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Solving the Traveling Salesman Problem with Exact Methods

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ABSTRACT

When it comes to operational research, we necessarily cross on our way the Traveling Salesman Problem (TSP). It is characterized by its simple appearance and its statement that does not demand a lot of mathematical knowledge to understand and a high level of reflection in order to find one of its solutions, it can be seen as a simple problem of a commercial traveler seeking the shortest tour to gain more energy and time and to have more money, but the TSP has a great importance in the formulation of complex problems. In this work we are interested to do a comparative study between the various formulations of the TSP: DFJ (Danzig-Fulkerson-Johnson), MTZ (Miller-Tucker-Zemlin) and DL (Desrochers-Laporte). For this we have made the resolution with exact methods.

Keywords: Branch and Cut method, Linear integer programming, Traveling Salesman Problem.

1. INTRODUCTION

The Combinatorial optimization problems (COP) are of considerable importance in the scientific and industrial world, among the best-known problems in COP, we can cite the traveling salesman problem (TSP). It was studied in the 18th century by a mathematician from Ireland named Sir William Rowam Hamilton, his statement is as follows: "a traveling salesman must visit once and only once a finite number of cities and return to its point of origin, find the order of cities visited that minimizes the total distance traveled by the traveler "(Fig.1). This problem was also treated by the British mathematician named Thomas Penyngton Kirkman, detailed discussion about the work of Hamilton & Kirkman can be seen from the book titled Graph Theory, Biggs et al **[1]**. It is believed that the general form of the TSP have been first studied by Kalr Menger **[2]** in Vienna and Harvard. The TSP was later promoted by Hassler, Whitney & Merrill at Princeton. A detailed description about the connection between Menger & Whitney, and the development of the TSP can be found in (Schrijver, **[3]**). The study of this problem has attracted many researchers from different fields, e.g., Mathematics, Operations Research, Physics, Biology, or Artificial Intelligence, and there is a vast amount of literature on it.

Hundreds of articles and several books have been written about the TSP. In particular, the three books of Lawler et al. **[4]** listed nearly 600 references**.** Gutin and Punnen **[5]**, Applegate et al**. [6],** Jünger et al. **[7]** provide a wealth of information on this problem. Many formulations of the TSP can be found in the literature for example, those of Orman and Williams **[8]**,Oncan et al. **[9].**Also those of Danzig et al. **[10]** are the first formulations of the TSP. In 1998, basing on the demonstration of Papadimitriou **[11]**,Arora showed that the TSP Euclidean is part of NP-hard optimization problems**[12],** if the number of cities increases, the complexity becomes extremely important .This complexity did not facilitate his study and if you analyze all the possible routes, for n cities, the number of possibilities is of (n-1)! , for 6 cities we have 120 possibilities, for 10 cities we have more than 362 000 possibilities, and for 60 cities we have 10^{80}, is the number of estimated atoms in the universe. This may explain why the problem has not been studied seriously before the arrival of computers in universities, but many researchers have treated it since the 20th century.

In the reality, the TSP is the basis of the all routing problems and has a multitude real-world application, Lenstra and Kan **[13],** Reinelt **[14]** cited a set of direct and indirect applications of the TSP in several industrial and technological fields. A classic example for these applications is the scheduling of a machine to drill holes in a circuit board or other object. In this case the holes to be drilled are the cities, and the cost of travel is the time it takes to move the drill head from one hole to the next. The technology for drilling varies from one industry to another, but whenever the travel time of the drilling device is a significant portion of the overall manufacturing process then the TSP can play a role in reducing costs. This problem can be also useful in crystallography to minimize time used to take measures to the X-ray **[15].**

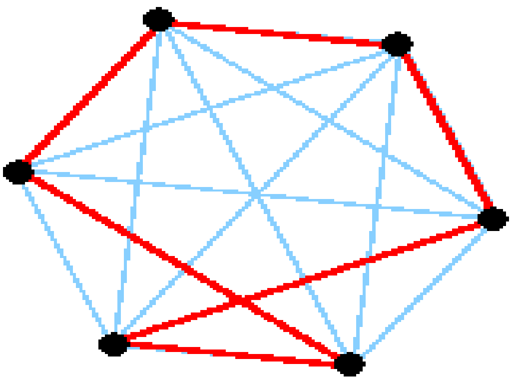

Figure 1: Solution of the traveling salesman problem: the red line is the shortest path that connects all the black points (cities).

