Hexagonal-8-QAM Constellation with Low PMEPR codes

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Abstract—A construction of hexagonal shaped 8-point QAM called H8QAM is shown. H8QAM constellation outperforms the conventional rectangular constellation and 8-PSK. Furthermore, construction of H8QAM Golay sequences having low peak-to-mean envelope power ratios (PMEPR) from two QPSK Golay sequences is presented. Various upper bounds on peak envelope powers of these sequences are evaluated.

I. INTRODUCTION

Quadrature amplitude modulation (QAM) is a very popular constellation in the literature [1], [2], [3]. QAM occurs in many arrangements and the one that is most commonly encountered in literature is square arrangement. Non-square QAMs and their advantages are known for a long time. They can be differentially detected with non-coherent techniques which are computationally simpler than coherent detection [4], [2] and they have lower average power. In this article, the focus is on Hexagonal-8-QAM (H8QAM) constellation and its application in orthogonal frequency-division multiplexing (OFDM). H8QAM is chosen among other 8-point constellations because of its unique construction suited for application in OFDM.

OFDM has become a most favored technique for LTE (long term evolution) standards due to susceptibility to signal dispersion under multipath conditions. A major drawback of employing OFDM is the high peak-to-mean envelope power ratio (PMEPR) of uncoded OFDM signals. High PMEPR leads to spectral growth of the OFDM signal in the form of intermodulation among subcarriers and out-of-band radiation. Therefore, expensive transmit amplifiers with large linear range must be used and portable device’s power consumption is high leading to short battery life.

Many solutions have been proposed to control PMEPR in OFDM [5]. One approach is designing codes that not only provide error correction but also reduce the PMEPR [6], [7], [8]. It has been known since the work of Popovic and Boyd [9], [10] that the use of Golay complementary sequences (GCS) [11] as codewords to control the modulation of carrier signals results in OFDM with PMEPR of at most 2. Davis and Jedwab [12] made a major theoretical advance, discovering that the large sets of binary length $2^n$ Golay complementary pairs described in [11] can be obtained from certain second-order cosets of the classical first order Reed-Muller Code. Special cases of these codes were given in [13], and the underlying theory was developed in [14]. However, the aforementioned codes are defined over the phase-shift keying (PSK) signal constellations. Codes based on popular constellations such as square-QAM were shown in [15], [16], [17]. Codes constructed from Golay complementary sequences are employed as pilot sequences by the European Telecommunications Standards Institute (ETSI) Broadband Radio Access Networks (BRAN) committee. The construction of low PMEPR codes for non-square QAM as a scaled set sum of two quadrature phase-shift keying (QPSK) constellations is given in [4]. