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# Performance of Adaptive Filtering Techniques Using the Fractional Fourier Transform for Non-Stationary Interference and Noise Suppression

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**ABSTRACT**

**The Fractional Fourier Transform (FrFT) performs better interference suppression than the fast Fourier Transform (FFT) when the signal-of-interest (SOI) or interference is nonstationary. Minimum mean-square error (MMSE) based filtering in the FrFT domain provides additional benefit in interference suppression in non-stationary environments. However, MMSE filtering requires computational covariance matrix inversion. Furthermore, non-stationary environments require fewer samples than needed to form the covariance matrix or to invoke most reduced rank techniques. Hence, MMSE-FrFT filtering results in errors. In this paper, we propose to apply the correlations subtraction architecture of the multistage Wiener filter (CSA-MWF) in the FrFT domain to overcome these problems. We compare the proposed MWF-FrFT algorithm to the MMSE-FrFT algorithm and to the conventional MMSE-FFT algorithm by simulation. Using a BPSK signal in chirp noise and Gaussian pulse interference as examples, we show bit error rates (BERs) with 2 − 4 dB less $E_b/N_0$ and just N = 4 samples per block.**

**Keywords: Adaptive Filtering, Fractional Fourier Transform, Minimum Mean-Square Error, Multistage Wiener Filter.**

## 1. INTRODUCTION

The Fractional Fourier Transform (FrFT) has a wide range of applications in the fields of optics, quantum mechanics, image processing, and communications. Also, its properties, as well as its relationship to other analysis methods such as the Wigner distribution ([7] and [8]) and the wavelet transform [10] are well understood. Specifically, it is a useful signal processing tool for separating a signal-of-interest (SOI) from interference and/or noise when the statistics of either are nonstationary, as is often the case [11]. The FrFT enables us to translate the received signal to an axis in the time-frequency plane where the SOI and interference may be separable, when they are not separable in the time domain or the frequency domain as provided by the conventional fast Fourier Transform (FFT).

The FrFT of a function f(x) of order a is defined as [11]

$$\mathbf{F}^a[f(x)] = \int_{-\infty}^{\infty} B_a(x,x')f(x')dx', \tag{1}$$

where the kernel $B_a(x,x')$ is defined as

$$B_a(x,x') = \frac{e^{i(\pi\hat{\phi}/4-\phi/2)}}{|sin\phi|^{1/2}} \times e^{i\pi(x^2cot\phi-2xx'csc\phi+x'^2cot\phi)}, \tag{2}$$

$\phi = a\pi/2$, and $\hat{\phi} = sgn[sin(\phi)]$. This applies to the range $0 < |\phi| < \pi$, or $0 < |a| < 2$. In discrete time, we can model the N × 1 FrFT of an N × 1 vector x as ,

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$$\mathbf{X}_a = \mathbf{F}^a\mathbf{x}. \tag{3}$$

where $\mathbf{F}^a$ is an N × N matrix whose elements are given by ([2] and [11])

$$\mathbf{F}^a[m,n] = \sum_{k=0,k\neq(N-1+(N)_2)}^{N} u_k[m]e^{-j\frac{\pi}{2}ka}u_k[n], \tag{4}$$

where $u_k[m]$ and $u_k[n]$ are the eigenvectors of the matrix S defined by [2]

$$\mathbf{S} = \begin{bmatrix} C_0 & 1 & 0 & \dots & 1 \\ 1 & C_1 & 1 & \dots & 0 \\ 0 & 1 & C_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \dots & C_{N-1} \end{bmatrix}, \tag{5}$$

and

$$C_n = 2cos(\frac{2\pi}{N}n) - 4. \tag{6}$$

An MMSE-FrFT solution for estimating an SOI in the presence of non-stationary interference and noise in the FrFT domain has been developed which does not rely on knowledge of the nature of the interfering signal or noise [14]. When the environment is non-stationary, it is necessary to perform this estimation with very few samples, i.e. before the statistics of the received signal change. If the number of samples used is large, then estimation errors occur. MMSE-based algorithms, however, are known to require a large number of samples in practice, which limits their performance in nonstationary scenarios [12]. In addition, any MMSE-based solution requires inversion of a covariance matrix, which is a computationally complex operation that limits its ability to perform in real-time [15]. In this paper, we implement a reduced rank version of the MMSE-FrFT solution using the well-known correlations subtraction architecture of the multistage Wiener filter (CSA-MWF) to improve performance and operate more efficiently in non-stationary environments. The computational savings of the CSA-MWF has already been documented in the literature, so we will not discuss that here (for example, see [15]).