2. DESCRIPTION OF TRAVELING SALESMAN PROBLEM:

The TSP is defined on a complete and directed graph G= (V, A), because we treat Asymmetric TSP (ATSP) (See section 3), the set V = {1 . . . n} is the vertex set (cities) with n is the number of cities, and $A = \{(i, j) : i, j \in V, i \neq j\}$ is an arc set; a cost matrix d_{ij} is defined on A. The cost matrix d_{ij} satisfies the triangle inequality whenever $d_{ij} \leq d_{ik} + d_{kj}$, for all i, j, k. In particular, this is the case of planar problems for which the vertices are points $P_i = (x_i, y_i)$ in the plane, and $d_{ij} = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2}$ is the Euclidean distance, the triangle inequality is also satisfied if d_{ij} is the length of a shortest path from i to j on G. The TSP then returns to the search for a Hamiltonian circuit (closed path passing exactly once by each vertex of the graph) of minimum length in G. In general, the TSP can be formulated by associating to each arc (i, j) of the visited city $(i = 1 \text{ to } n, j = 1 \text{ to } n, i \neq j)$ a binary succession variable x_{ij}, which takes 1 if city j is visited immediately after city i in the tour and which otherwise takes 0, with $N^+(i)$ that are the successors of i.The basic idea to formulate mathematically the Traveling Salesman Problem consists , starting from, a problem of assignment: each city i is assigned to city j which must follow it in the tour , then we get the formulation:

$$\min \sum_{i \neq j} d_{ij} x_{ij} \tag{1}$$

Subject to

$$\sum_{j \in N^+(i)} x_{ij} = 1 \quad \forall i \in V \tag{2}$$

$$\sum_{i \in N^+(j)} x_{ij} = 1 \quad \forall j \in V \tag{3}$$

$$x_{ij} \in \{0;1\} \quad \forall (i, j) \in A \tag{4}$$

In this formulation, (1) defines the objective function , the constraints (2) and (3) reflect that each city should be visited exactly once, (4) is an integrality constraint. This formulation is not valid in the sense that a solution does not match necessarily to a complete tour : In general it consists in several cycles (subtours) , it is obligatory to supplement this formulation so as to prohibit subtours, this can be done conventionally in two ways, which were proposed respectively by Danzig, Fulkerson & Johnson **[16]** , who were among the first to do the modelling and on other side Miller, Tucker & Zemlin **[17]** have rewritten the problem in a different way, to get two formulations of the TSP: the DFJ Formulation (Danzig, Fulkerson, & Johnson) and the MTZ formulation (Miller, Tucker &Zemlin), after a few years Desrochers and Laporte have strengthened the MTZ formulation by proposing the formulation DL **[18].**

3. CLASSIFICATION OF TRAVELING SALESMAN PROBLEM:

Broadly, the TSP is classified as Symmetric Travelling Salesman Problem (STSP), Asymmetric Travelling Salesman Problem (ATSP), and Multi Travelling Salesman Problem (MTSP) **[19].** The STSP is the problem of finding a minimal length closed tour that visits each city once; in this case there are some arcs (i, j) which belong to A, for which

$d_{ij} = d_{ji}$,if for at least one (i, j) $d_{ij} \neq d_{ji}$, then the TSP becomes an ATSP, the MTSP consists of finding tours for all m salesmen, who all start and end at the depot, such that each intermediate node is visited exactly once and the total cost of visiting all nodes is minimized. Possible variations of the problem are as follows: Single or multiple depots, Number of salesmen, Timeframe, Other constraints…etc. For each variation we obtain a new variant of the TSP for example when some nodes need to be visited in a particular time periods that are called time windows which are an extension of the MTSP, and referred as multiple traveling salesman problem with specified timeframe, in this case we have a MTSPTW (Multiple Traveling Salesman Problem with Time Windows) **[20].**

