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Exact Solution of Fluid Flow through Porous Media with Variable Permeability for a Given Vorticity Distribution

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ABSTRACT

Exact solutions are sought in this work for the flow variables involved in the flow through a porous medium with variable permeability. It is assumed that the vorticity of the flow is proportional to the streamfunction. A derivation is provided for the permeability distribution when the Darcy-Lapwood-Brinkman equation is employed.

Keywords: Variable permeability, exact solutions, Brinkman equation.

1. INTRODUCTION

Variable permeability considerations in the study of flow through porous media offer a more realistic approach in the flow through natural porous settings and in the simulation of flow through porous layers, [1-3]. In fact, considerations of flow in the transition zone mandates taking a non-constant permeability in order to avoid permeability discontinuity at the interface between layers, and to circumvent invalidity arguments of Brinkman's equation, (*cf.* [4-8] and the references therein). However, when the permeability is a variable function of position then an additional variable is introduced into the governing equations. This results in an under-determined system of more unknowns than equations unless, of course, a condition on the permeability is introduced or the permeability function is defined externally or specified, [9].

When a fluid flow model such as the Darcy-Lapwood-Brinkman model is used, where its structure is similar to that of the Navier-Stokes equations, one must deal with the inherent nonlinearity that arises due to the convective inertial terms. Methods available for approximating the Navier-Stokes equations are also applicable to the Darcy-Lapwood-Brinkman model. A number of methods have been available to linearize the Navier-Stokes equations, or to solve the Navier-Stokes equations under simplifying assumptions or under the assumptions of special types of flow (*cf.* [10-24] and the references therein). An approach that has received considerable attention is the assumption of vorticity being proportional to the stream function of the two-dimensional flow, [18-20]. This approach and other methods have been used successfully to study various fluid flows, [10-16]. We will employ the assumption of vorticity proportional to the streamfunction in the current work. We thus consider the two-dimensional flow of an incompressible fluid through a porous medium with variable permeability. We obtain an exact solution to the flow equations for a given vorticity distribution. We will assume that the vorticity distribution is proportional to the stream function of the flow.

2. GOVERNING EQUATIONS

Equations governing the steady, two-dimensional flow of a viscous fluid through a porous medium with variable permeability are given by the continuity equation and the Darcy-Laprood-Brinkman equation, given respectively by:

Continuity Equation:

$$\nabla \bullet \vec{v} = 0 \qquad \ldots(1)$$

Momentum Equations:

$$\rho(\vec{v}\bullet\nabla)\vec{v} = -\nabla p + \mu^{*}\nabla^{2}\vec{v} - \frac{\mu}{k}\vec{v} \qquad \text{…(2)}$$

where $\vec{v}$ is the velocity vector, ρ is the fluid density, μ is the base viscosity of the fluid, μ^{*} is the effective velocity of the fluid in the porous medium, p is the pressure and k is the permeability. Without loss of generality of the method of solution introduced in this work, we will take $\mu^{*} = \mu$.

For the 2-dimensional flow at hand we take $\vec{v} = (u, v)$, and equations (1) and (2) can be written as:

$$u_x + v_y = 0 \qquad \text{…(3)}$$

$$uu_x + vu_y = -P_x + \frac{\mu}{\rho}\nabla^2 u - \frac{\mu}{\rho k}u \qquad \text{…(4)}$$

$$uv_x + vv_y = -P_y + \frac{\mu}{\rho}\nabla^2 v - \frac{\mu}{\rho k}v \qquad \text{...(5)}$$

where $P = \dfrac{p}{\rho}$.

Equations (3), (4) and (5) represent a system of three scalar equations in the unknowns u, v, P as functions of *x* and *y*. The variable permeability, *k*, is also an unknown function of *x* and *y*. This results in a system of equations that is underdetermined. We must therefore devise a method of solution where the permeability is determined by satisfaction of a permeability condition. This condition is derived based on the integrability condition. However, we first introduce the vorticity and streamfunction of the flow.

Continuity equation (3) implies the existence of the streamfunction ψ such that:

$$u = \psi_y \qquad \text{…(6)}$$

and

$$v = -\psi_x \qquad \text{…(7)}$$

and vorticity, ω, is defined as:

$$\omega = \nabla \times \vec{u} = v_x - u_y. \qquad \text{…(8)}$$

Using (6) and (7) in (8), we obtain the streamfunction equation:

$$\omega = -\psi_{xx} - \psi_{yy} = -\nabla^2\psi. \qquad \text{…(9)}$$

Now, using (6) and (7), equations (4) and (5) take the following forms, respectively

$$P_x + \psi_y\psi_{yx} - \psi_x\psi_{yy} = \frac{\mu}{\rho}[(\psi_{xx} + \psi_{yy})_y] - \frac{\mu}{\rho k}\psi_y. \qquad \text{…(10)}$$

$$P_y - \psi_y\psi_{xx} + \psi_x\psi_{xy} = -\frac{\mu}{\rho}[(\psi_{xx} + \psi_{yy})_x] + \frac{\mu}{\rho k}\psi_x. \qquad \text{…(11)}$$

Multiplying (9) by ψ_x and subtracting from (10) and rearranging, we obtain:

$$P_x + \psi_x\psi_{xx} + \psi_y\psi_{yx} = -\psi_x\omega + \frac{\mu}{\rho}[(\psi_{xx} + \psi_{yy})_y] - \frac{\mu}{\rho k}\psi_y. \qquad \text{…(12)}$$

Multiplying (9) by ψ_y and subtracting from (11) and rearranging, we obtain:

$$P_y + \psi_x \psi_{xy} + \psi_y \psi_{yy} = -\psi_y \omega - \frac{\mu}{\rho}[(\psi_{xx} + \psi_{yy})_x] + \frac{\mu}{\rho k}\psi_x . \qquad \text{…(13)}$$