An outline of the paper is as follows: Section 2 describes the adaptive filtering problem, now in the Fractional Fourier Transform (FrFT) domain. Section 3 presents the full rank minimum mean-square error (MMSE) solution in the FrFT domain proposed in [14] and the conventional MMSE-based FFT solution, termed MMSE-FrFT and MMSE-FFT, respectively. Section 4 describes the proposed reduced rank solution using a multistage Wiener filter (MWF) in the FrFT domain, called MWF-FrFT. Section 5 presents simulation results to show that both the MMSE-FrFT and MWF-FrFT provide performance improvement in estimating signals in noise vs. the conventional MMSE-FFT based technique, thus showing the benefit of the FrFT. Furthermore, we show that the MWF-FrFT improves upon the MMSE-FrFT, thus showing the performance benefit of rank reduction. Finally, conclusions and remarks on future work are given in Section 6.

## 2. PROBLEM FORMULATION

Without loss of generality, we consider a digital binary sequence whose elements are in $(-1,+1)$ that we would like to estimate in the presence of non-stationary interference and non-stationary noise. Here, we ignore the carrier, and hence model the SOI as a baseband binary phase shift keying (BPSK) signal. The number of bits per block is denoted $N_1$, and if we oversample each bit by a factor of $SPB$ (samples per bit), the number of samples per block in the BPSK signal is $N = N_1SPB$, and the signal is denoted in vector form as the $N \times 1$ vector $\mathbf{x}_1(i)$. The SOI $\mathbf{x}_1(i)$ is corrupted by a non-stationary interferer $\mathbf{x}_2(i)$ and a non-stationary noise signal $\mathbf{x}_3(i)$, both of which we describe in Section 5, and by an additive white Gaussian noise signal $\mathbf{n}(i)$. Here, index $i$ denotes the $i^{th}$ block, where $i = 1,2,,M$, and $M$ is the total number of blocks that we process. The received signal $\mathbf{y}(i)$ is then,

$$\mathbf{y}(i) = \mathbf{x}_1(i) + \mathbf{x}_2(i) + \mathbf{x}_3(i) + \mathbf{n}(i). \tag{7}$$

We obtain an estimate of the transmitted signal $\mathbf{x}_1(i)$, denoted $\hat{\mathbf{x}}_1(i)$, by first transforming the received signal to the FrFT domain, applying an adaptive filter, and taking the inverse FrFT. This is written as [14]

$$\hat{\mathbf{x}}_1(i) = \mathbf{F}^{-a}\mathbf{G}\mathbf{F}^a\mathbf{y}(i), \tag{8}$$

where $\mathbf{F}^{a}$ and $\mathbf{F}^{-a}$ are the $N \times N$ FrFT and inverse FrFT matrices of order 'a', respectively, and

$$\mathbf{g} = diag(\mathbf{G}) = (g_0, g_1, \ldots, g_{N-1}) \tag{9}$$

is an $N \times 1$ set of optimum filter coefficients to be found such that the mean-square error between the desired signal $\mathbf{x}_1(i)$ and its estimate $\hat{\boldsymbol{x}}_1(i)$ is minimized. That is, we minimize

$$J(\mathbf{g}) = \frac{1}{M}\sum_{i=1}^{M} ||\mathbf{F}^{-a}\mathbf{G}\mathbf{F}^{a}\mathbf{y}(i) - \mathbf{x}_1(i)||^2, \tag{10}$$

The notation $diag(\mathbf{G}) = (g_0, g_1, \ldots, g_{N-1})$ means that matrix $\mathbf{G}$ has the scalar coefficients $g_0$, $g_1$, ..., $g_{N-1}$ as its diagonal elements, with all other elements equal to zero.

