	**741**	**44**	**564**	**35**	**367**
Total		**2723**	**5391**	**2033**	**4115**	**1930**	**3779**

Fail: The algorithm fail to converge.

Problems numbers indicant for: 1. is the Trigonometric, 2. is the Extended White & Holst, 3. is the Extended Tridiagonal 1, 4. is the Extended Powell, 5. is the Quadratic Diagonal Perturbed, 6. is the Extended Wood, 7. is the Extended Hiebert, 8. is the Extended Quadratic Penalty, 9. is the Extended Tridiagonal 2, 10. is the TRIDIA (CUTE), 11. is the DIXMAANE (CUTE), 12. is the Partial Perturbed Quadratic, 13. is the ARWHEAD (CUTE), 14. is the LIARWHD (CUTE), 15. is the Generalized Tridiagonal 1 .

REFERENCES

[1]. Andrei N. (2008), An Unconstrained Optimization test function collection. Adv. Model. Optimization. 10. pp.147-161.

[2]. Dai Y. H. and Yuan Y., (1999). A nonlinear conjugate gradient method with a strong global convergence property, SIAM J. optimization, pp. 177-182.

[3]. Fletcher R. and Reeves C. M.,(1964). Funtion minimization by conjagate gradients, Computer Journal 7, pp. 149-154.

[4]. Fletcher R., (1989). Practical Method of Optimization (2nd Edition), John Wiley and Sons, New York .

[5]. Hestenes M. R. and Stiefel E., (1952). Methods of conjugate gradients for solving linear systems, Journal of Research of National Burean of Standard,49, pp. 409-436.

[6]. Hideaki I., and Yasushi N., (2011)," Conjugate gradient methods using value of objective function for unconstrained optimization Optimization Letters, V.6, Issue 5, pp. 941-955.

[7]. Hager W.W. and Zhang H., (2006),'A survey of nonlinear conjugate gradient methods, Pacific journal of Optimization, 2, pp.35–58.

[8]. Polak E. and Ribiere G., (1969). Note sur la Convergence de Directions Conjugate, Revue Francaise Informant, Reserche. Opertionelle16, pp. 35-43.

[9]. Snezana S. D., (2017),' New hybrid conjugate gradient method as a convex combination of LS and CD methods, Filomat ,6 ,pp. 1813–1825.

[10]. Liu Y. and Storey C. (1991) ' Efficient generalized conjugate gradients algorithms' Part 1 : Theory. J. Optimization Theory and Applications, 69, pp. 129-137.