Modified Black-Scholes Model based on Monte Carlo Simulation

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Abstract: Understanding stock market dynamics is a challenging problem. We have proposed a novel approach for Black-Scholes model with periodic interest rate using Monte Carlo simulation. In this paper, we have also analyzed the First Passage Time (FPT) distribution under some given strike price. We have applied Monte Carlo simulation to the underlying Stochastic Differential Equation (SDE), which produces nonlinear dynamics of stock price. Computer simulation and comparison with standard example is also provided to support our approach.

Keywords: Black-Scholes, First Passage Time, No-Arbitrage, Periodic Rates, Monte Carlo Simulation.

Introduction

From last few decades, understanding the dynamics of stock market is a challenging as well exciting problem. Several mathematical and computational techniques have been developed to contemplate the characteristics of stock market. But mathematical formulation is intractable in nature, so clearly, to capture the variations of stock market one has to apply the well-known Monte Carlo simulation. It is a common practice to model stock market dynamics by the well-known Black-Scholes model \cite{Black1973}. Several researchers \cite{2,3,4,5,6} have proposed various models for better understanding of the nature of stock price movement. Researchers also developed some computational schemes for simulation of stock prices and related quantities of interest. The limitations of Black-Scholes model are that it assumes the interest rate as time independent; however, in real world, markets interest rates are known to be functions of time. In this paper, we have considered the time dependent interest rate Black-Scholes model. The basis for interest rate modeling depends on effects of business cycles on interest rate. Using this modified Black-Scholes model, we are interested in examining the effects of seasonal periodicities in the temporal evolution of stock price dynamics. This model is using analytical results and Monte Carlo simulation of stock price dynamics for the corresponding SDEs. During the analysis of the model, we have obtained conditions of no-arbitrage.

Black-Scholes model and Monte Carlo Simulation

The Black-Scholes model has been the most important mathematical model to describe the stock price dynamics. This model especially calculates the fair option pricing in stock market, which is resembled as observed pricing. The generalized Black-Scholes model has been suggested in the literature\cite{2}. Black-Scholes model is represented by the SDE as given below:

\begin{equation}
\frac{dS}{S} = \mu dt + \sigma dW(t), \quad S(t = 0) = S_0
\end{equation}

here, W(t) is the Weiner process. The solution of this equation \text{(1)} is given by

\[ S(t) = S_0 e^{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma \sqrt{t} Z} \]

The corresponding probability density function for is

\begin{equation}
P(S, S_0, t) = f(S, t) = \begin{cases}
\frac{\exp\left(-\frac{\log(S/S_0) - \left(\mu - \frac{1}{2}\sigma^2\right)t}{2\sigma^2 t}\right)}{S \sigma \sqrt{2\pi t}}, & S \geq 0 \\
0, & S < 0
\end{cases}
\end{equation}

Monte Carlo method is a simulation technique, which uses random numbers and these numbers are independent and identically distributed. This method is generally used in optimization, numerical integration, and simulating financial market. The analytical solution of a vast class of SDEs is not possible; therefore we resort to Monte Carlo simulation of SDEs. Referencing [6,7,8,9], we have used the following discretized scheme for simulation

\[
S(t_{i+1}) = S(t_i) + \mu(t_i)S(t_i)h + \sigma(t_i)S(t_i)\sqrt{h} Z_{i+1},
\]

where \( Z_{i+1} \sim N(0,1) \) and \( Z_1, Z_2, \ldots \) are independent normal variates and \( h \) is the step size.

**Modified Black−Scholes Model**

Our proposed modified Black-Scholes model considers time dependent interest rates which is oscillatory. The underlying stock \( S(t) \) under time dependent interest rate is modeled as given below:

\[
dS(t) = \mu(t)S(t)dt + \sigma(t)S(t)dW(t), \quad S(t = 0) = S_0 \quad (4)
\]

\[
d\beta(t) = r(t)\beta(t)dt, \quad (5)
\]

where, \( \mu(t) \) is the time dependent drift, \( W(t) \) represents Wiener process \( \sigma \) is the market volatility, and \( r(t) \) is the corresponding sinusoidal interest rate represented as:

\[
r(t) = r_0 + \bar{r}\cos(\omega t), \quad r_0 > \bar{r} \quad (6)
\]
where, \( r_0 \) is the mean interest rate and is \( r_0 = \frac{\pi}{\alpha} \sin(\alpha t) \) the interest rate which is periodic with frequency \( \alpha \). For further calculations, we will use the time-average interest rate given below:

\[
\bar{r}(t) = r_0 - \frac{\pi}{\alpha} \sin(\alpha t),
\]

(7)

Interestingly, with the deterministic time structure in the parameters, under the no-arbitrage condition the proposed model is found to follow the same pde as originally proposed by Black-Scholes[3]. The pde is as under:

\[
\frac{\partial f(t, S(t))}{\partial t} = r(t) f(t, S(t)) - \left( r(t)S(t) \right) \frac{\partial f(t, S(t))}{\partial S} - \left( \frac{\sigma^2 (S(t))^2}{2} \right) \frac{\partial^2 f(t, S(t))}{\partial S^2}
\]

(8)

Thus, under no-arbitrage condition in the Black-Scholes model, the time-dependent drift of the underlying stock is equal to the interest rate of the bond, we have

\[
\mu(t) = r(t).
\]

Therefore, it is easy to see that the stochastic evolution of the underlying stock price follows lognormal process with oscillatory mean. The pdf of the process is:

\[
p(S, t) = \begin{cases} 
\frac{1}{S \sigma \sqrt{2\pi t}} \exp \left[ -\frac{\log \left( \frac{S}{S_0} \right) - \left( \frac{r(t) - \frac{\sigma^2}{2} \bar{r}(t)}{\sigma^2} \right) t}{2 \sigma^2 t} \right], & S \geq 0 \\
0, & S < 0 
\end{cases}
\]

Computer Simulation and Comparisons

As it is interesting to note that we have explicit expression for the time dependent density function for stock prices. The density function represents lognormal process with periodic drift. The stochastic evolution of the stock price has been adopted as a first passage time (FPT) problem [10]. When the price of the underlying asset \( S(t) \) with initial price as \( S_0 \) reaches the strike price (K), the option is exercised.

The FPT of stock price is accordingly defined as:

\[
FPT = \inf \{ t \geq 0 : S(t) > K, S_0 < K \}
\]

Figure 3: Histogram of FPT distribution for model driven by periodic interest rate.

\( S_0 = 10, \sigma_0 = 0.05, \omega = 0.2, r_0 = 0.1, \bar{r} = 1, \sigma = 0.05 \)

Figure 4: FPT distribution for model driven by periodic interest rate.
Figure 5: Histogram of FPT distribution without periodic interest rate.

Figure 6: Probability Density of FPT without periodic interest rate.

Conclusion

In this paper, we have presented a novel approach for the dynamic evaluation of stock prices. The proposed model is formulated as a SDE having periodic interest rate. It provides a greater insight, because it recovers the existing lognormal process of stock market behavior in the absence of oscillating drift in our model. Also, it is interesting to find out that the strike price can be derived from a first pass passage time nature of stock market under the rigorous Monte Carlos simulation. Although, some future work is still possible and one can find option price based on our model may be helpful for portfolio manager, hedge funder, and investors.

References