## 3. FULL RANK MMSE-FRFT SOLUTION

It is well known that the optimum set of filter coefficients $\mathbf{g}_0$ that minimizes the cost function in Eq. (10) can be obtained by setting the partial derivative of the cost function to zero [14]. That is, compute $\mathbf{g}_0$ such that

$$\frac{\partial J(\mathbf{g})}{\partial \mathbf{g}}|_{\mathbf{g}=\mathbf{g}_0} = 0 \tag{11}$$

This is the MMSE-FrFT solution, given by [14]

$$\mathbf{g}_{0,MMSE-FrFT} = \frac{1}{2}\mathbf{Q}^{-1}\mathbf{b}, \tag{12}$$

where

$$\mathbf{Q} = \sum_{i=1}^{M}\mathbf{Q}(i), \tag{13}$$

$$\mathbf{b} = \sum_{i=1}^{M}\mathbf{b}(i), \tag{14}$$

$$\mathbf{Q}(i) = (\mathbf{F}^{-a}\mathbf{Z}(i))^H(\mathbf{F}^{-a}\mathbf{Z}(i)), \tag{15}$$

$$\mathbf{z}(i) = [z_0(i)\ z_1(i)\ \ldots\ z_{N-1}(i)]^T = diag(\mathbf{Z}(i)) = \mathbf{F}^{a}\mathbf{y}(i), \tag{16}$$

and

$$\mathbf{b}(i) = (-2Re(\mathbf{x}(i)^H\mathbf{F}^{-a}\mathbf{Z}(i)))^T. \tag{17}$$

Note that the LMS-FrFT solution presented in [9] will perform comparably to the MMSE-FrFT algorithm over time, hence we do not include it in our simulations. Note also that the MMSE-FFT solution is obtained simply by setting a = 1 in calculating $\mathbf{g}_0$ from Eqs. (12)–(17). This solution simply becomes one of applying the optimum filter given by Eq. (12) in the frequency domain, since $\mathbf{F}^{1}$ reduces to an FFT and $\mathbf{F}^{-1}$ is an inverse FFT (IFFT). In other words,

$$\mathbf{g}_{0,MMSE-FFT} = \mathbf{g}_{0,MMSE-FrFT}|_{a=1}. \tag{18}$$

## 4. PROPOSED REDUCED RANK MWF-FRFT SOLUTION

The full rank solution implemented in Eq. (12) can be implemented efficiently using the correlations subtraction architecture of the multistage Wiener filter (CSA-MWF). The MWF was first introduced in [3] − [6] and it offers the advantages that it often exceeds MMSE performance without any computationally complex matrix inversion or eigen-decompositions, and we show here that these advantages exist in the FrFT domain as well. The efficient CSA implementation of the MWF was first presented in [13]. The recursion equations for the CSA-MWF are shown in Table 1. Rank reduction is achieved because we can set D < N. We initialize the filter in the conventional way, except that we transform all the variables to the FrFT domain first. So, we let

$$d_0(i) = (\mathbf{F}^{a}\mathbf{x}(i))^T, \tag{19}$$

and

$$\mathbf{x}_0(i) = \mathbf{Z}(i). \tag{20}$$

The CSA-MWF computes the D scalar weights $w_j$, j = 1,2,...,D, from which we form the optimum filter

$$\mathbf{g}_{0,MWF-FrFT} = w_1\mathbf{h}_1 - w_1w_2\mathbf{h}_2 + ... - (-1)^D w_1w_2...w_D\mathbf{h}_D. \quad (21)$$

**Table 1: Recursion Equations for the CSA-MWF**

| |
|---|
| Initialization: $d_0(i)$ and $\mathbf{x}_0(i)$ |
| Forward Recursion: For $j = 1, 2, ..., D$: <br> $\mathbf{h}_j = \frac{\sum_\Omega\{d^*_{j-1}(i)\mathbf{x}_{j-1}(i)\}}{\|\sum_\Omega\{d^*_{j-1}(i)\mathbf{x}_{j-1}(i)\}\|}$ <br> $d_j(i) = \mathbf{h}_j^H\mathbf{x}_{j-1}(i)$ <br> $\mathbf{x}_j(i) = \mathbf{x}_{j-1}(i) - \mathbf{h}_jd_j(i)$ |
| Backward Recursion: $\epsilon_D(i) = d_D(i)$ <br> For $j = D, D-1, ..., 1$: <br> $w_j = \frac{\sum_\Omega\{d^*_{j-1}(i)\epsilon_j(i)\}}{\sum_\Omega\{\|\epsilon_j(i)\|^2\}}$ <br> $\epsilon_{j-1}(i) = d_{j-1}(i) - w^*_j\epsilon_j(i)$ |

## 5. SIMULATIONS

We present simulation examples to compare the performance of the three adaptive filtering techniques: MMSE-FrFT, MMSE-FFT, and MWF-FrFT and demonstrate the performance benefits of the MWF-FrFT method. We compute the filter coefficients using Eqs. (12), (18), and (21), respectively, and then apply them to Eq. (8) to compute the bit estimates. These are compared to the true bit to determine if an error occurs, so we can compute a bit error rate (BER). In the first example we assume there is no interfering signal, so that $\mathbf{x}_2 = \mathbf{0}$. The desired BPSK signal is corrupted by a non-stationary chirp noise signal given in vector form by (see Example 3 in [7])

$$\mathbf{x}_3 = e^{-j1.73\pi((0:N-1)/f_s)^2}, \quad (22)$$

where $f_s$ is an arbitrary sampling rate and we have dropped the block index i for convenience. We let $CIR_3$, the ratio of the desired signal power to the chirp noise signal power be −5 dB. Hence, the chirp noise is much stronger than the desired signal. Here, due to the non-stationarity of the interference, we choose a very small block size for best performance, so we let $N_1 = 2$, SPB = 2, and therefore N = 4. We let the rank of the MWF-FrFT algorithm be D = 1, since there is only a single desired signal and we have a training sequence. The performance is not very sensitive to rank, so we choose D = 1 for faster implementation. Since the sample size (N = 4) is so small, the choice of rank is $1 \leq D \leq 4$. AWGN is also present, and we plot the BER as a function of $E_b/N_0$ in Fig. 1. At low $E_b/N_0$, all three techniques perform comparably, since the techniques are not designed to handle (stationary) AWGN. However, at high $E_b/N_0$, we see that the MWF-FrFT technique provides about 1 − 2 dB performance improvement over the MMSE-FrFT method, which in turn is about 2 − 3 dB better than the conventional MMSE-FFT. Given that MWF-FrFT does not require inversion of the covariance matrix **Q**, there is a computational savings as well.

Some additional noteworthy points follow. First, note that because we are already using such a small sample size (N = 4 in this case), we cannot apply other rank reduction techniques based on eigen-decomposition, such as principal components. Since the full rank is 4, there is not much rank reduction that can be done and hence no benefit to doing so. This is the reason we choose to apply only the MWF-FrFT reduced rank method and not any other methods. Second, we note that in order to compute the optimum filter coefficients, we need an estimate of the FrFT rotational parameter 'a'. This is done in the conventional way, by computing the value of 'a' that provides the best estimate first, and then applying the value of 'a' over the Monte Carlo trials to compute the BER [7]. Third, we note that the optimum value of 'a' is not necessarily the same for the MMSE-FrFT and MWF-FrFT techniques and therefore must be computed separately for each. The development of techniques for estimating 'a' using analytic or other a priori methods is the subject of ongoing research.

In the second example, we set the chirp noise to $\mathbf{x}_3 = \mathbf{0}$ but we apply an interfering signal $\mathbf{x}_2$ which is modeled as a Gaussian pulse with random amplitude and phase, that takes the form [14]

$$\mathbf{x}_2 = \mathbf{A}e^{-\pi((0:N-1)/f_s-\mathbf{s})}, \quad (23)$$

where **A** and **s** are N × 1 vectors of randomly generated amplitudes and phases uniformly distributed in (0.5,1.5). Here we let $CIR_2 = -5$ dB. Hence, the interferer is stronger than the desired signal by 5 dB. We again include AWGN so we can plot the BER as a function of $E_b/N_0$, shown in Fig. 2. We see greater improvement in the MWF-FrFT method versus the MMSE-FrFT and MMSE-FFT as in the previous example. We also note that both FrFT techniques perform better than in the previous example. This is simply due to the nature of the interference and the ability of the FrFT to separate the interfering signal from the desired signal by transforming to the appropriate domain 'a'. This is also the subject of future study, and it is expected that examining the Wigner distribution ([1] and [7]) of different signal types will enable a better understanding of which signal types can be more easily separated.