4. MATHEMATICAL FORMULATIONS OF TRAVELING SALESMAN PROBLEM:

A. The Danzig-Fulkerson-Johnson formulation:

The DFJ formulation and many ATSP formulations consist of an assignment problem with integrality constraint and subtour elimination constraints **(SECs) [10],** they use a binary variable x_{ij} equal to 1 if and only if arc (i, j) belongs to the optimal solution and otherwise it would be equal to 0, **S** is the sub set of vertices (cities) in the sub tour. The basic model is as follows:

$$\min \sum_{i \neq j} d_{ij} x_{ij} \tag{5}$$

Subject to

$$\sum_{j=1}^{n} x_{ij} = 1 \quad (\forall i \in V, i \neq j) \tag{6}$$

$$\sum_{i=1}^{n} x_{ij} = 1 \quad (\forall j \in V, i \neq j) \tag{7}$$

$$\sum_{i,j \in S} x_{ij} \leq |S| - 1 \quad (S \subset V, 2 \leq |S| \leq n-2) \tag{8}$$

$$x_{ij} \in \{0;1\} \quad \forall (i, j) \in A \tag{9}$$

(5) represents the objective function, (6) and (7) represent the assignment constraints, (8) is the subtour elimination constraints (9) is a integrality constraints**.** This formulation becomes difficult in practice, as soon as the size of the model increases, and it is due to the exponential number of subtour elimination constraints generated, this formulation of the TSP contains $n(n-1)$ variables and $2^{n-1} + n - 2$ constraints.

B. The Miller -Tucker-Zemlin formulation:

This formulation was originally proposed by Miller et all, for a traveling salesman problem , where the number of vertices of each route is limited **[17]**, they use one binary variable x_{ij} and for each arc $(i, j) \in A$, x_{ij} which takes 1 if the city j is visited immediately after i in the tour , otherwise it takes 0, they uses U_i variables to define the order in which each vertex i is visited on a tour, so the MTZ formulation of the TSP is :

$$\min \sum_{i \neq j} d_{ij} x_{ij} \tag{10}$$

Subject to

$$\sum_{j=1}^{n} x_{ij} = 1 \quad (\forall i \in V, i \neq j) \tag{11}$$

$$\sum_{i=1}^{n} x_{ij} = 1 \quad (\forall j \in V, i \neq j) \tag{12}$$

$$x_{ij} \in \{0;1\} \quad \forall (i, j) \in A \tag{13}$$

$$U_i - U_j + n x_{ij} \leq n - 1 \quad , \quad i, j = 2..n \tag{14}$$

$$1 \leq U_i \leq n - 1 \quad i = 2..n \tag{15}$$

(10) Represents the objective function, (11) and (12) are the assignment constraints, (13) is the integrality constraint, (14) are the subtour elimination constraints, (15) is the integrality constraint. This TSP formulation contains $(n-1)(n+1)$ variables, and $n^2 - n + 2$ constraints.

C. The Desrochers-Laporte formulation:

Among the advantages of the MTZ formulation is that the subtour elimination constraints (14) and (15) can be incorporated into other types of problem formulations together with stronger constraints **[18]**. Motivated by these facts, Desrochers and Laporte have lifted subtour elimination constraints (14) and (15) to obtain the stronger forms:

$$\min \sum_{i \neq j} d_{ij} x_{ij} \tag{16}$$

Subject to

$$\sum_{j=1}^{n} x_{ij} = 1 \quad (\forall i \in V, i \neq j) \tag{17}$$

$$\sum_{i=1}^{n} x_{ij} = 1 \quad (\forall j \in V, i \neq j) \tag{18}$$

$$x_{ij} \in \{0;1\} \quad \forall (i, j) \in A \tag{19}$$

$$U_i - U_j + (n-1) x_{ij} + (n-3) x_{ji} \leq n-2 \quad , \quad \text{i,j} = 2..n \tag{20}$$

$$1 \leq U_i \leq n-1 \quad , \quad \text{i} = 2..n \tag{21}$$

(16) is the objective function, the assignment constraints (17) and (18) ensure that each city is visited one and only once, (19)and (21) are the integrality constraints, (20) is the subtour elimination constraints .