Now, defining the following generalized pressure function

$$L = P + \frac{1}{2}(u^2 + v^2) = P + \frac{1}{2}(\psi_x{}^2 + \psi_y{}^2) \qquad \text{…(14)}$$

and differentiating (14) once with respect to x and once with respect to y, we obtain

$$L_x = p_x + (\psi_x \psi_{xx} + \psi_y \psi_{yx}) \qquad \text{…(15)}$$

$$L_y = p_y + (\psi_x \psi_{xy} + \psi_y \psi_{yy}) \qquad \text{…(16)}$$

Equation (15) is the LHS of (12) and equation (16) is the LHS of (13). Equations (12) and (13) are thus written respectively as

$$L_x = -\psi_x \omega + \frac{\mu}{\rho}[(\psi_{xx} + \psi_{yy})_y] - \frac{\mu}{\rho k}\psi_y \qquad \text{…(17)}$$

$$L_y = -\psi_y \omega - \frac{\mu}{\rho}[(\psi_{xx} + \psi_{yy})_x] + \frac{\mu}{\rho k}\psi_x \qquad \text{…(18)}$$

The governing equations are thus (17) and (18), in the unknowns ω and ψ, with the vorticity defined by (9). Continuity equation (3) is automatically satisfied by virtue of introducing the streamfunction in (6) and (7). Once ω and ψ are determined, velocity components can be calculated from (6) and (7). The pressure, $P(x, y)$, can then be determined from (14) once the generalized pressure function, $L(x, y)$, is determined from (17) and (18).

3. METHOD OF SOLUTION

3.1. Integrability Condition and Permeability Equation

In this work we assume that the voticity is proportional to the streamfunction of the flow. We thus assume that:

$$\omega = -\alpha \psi \qquad \text{…(19)}$$

where α is a nonzero constant.

Using (19) in (9), (17) and (18), we obtain, respectively:

$$\psi_{xx} + \psi_{yy} = \alpha \psi \qquad \text{…(20)}$$

$$L_x = -\psi_x \omega + [\frac{\alpha\mu}{\rho} - \frac{\mu}{\rho k}]\psi_y = -\psi_x \omega + A\psi_y \qquad \text{…(21)}$$

$$L_y = -\psi_y \omega + [\frac{\mu}{\rho k} - \frac{\alpha\mu}{\rho}]\psi_x = -\psi_y \omega - A\psi_x \qquad \text{…(22)}$$

where

$$A = [\frac{\alpha\mu}{\rho} - \frac{\mu}{\rho k}] . \qquad \text{…(23)}$$

From (21) and (22) we obtain:

$$L_{xy} = -\psi_{xy}\omega - \psi_x \omega_y + A_y \psi_y + A\psi_{yy} \quad \ldots(24)$$

$$L_{yx} = -\psi_{yx}\omega - \psi_y \omega_x - A_x \psi_x - A\psi_{xx}. \quad \ldots(25)$$

Setting $L_{xy} = L_{yx}$ yields the following integrability condition:

$$-\psi_x \omega_y + \psi_y \omega_x + A_y \psi_y + A_x \psi_x + A\alpha\psi = 0. \quad \ldots(26)$$

Integrability condition (26) must be met if (19) is to hold and (17) and (18) are satisfied.

Now, from (19) we get

$$\omega_x = -\alpha\psi_x \quad \ldots(27)$$

$$\omega_y = -\alpha\psi_y. \quad \ldots(28)$$

Upon using (27) and (28) in (26), we obtain

$$A_y \psi_y + A_x \psi_x + A\alpha\psi = 0. \quad \ldots(29)$$

Equation (29) is to be solved for A (hence the permeability function) once the form of ψ is determined.

3.2. Determination of Streamfunction, Vorticity and Velocity Components

In order to determine ψ, we rely on equation (20), which is a Helmholtz equation that admits plane wave solution of the form, [10-16]:

$$\psi(x, y) = \varphi(\xi) \quad \ldots(30)$$

where

$$\xi = x\cos\vartheta + y\sin\vartheta; \ -\pi \le \vartheta < \pi. \quad \ldots(31)$$

From (31) we obtain the following derivatives:

$$\xi_x = \cos\vartheta; \xi_y = \sin\vartheta; \xi_{xx} = 0; \xi_{yy} = 0. \quad \ldots(32)$$

Using the chain rule, the following derivatives of ψ are established:

$$\psi_x = \varphi'(\xi)\cos\vartheta \quad \ldots(33)$$
$$\psi_y = \varphi'(\xi)\sin\vartheta \quad \ldots(34)$$
$$\psi_{xx} = \varphi''(\xi)\cos^2\vartheta \quad \ldots(35)$$
$$\psi_{yy} = \varphi''(\xi)\sin^2\vartheta \quad \ldots(36)$$

Using (30), (35) and (36) in (20), we obtain

$$\varphi''(\xi) = \alpha\varphi(\xi). \quad \ldots(37)$$

Equation (37) is a homogeneous, second order ordinary differential equation with auxiliary equation given by

$$r^2 - \alpha = 0. \quad \ldots(38)$$

Solution to (38) gives rise to the following cases:

1) $\alpha = -n^2; n > 0$
2) $\alpha = m^2; m > 0$

Case 1: When $\alpha = -n^2; n > 0$

Solution to (37) takes the form

$$\varphi(\xi) = C_1(\vartheta)\cos(n\xi + C_2(\vartheta)) \quad \ldots(39)$$

where $C_1(\vartheta)$ and $C_2(\vartheta)$ are real constants that depend on $\vartheta \in [-\pi, \pi)$.

Using (30), (331) and (39), we obtain

$$\psi(x, y) = C_1(\vartheta)\cos(n(x\cos\vartheta + y\sin\vartheta) + C_2(\vartheta)). \quad \ldots(40)$$

Velocity components are then obtained from (6), (7) and (40), respectively as

$$u(x, y) = \psi_y(x, y) = -C_1(\vartheta)n\sin\vartheta\sin(n(x\cos\vartheta + y\sin\vartheta) + C_2(\vartheta)) \quad \ldots(41)$$

$$v(x, y) = -\psi_x = C_1(\vartheta)n\cos\vartheta\sin(n(x\cos\vartheta + y\sin\vartheta) + C_2(\vartheta)) \quad \ldots(42)$$

and the vorticity is obtained using (19) and (40) as

$$\omega = -\alpha C_1(\vartheta)\cos(n(x\cos\vartheta + y\sin\vartheta) + C_2(\vartheta)). \quad \ldots(43)$$