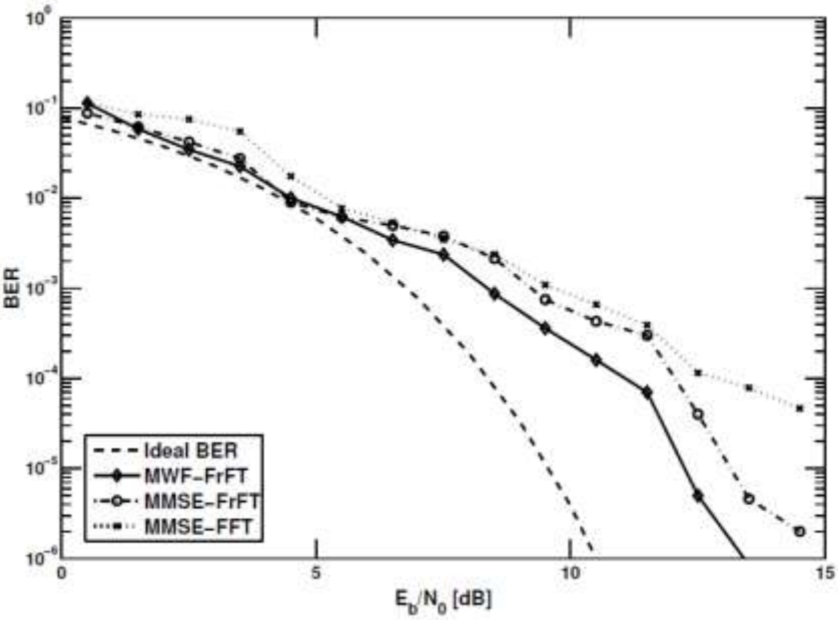


**Fig. 1**. $E_b/N_0$ [dB] vs. BER; BPSK Signal in AWGN; no Interferer ($CIR_2 = \infty$); Chirp Noise ($CIR_3 = -5$ dB)

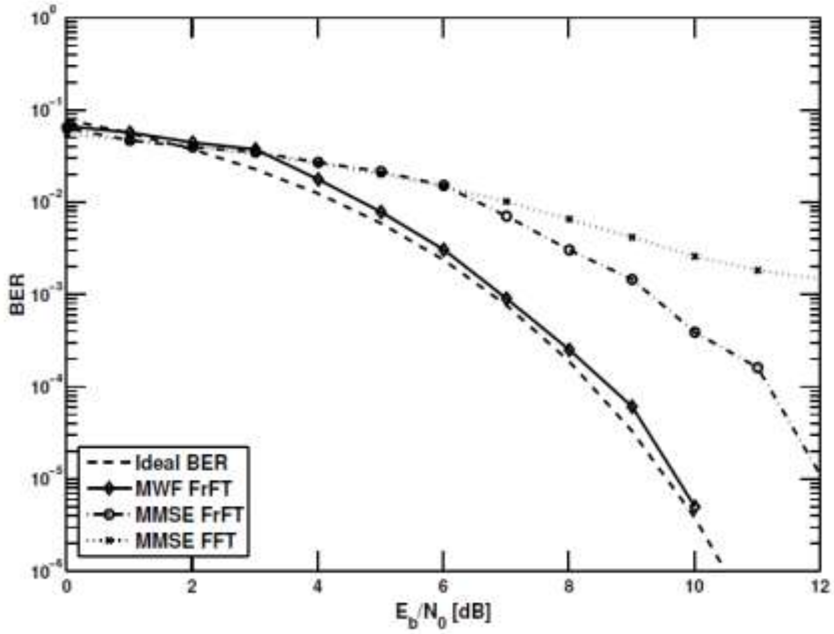


**Fig. 2**. $E_b/N_0$ [dB] vs. BER; BPSK Signal in AWGN; Gaussian Pulse Interferer ($CIR_2 = -5$ dB); no Chirp Noise ($CIR_3 = \infty$ dB)