5. SOLUTION METHODS FOR THE TRAVELING SALESMAN PROBLEM :

The methods of solving the TSP can be divided into two classes: the exact methods and approached methods, the exact methods make it possible to find the optimal solution, but their complexity is exponential, approached methods (Heuristic) get good solutions but give no guarantee on the optimality of the solution. In our case, for the resolution we used exact methods **[21]** , using a Integer Linear Programming Solver **(ILP-Solver)** , CPLEX version 12.6.3 which by default uses the Branch and Cut method , it is an accurate method for solving problems of linear optimization in integers, where we use the method of separation and evaluation (Branch and Bound) and the cutting plane method **.**

The Branch and Bound method uses two concepts: the branching that divides a set of solutions into subsets, the evaluation which consists in limiting or diminishing the solutions, and the cutting plane method used to find an integer solution of a linear optimization problem, the principle of this method is to add constraints to the linear program to refine it, and to bring it closer to integral solutions **[22].**

For the resolution of the MTZ formulation, we used the method Branch and Cut and for the DFJ given the exponential number of constraint of the order 2^n, it is necessary to proceed by generation of constraints **[23]** that consists to add to each iteration a set of subtour elimination constraints **(SECs)** until we find an optimal solution. First, we relax all SEC **(8)** from the model and solve the remaining ILP model. Then we check if the obtained integer solution contains subtours. If not, the solution is an optimal TSP tour. Otherwise, we find all subtours in the integral solution (which can be done by a simple scan) and add the corresponding **SEC** to the model, each of them represented by the subset of vertices in the corresponding subtour. The resulting enlarged ILP model is solved again to optimality (see algorithm 1).

Input: TSP instance;

Output: Optimal TSP tour;

1: define current model did (5), (6), (7), (9);
2: **repeat;**
3: solve the current model to optimality by ILP solver.
4: **if** solution contains no. sub tour **then**;
5: set the solution as optimal tour;
6: **else;**
7: find all sub-tours of the solution and add the corresponding SEC into the model.
8: **end if;**
9: **until** optimal tour found.

Algorithm 1: Main idea of generation constraints

6. RESULTS AND DISCUSSTION

Experiments were performed on a processor Intel® Celeron® CPU 847 of 1.10 GHz, 1.10 GHz with 4 GB Memory. The solution has been provided by IBM ILOG CPLEX version 12.6.3; we have developed programs in OPL language (Language Modeler of the CPLEX Solver). To compare the formulations MTZ, DL and DFJ we have generated small random instances similar to some of the TSPLIB instances to obtain 10 complete graphs from 7 to 45 vertices. These vertices have taken between 0 and 99 coordinates randomly; the distances between vertices are the Euclidean distances (d_{ij}). We have calculated the Continuous Relaxation Value **(R)** (obtained by relaxing the integrality constraints), and the Optimal Value **(Vopt)**, with **T(s) is** the calculation time of the Optimal Value in second, **(NC)** is the number of generated constraints and **n** is the number of cities . Table 1 shows the results.

Table 1: Comparison of formulations DFJ, MTZ and DL (Optimal Value (Vopt), Solution time T(s), Continuous Relaxation Value(R))

n	MTZ				DL				DFJ				
	Vopt	R	%	T(s)	Vopt	R	%	T(s)	Vopt	R	%	T(s)	NC
7	282.29	207.79	73.60	0.71	282.29	274.74	97.32	0.49	282.29	282.29	100.00	0.26	10
10	337.67	256.78	76.04	0.87	337.67	263.76	78.11	0.53	337.67	322.5	95.50	0.37	11
18	406.8	340.71	83.75	2.94	406.8	360.98	88.73	2.13	406.8	406.15	99.84	0.39	27
20	412.79	344.37	83.42	2.51	412.79	366.97	88.89	2.44	412.79	411.13	99.59	0.47	35
26	469.45	397.37	84.64	41.16	469.45	439.57	93.63	27.79	469.45	439.57	93.63	0.57	38
29	482.43	377.42	78.23	182.7	482.43	446.24	92.49	174.56	482.43	446.24	92.49	4.78	43
31	502.54	404.38	80.46	187.1	502.54	478.39	95.19	180.16	502.54	483.61	96.23	2.32	53
35	-	-	-	-	-	-	-	-	518.961	494.99	95.38	3.71	58
40	-	-	-	-	-	-	-	-	532.448	505.00	94.84	7.64	88
45	-	-	-	-	-	-	-	-	537.68	527.00	98.14	11.2	68