Case 2: When $\alpha = m^2; m > 0$

Solution to (37) takes the form

$$\varphi(\xi) = B_1(\vartheta)\exp[m\xi] + B_2(\vartheta)\exp[-m\xi] \quad \ldots(44)$$

where $B_1(\vartheta)$ and $B_2(\vartheta)$ are real constants that depend on $\vartheta \in [-\pi, \pi)$.
Using (30), (31) and (44), we obtain

$$\psi(x, y) = B_1(\vartheta)\exp[m(x\cos\vartheta + y\sin\vartheta)] + B_2(\vartheta)\exp[-m(x\cos\vartheta + y\sin\vartheta)]. \quad \ldots(45)$$

Velocity components are then obtained from (6), (7) and (45), respectively as

$$u(x, y) = mB_1(\vartheta)\sin\vartheta\exp[m(x\cos\vartheta + y\sin\vartheta)] - mB_2(\vartheta)\sin\vartheta\exp[-m(x\cos\vartheta + y\sin\vartheta)] \quad \ldots(46)$$

$$v(x, y) = -mB_1(\vartheta)\cos\vartheta\exp[m(x\cos\vartheta + y\sin\vartheta)] + mB_2(\vartheta)\cos\vartheta\exp[-m(x\cos\vartheta + y\sin\vartheta)] \quad \ldots(47)$$

and the vorticity is obtained using (19) and (45) as

$$\omega = -\alpha[B_1(\vartheta)\exp[m(x\cos\vartheta + y\sin\vartheta)] + B_2(\vartheta)\exp[-m(x\cos\vartheta + y\sin\vartheta]]. \quad \ldots(48)$$

3.3. Determination of the Permeability Function

Equation (29) must be satisfied by the streamfunction and the permeability function. With the knowledge of the streamfunction, for the two cases discussed above and given by equations (40) and (45), we substitute the streamfunction expressions in equation (29) and determine the permeability function.

Using (33) and (34), equation (29) takes the form

$$A_y[\varphi'(\xi)\sin\vartheta]+A_x[\varphi'(\xi)\cos\vartheta]+A\alpha[\varphi(\xi)]=0. \qquad \ldots(49)$$

Case 1:

Using (31) and (39), we write (54) as

$$\begin{aligned}&A_y[-nC_1(\vartheta)\sin(n\xi+C_2(\vartheta))\sin\vartheta]+A_x[-nC_1(\vartheta)\sin(n\xi+C_2(\vartheta))\cos\vartheta]\\&+A\alpha[C_1(\vartheta)\cos(n\xi+C_2(\vartheta))]=0\end{aligned}. \qquad \ldots(50)$$

Dividing (50) by $C_1(\vartheta)\sin(n\xi+C_2(\vartheta))$, we obtain

$$A_y[-n\sin\vartheta]+A_x[-n\cos\vartheta]+A\alpha[\cot(n\xi+C_2(\vartheta))]=0. \qquad \ldots(51)$$

Now, using the chain rule, we obtain

$$A_x=A_\xi\cos\vartheta \qquad \ldots(52)$$
$$A_y=A_\xi\sin\vartheta. \qquad \ldots(53)$$

Using (52) and (53) in (51), and simplifying, we obtain

$$A_\xi-A\frac{\alpha}{n}[\cot(n\xi+C_2(\vartheta))]=0. \qquad \ldots(54)$$

Using separation of variables, we write (54) as

$$\frac{dA}{A}=\frac{\alpha}{n}[\cot(n\xi+C_2(\vartheta))]d\xi. \qquad \ldots(55)$$

Solution to (55) is given by

$$\ln A=\frac{\alpha}{n^2}\ln[\sin(n\xi+C_2(\vartheta))]+\ln C_3(\vartheta) \qquad \ldots(56)$$

which we write as

$$A=C_3(\vartheta)[\sin(n\xi+C_2(\vartheta))]^{\frac{\alpha}{n^2}}=C_3(\vartheta)[\sin(n((x\cos\vartheta+y\sin\vartheta))+C_2(\vartheta))]^{\frac{\alpha}{n^2}} \qquad \ldots(57)$$

where $C_3(\vartheta)$ is an arbitrary function of $\vartheta\in[-\pi,\pi)$.

Using (23) and (57), we obtain the permeability function as:

$$k(x,y)=\frac{\mu}{\alpha\mu-\rho A}=\frac{\mu}{\alpha\mu-\rho\left\{C_3(\vartheta)[\sin(n((x\cos\vartheta+y\sin\vartheta))+C_2(\vartheta))]^{\frac{\alpha}{n^2}}\right\}}. \qquad \ldots(58)$$

Case 2:

Using (31) and (44), we write (49) as

$$A_y \sin\vartheta[mB_1(\vartheta)\exp(m\xi) - mB_2(\vartheta)\exp(-m\xi)] + A_x \cos\vartheta[mB_1(\vartheta)\exp(m\xi) - mB_2(\vartheta)\exp(-m\xi)] + A\alpha[B_1(\vartheta)\exp(m\xi) + B_2(\vartheta)\exp(-m\xi)] = 0 \quad ...(59)$$

Using (52) and (53), we write (59) as:

$$mA_\xi[B_1(\vartheta)\exp(m\xi) - B_2(\vartheta)\exp(-m\xi)] + A\alpha[B_1(\vartheta)\exp(m\xi) + B_2(\vartheta)\exp(-m\xi)] = 0 \quad ...(60)$$

Equation (60) is variable separable and can be written as

$$\frac{dA}{A} = -\frac{\alpha}{m}\frac{[B_1(\vartheta)\exp(m\xi) + B_2(\vartheta)\exp(-m\xi)]}{[B_1(\vartheta)\exp(m\xi) - B_2(\vartheta)\exp(-m\xi)]}d\xi \quad ...(61)$$

whose solution is given by

$$\ln A = -\frac{\alpha}{m^2}\ln[B_1(\vartheta)\exp(m\xi) - B_2(\vartheta)\exp(-m\xi)] + \ln B_3(\vartheta) \quad ...(62)$$

or

$$A = B_3(\vartheta)[B_1(\vartheta)\exp(m\xi) - B_2(\vartheta)\exp(m\xi)]^{-\frac{\alpha}{m^2}}. \quad ...(63)$$

Permeability function is then obtained by substituting (63) in (23) to obtain

$$k(x,y) = \frac{\mu}{\alpha\mu - \rho A}$$

$$= \frac{\mu}{\alpha\mu - \rho\left\{B_3(\vartheta)[B_1(\vartheta)\exp(m(x\cos\vartheta + y\sin\vartheta)) - B_2(\vartheta)\exp(m(x\cos\vartheta + y\sin\vartheta))]^{-\frac{\alpha}{m^2}}\right\}}. \quad ...(64)$$