In the third example we apply both the interfering signal $\mathbf{x}_2$ and the chirp noise signal $\mathbf{x}_3$, plus AWGN. Now we let $CIR_2 = -5$ dB, and $CIR_3 = 5$ dB. So, again the interferer is much stronger than the desired signal, but the chirp noise is not. We again choose a very small block size for best performance, so we let $N_1 = 2$, SPB = 2, and therefore N = 4. We again let the rank of the MWF-FrFT algorithm be D = 1. The performance is still not very sensitive to rank. From the BER vs. $E_b/N_0$ plot in Fig. 3, we see that the MWF-FrFT algorithm provides up to 3 dB performance improvement over the MMSE-FrFT, at reduced complexity. Due to the non-stationary interference and noise which the MMSE-FFT algorithm is not designed to handle, it fails to perform well, even as the $E_b/N_0$ is increased. Since now there is also chirp noise present, performance degrades compared to the last example, but since $CIR_3$ is fairly reasonable, 5 dB, degradation is less than 1 dB. This example demonstrates the robustness of the reduced rank technique. As we decrease $CIR_3$, the degradation in performance is graceful.

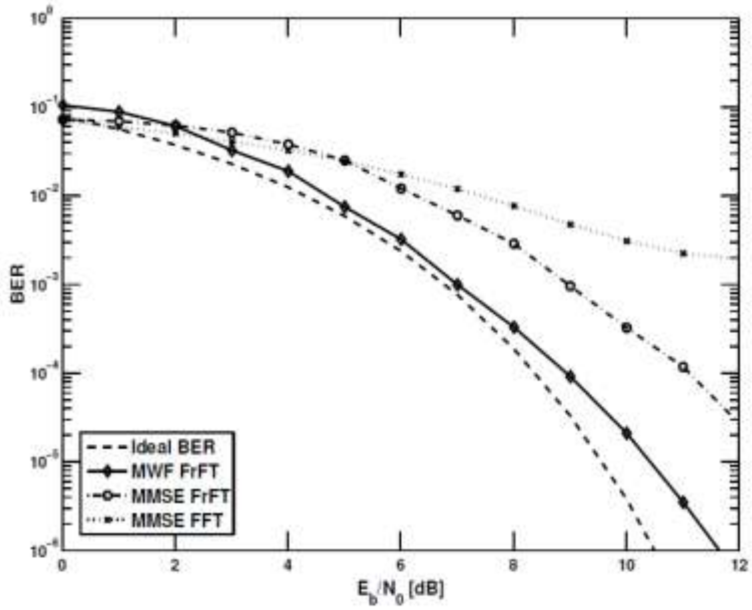


**Fig. 3** $E_b/N_0$ [dB] vs. BER; BPSK Signal in AWGN; Gaussian Pulse Interferer ($CIR_2 = -5$ dB); Chirp Noise ($CIR_3 = 5$ dB)

## 6. CONCLUSION

In this paper, we study the performance of adaptive filtering algorithms in the Fractional Fourier domain when a desired BPSK signal is corrupted by a non-stationary environment. We first apply a Fractional Fourier Transform (FrFT) to the signals and then seek the optimum filter coefficients that produce the minimum mean-square error (MMSE) between the desired signal and its estimate. We compare the full rank MMSE-FrFT method presented in [14] to a reduced rank technique based on the multistage Wiener filter, called MWF-FrFT. We also compare these to the conventional FFT-based MMSE methods (MMSE-FFT) to show the benefit of the fractional Fourier domain when interference and noise are non-stationary, and further show the performance benefit of the reduced rank MWF-FrFT, which has a computational benefit also compared to the full rank MMSE-FrFT. We show by simulation that the reduced rank method in the FrFT domain reduces the required $E_b/N_0$ by as much as 3 dB in the presence of a chirp interferer and Gaussian pulse noise signal, in addition to AWGN. The FrFT rotational parameter 'a' is found by simulation first, and future work includes researching analytical methods to determine 'a' to make these methods more feasible for real-time system implementation.

## 7. ACKNOWLEDGMENT

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