It is apparent from the results that:

- The Continuous Relaxation Value **(R)** of the DFJ formulation is better than that of the MTZ and the DL formulations (Fig.2).
- The Calculation time **T(s)** of the DFJ formulation is better than that of the MTZ and the DL (Fig.3).
- From 35 cities, MTZ and DL formulations become very slow in time calculation.(We determined a maximum time to take our results T_{max} = 10 min, after 10 min, the MTZ and DL formulation did not yet give an optimal solution)

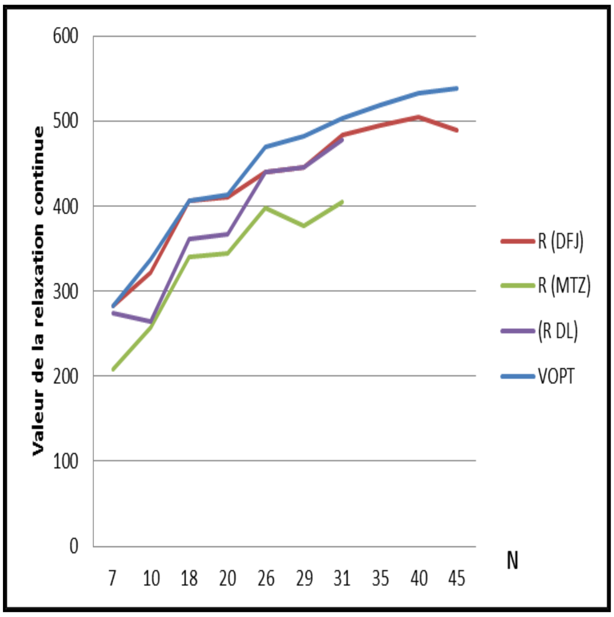

Figure 2.The Continuous Relaxation Value (R) of the DFJ, MTZ and DL formulations

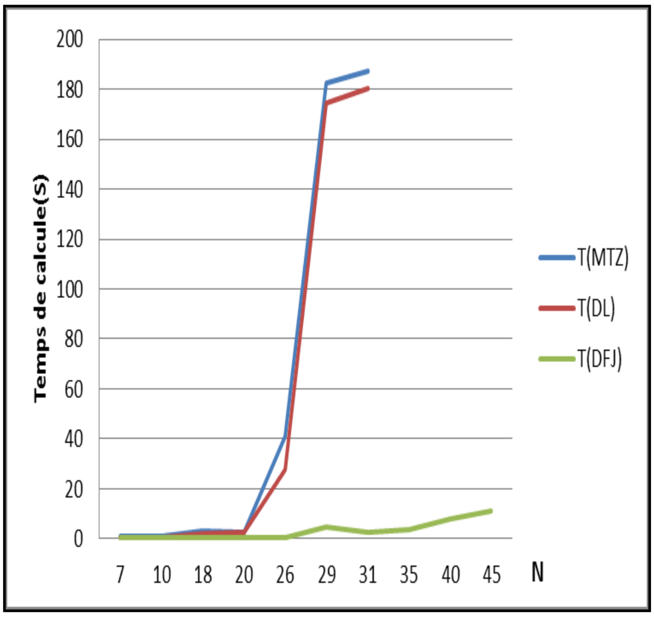

Figure 3. The Calculation time of the DFJ, MTZ and DL formulations

CONCLUSION

The study leads to the following conclusions:

- The MTZ formulation is seductive because it is easy to implement but it gives a low continuous relaxation.
- In practice it is more interesting to use the DFJ formulation, even this requires to implement the generation of constraints.
- The DL formulation, although significantly strengthens the MTZ formulation, does not compete with the DFJ formulation.

So finally we can say that in practice, the DFJ is better than that the MTZ and DL formulation.

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