3.4. Determination of Pressure

Equations (21) and (22) take the following forms in terms of the variable ξ :

$$L_\xi \cos\vartheta = \alpha\cos\vartheta[\varphi'(\xi)][\varphi(\xi)] + A[\varphi'(\xi)\sin\vartheta] \quad ...(65)$$

$$L_\xi \sin\vartheta = \alpha\sin\vartheta[\varphi'(\xi)][\varphi(\xi)] - A[\varphi'(\xi)\cos\vartheta]. \quad ...(66)$$

Multiply (65) by $\cos\vartheta$ and (66) by $\sin\vartheta$, wand adding, e get

$$L_\xi = \alpha[\varphi'(\xi)][\varphi(\xi)]. \quad ...(67)$$

Integrating (73) we get

$$L(\xi) = \frac{\alpha}{2}[\varphi(\xi)]^2 + C_4(\vartheta) \quad ...(68)$$

where $C_4(\vartheta)$ is an arbitrary function of $\vartheta \in [-\pi, \pi)$.

The pressure is then determined from equation (14) as

$$P = L - \frac{1}{2}(u^2 + v^2) \quad ...(69)$$

where L is given in (68) and u and v are given in terms of ξ as:

$$v = -\varphi'(\xi)\cos\vartheta \qquad \ldots(70)$$
$$u = \varphi'(\xi)\sin\vartheta. \qquad \ldots(71)$$

Using (68), (70) and (71) in (69), we obtain the following expression for pressure:

$$P = \frac{\alpha}{2}[\varphi(\xi)]^2 - \frac{1}{2}[\varphi'(\xi)]^2 + C_4(\vartheta). \qquad \ldots(72)$$

Corresponding to solutions (39) and (44), the pressure function takes the following forms. In case 1, using (39) in (72) gives:

$$P(\xi) = \frac{\alpha}{2}[C_1(\vartheta)\cos(n\xi + C_2(\vartheta))]^2 - \frac{n^2}{2}[C_1(\vartheta)\sin(n\xi + C_2(\vartheta))]^2 + C_4(\vartheta) \qquad \ldots(73)$$

and, upon using (31), we obtain

$$P(x,y) = \frac{\alpha}{2}[C_1(\vartheta)\cos(n(x\cos\vartheta + y\sin\vartheta) + C_2(\vartheta))]^2 - \frac{n^2}{2}[C_1(\vartheta)\sin(n(x\cos\vartheta + y\sin\vartheta) + C_2(\vartheta))]^2 + C_4(\vartheta). \qquad \ldots(74)$$

In Case 2, using (44) in (72) gives:

$$P(\xi) = \frac{\alpha}{2}[B_1(\vartheta)\exp(m\xi) + B_2(\vartheta)\exp(-m\xi)]^2 - \frac{m^2}{2}[B_1(\vartheta)\exp(m\xi) - B_2(\vartheta)e(-m\xi)]^2 + C_4(\vartheta) \qquad \ldots(75)$$

and, upon using (31), we obtain

$$P(x,y) = \frac{\alpha}{2}[B_1(\vartheta)\exp[m(x\cos\vartheta + y\sin\vartheta)]^2 + B_2(\vartheta)\exp[m(x\cos\vartheta + y\sin\vartheta)]] - \frac{m^2}{2}\{B_1(\vartheta)\exp[m(x\cos\vartheta + y\sin\vartheta)] - B_2(\vartheta)\exp[-m(x\cos\vartheta + y\sin\vartheta)]\}^2 + C_4(\vartheta). \qquad \ldots(76)$$

3.5. Summary of Solutions

Flow variables based on the two cases of solution are summarized as follows, where we keep their equation numbers as they appeared in the text.

Case 1:

$$\psi(x,y) = C_1(\vartheta)\cos(n(x\cos\vartheta + y\sin\vartheta) + C_2(\vartheta)). \qquad \ldots(40)$$

$$u(x,y) = \psi_y(x,y) = -C_1(\vartheta)n\sin\vartheta\sin(n(x\cos\vartheta + y\sin\vartheta) + C_2(\vartheta)) \qquad \ldots(41)$$

$$v(x,y) = -\psi_x = C_1(\vartheta)n\cos\vartheta\sin(n(x\cos\vartheta + y\sin\vartheta) + C_2(\vartheta)) \qquad \ldots(42)$$

$$\omega = -\alpha C_1(\vartheta)\cos(n(x\cos\vartheta + y\sin\vartheta) + C_2(\vartheta)). \qquad \ldots(43)$$

$$k(x,y) = \frac{\mu}{\alpha\mu - \rho A} = \frac{\mu}{\alpha\mu - \rho\left\{C_3(\vartheta)[\sin(n((x\cos\vartheta + y\sin\vartheta)) + C_2(\vartheta))]^{\frac{\alpha}{n^2}}\right\}}. \qquad \ldots(58)$$

$$P(x,y) = \frac{\alpha}{2}[C_1(\vartheta)\cos(n(x\cos\vartheta + y\sin\vartheta) + C_2(\vartheta))]^2 - \frac{n^2}{2}[C_1(\vartheta)\sin(n(x\cos\vartheta + y\sin\vartheta) + C_2(\vartheta))]^2 + C_4(\vartheta). \qquad \ldots(74)$$

Case 2:

$$\psi(x,y)=B_1(\vartheta)\exp[m(x\cos\vartheta+y\sin\vartheta)]+B_2(\vartheta)\exp[-m(x\cos\vartheta+y\sin\vartheta)]. \quad \ldots(45)$$

$$u(x,y)=mB_1(\vartheta)\sin\vartheta\exp[m(x\cos\vartheta+y\sin\vartheta)]-mB_2(\vartheta)\sin\vartheta\exp[-m(x\cos\vartheta+y\sin\vartheta)] \quad \ldots(46)$$

$$v(x,y)=-mB_1(\vartheta)\cos\vartheta\exp[m(x\cos\vartheta+y\sin\vartheta)]+mB_2(\vartheta)\cos\vartheta\exp[-m(x\cos\vartheta+y\sin\vartheta)] \quad \ldots(47)$$

$$\omega=-\alpha[B_1(\vartheta)\exp[m(x\cos\vartheta+y\sin\vartheta)]+B_2(\vartheta)\exp[-m(x\cos\vartheta+y\sin\vartheta]]. \quad \ldots(48)$$

$$k(x,y)=\frac{\mu}{\alpha\mu-\rho A}$$
$$=\frac{\mu}{\alpha\mu-\rho\left\{B_3(\vartheta)[B_1(\vartheta)\exp(m(x\cos\vartheta+y\sin\vartheta))-B_2(\vartheta)\exp(m(x\cos\vartheta+y\sin\vartheta))]^{-\frac{\alpha}{m^2}}\right\}}. \quad \ldots(64)$$

$$P(x,y)=\frac{\alpha}{2}[B_1(\vartheta)\exp[m(x\cos\vartheta+y\sin\vartheta)]^2+B_2(\vartheta)\exp[m(x\cos\vartheta+y\sin\vartheta)]]$$
$$-\frac{m^2}{2}\{B_1(\vartheta)\exp[m(x\cos\vartheta+y\sin\vartheta)]-B_2(\vartheta)\exp[-m(x\cos\vartheta+y\sin\vartheta)]\}^2+C_4(\vartheta). \quad \ldots(76)$$

The above solutions do not involve the permeability functions explicitly. This is due to the fact that the permeability function was derived based on an ingerability condition that in terms of a pre-determined streamfunction that was determined independent of the permeability. The above solutions for the streamfunction, vorticity, velocity and pressure are valid for both cases of constant permeability and variable permeability. In fact, they are valid for all permeability range (in particular, as permeability approached infinity and the flow reduces to the Navier-Stokes flow). In other words, these are the same solutions to the Navier-Stokes flow when vorticity is proportional to the streamfunction. We therefore interpret the permeability functions as the needed permeability distributions to generate the flow variables given in the above equations.

Typical three-dimensional plots for the above solutions are given below to illustrate the effects of the parameters appearing in **Case 1** and **Case 2**. **Figures 1.1-1.8** are for the **Case 1** results and **Figures 2.1-2.8** are for **Case 2**.

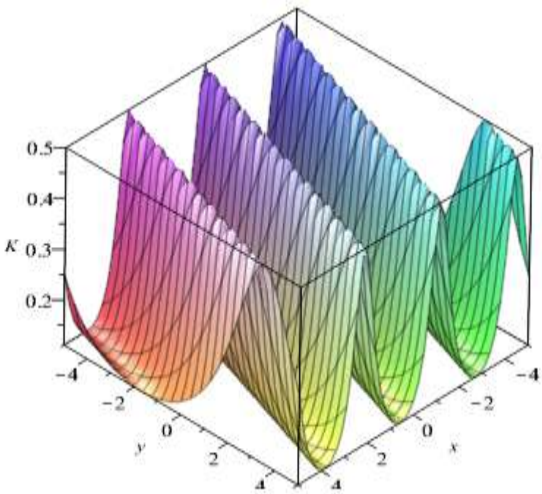

Fig. 1.1. Case 1 Permeability Distribution

$\alpha=1,\ D_1(\vartheta)=-7,\ n=1,\ \vartheta=-\pi/3,\ C_2(\vartheta)=-1$

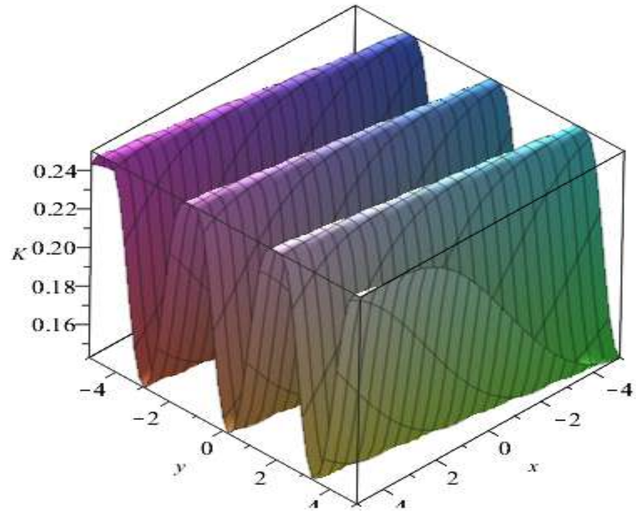

Fig. 1.2. Case 1 Permeability Distribution

$\alpha = 1,\ D_1(\vartheta) = -3,\ n = 1,\ \vartheta = -\pi/6,\ C_2(\vartheta) = -1$

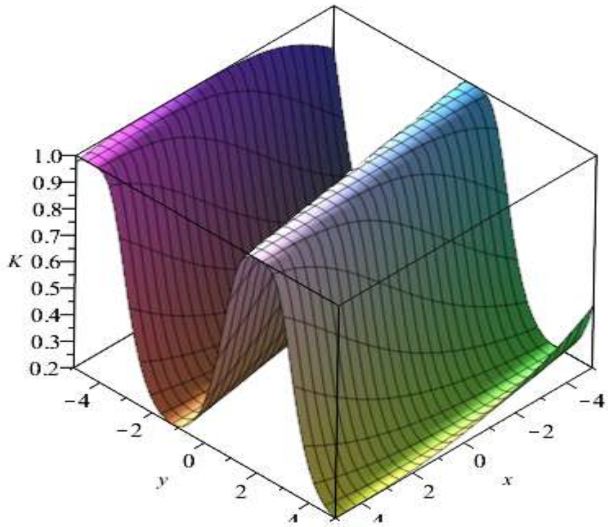

Fig. 1.3. Case 1 Permeability Distribution

$\alpha = 1,\ D_1(\vartheta) = -4,\ n = 0.5,\ \vartheta = \pi/6,\ C_2(\vartheta) = -1$

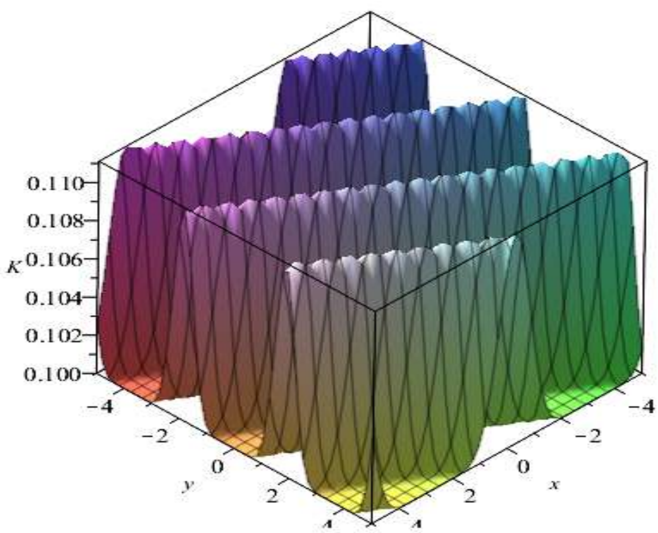

Fig. 1.4. Case 1 Permeability Distribution

$\alpha = 10,\ D_1(\vartheta) = 1,\ n = 1,\ \vartheta = \pi/4,\ C_2(\vartheta) = -1$

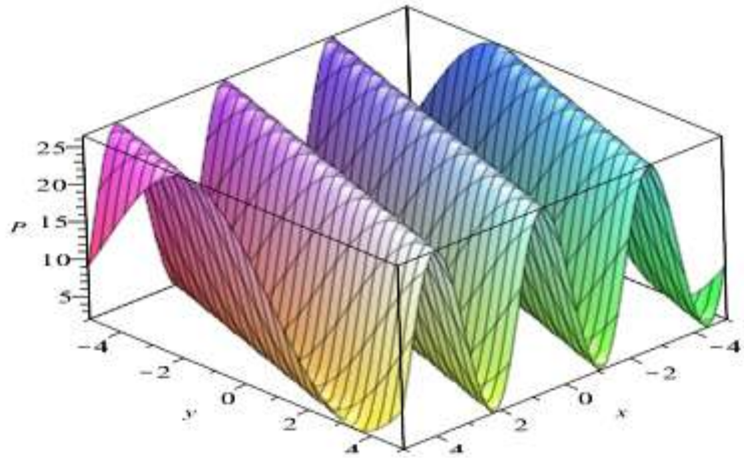

Fig. 1.5. Case 1 Pressure

$\alpha = 2$, $C_1(\vartheta) = -7$, $n = 1$, $\vartheta = -\pi/3$, $C_2(\vartheta) = -1$, $C_4(\vartheta) = 2$

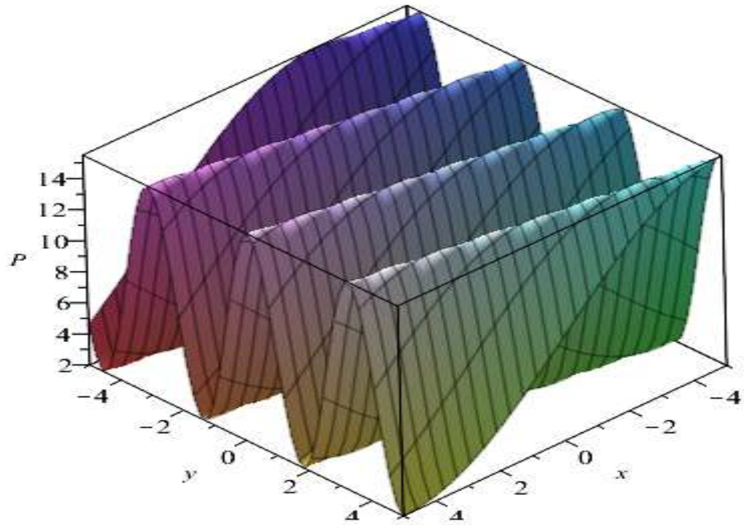

Fig. 1.6. Case 1 Pressure

$\alpha = 4$, $C_1(\vartheta) = -3$, $n = 1$, $\vartheta = -\pi/6$, $C_2(\vartheta) = -1$, $C_4(\vartheta) = 2$

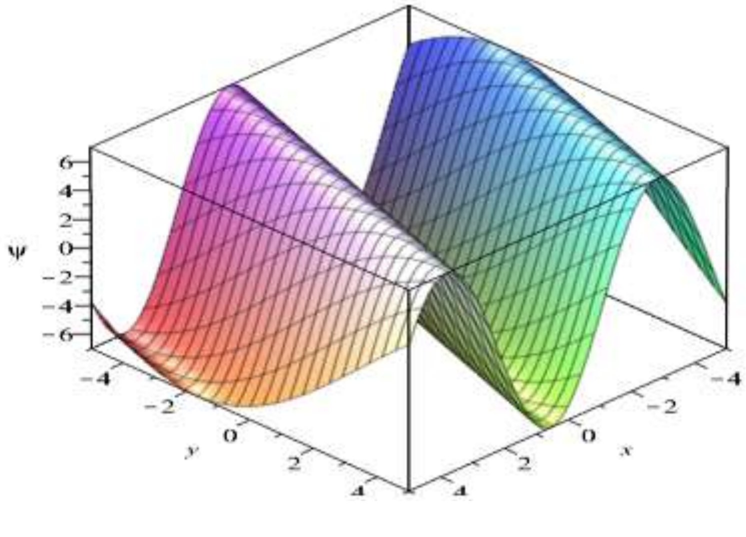

Fig. 1.7. Case 1 Streamsurface

$\alpha = 1$, $D_1(\vartheta) = -7$, $n = 1$, $\vartheta = -\pi/3$, $C_2(\vartheta) = -1$

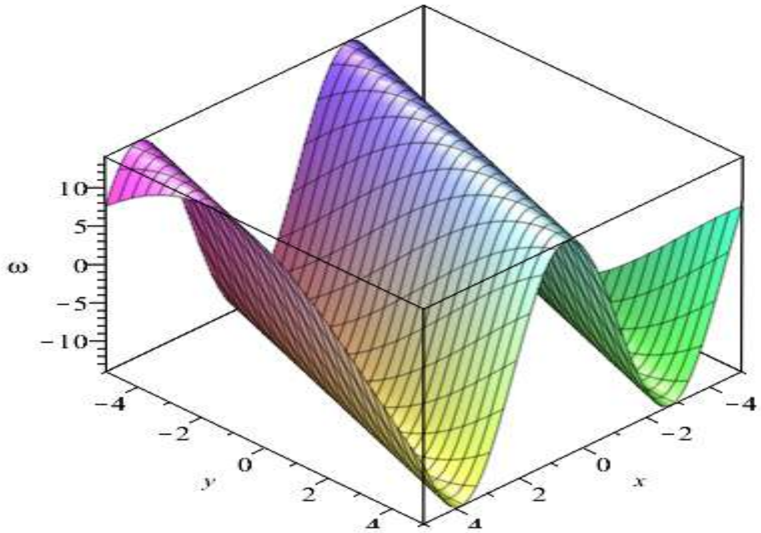

Fig. 1.8. Case 1 Vorticity Distribution

$\alpha = 1,\ C_1(\vartheta) = -7,\ n = 1,\ \vartheta = -\pi/3,\ C_2(\vartheta) = -1$

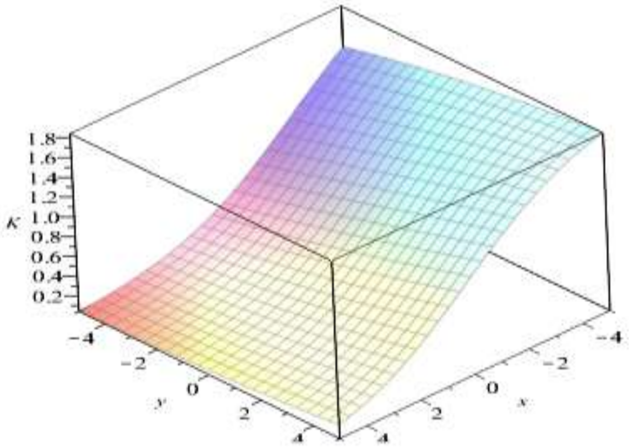

Fig. 2.1. Case 2 Permeability Distribution

$\alpha = 0.5,\ B_1(\vartheta) = 0.5,\ B_2(\vartheta) = -0.5,\ m = 1, \vartheta = -\pi/3, D_2(\vartheta) = -1$

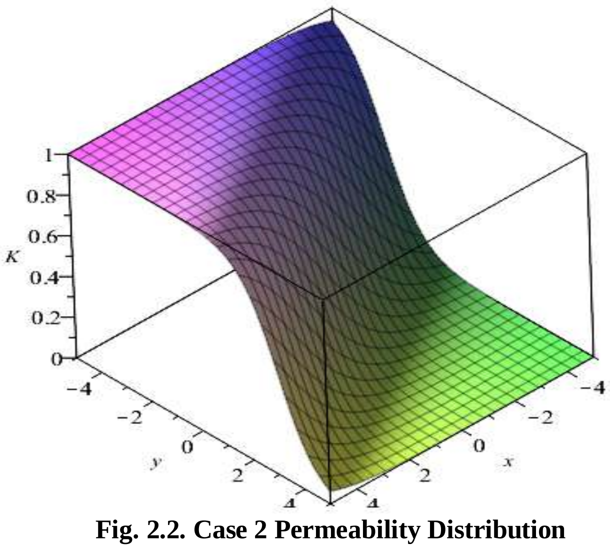

Fig. 2.2. Case 2 Permeability Distribution

$\alpha = 1,\ B_1(\vartheta) = 0.5,\ B_2(\vartheta) = -0.5,\ m = 0.6, \vartheta = -\pi/4,\ D_2(\vartheta) = -1$

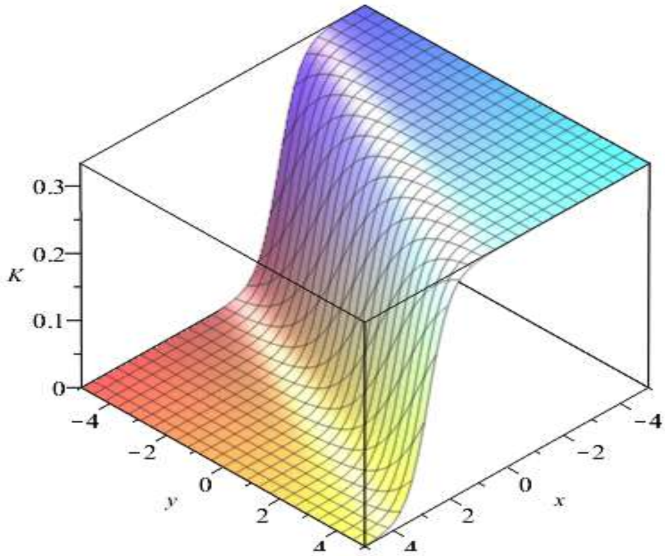

Fig. 2.3. Case 2 Permeability Distribution

$\alpha = 3$, $B_1(\vartheta) = 0.5$, $B_2(\vartheta) = -0.5$, $m = 1$, $\vartheta = -7\pi/36$, $D_2(\vartheta) = -1$

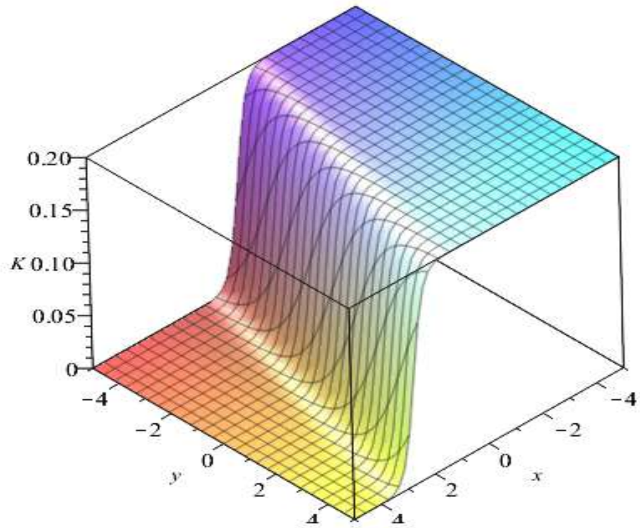

Fig. 2.4. Case 2 Permeability Distribution

$\alpha = 5$, $B_1(\vartheta) = 1$, $B_2(\vartheta) = -1$, $m = 0.9$, $\vartheta = -\pi/3$, $D_2(\vartheta) = -1$

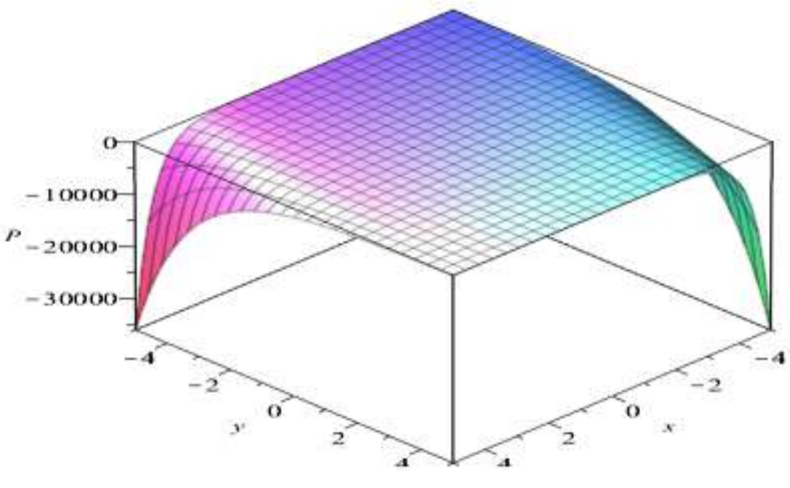

Fig. 2.5. Case 2 Pressure

$\alpha = 0.5$, $B_1(\vartheta) = 0.5$, $B_2(\vartheta) = -0.5$, $m = 1$, $\vartheta = -\pi/3$, $C_4(\vartheta) = -1$

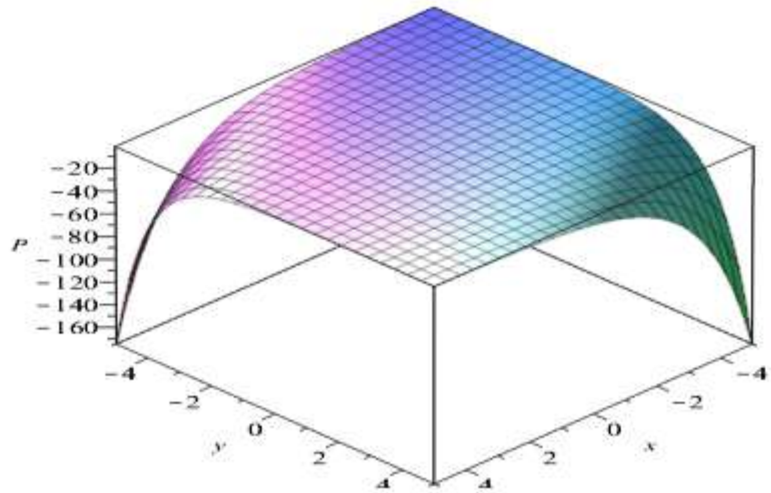

Fig. 2.6. Case 2 Pressure

$\alpha = 1,\ B_1(\vartheta) = 0.5,\ B_2(\vartheta) = -0.5,\ m = 0.6,\ \vartheta = -\pi/4,\ C_4(\vartheta) = -1$

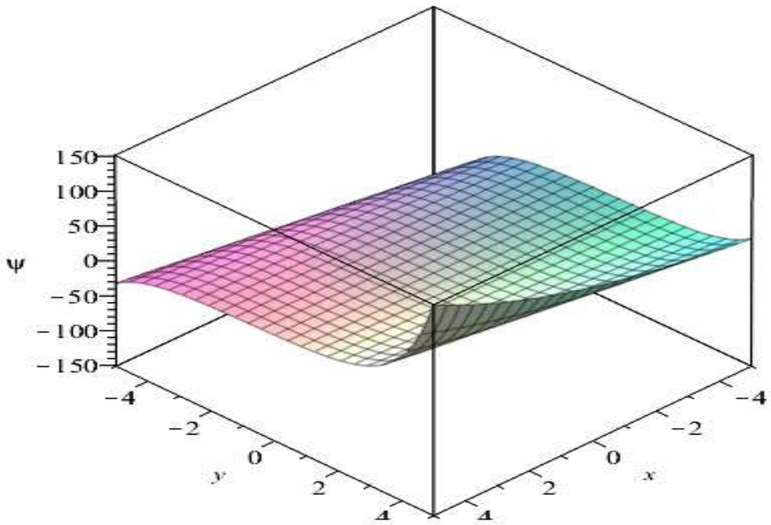

Fig. 2.7. Case 2 Streamsurfaces

$\alpha = 4,\ B_1(\vartheta) = 0.5,\ B_2(\vartheta) = -0.5,\ m = 1,\ \vartheta = -\pi/6$

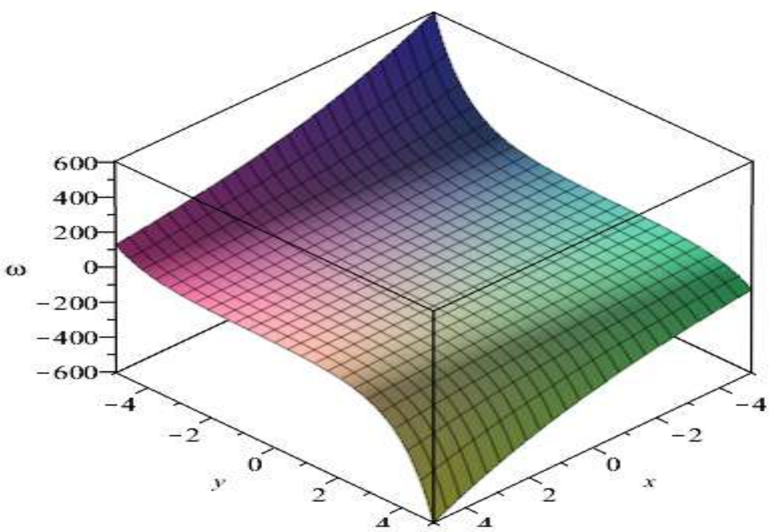

Fig. 2.8. Case 2 Vorticity Distribution

$\alpha = 4,\ B_1(\vartheta) = 0.5,\ B_2(\vartheta) = -0.5,\ m = 1,\ \vartheta = -\pi/6$

4. CONCLUSION

The main theme of this work has been the devising of a method to obtain the permeability distribution in a variable permeability porous medium. To accomplish this, exact solutions were obtained under the assumption of vorticity being a function of the streamfunction of the flow. Expressions for the permeability, pressure, velocity components, streamfunction and vorticity were successfully obtained. Three-dimensional figures are provided to illustrate the distributions obtained.

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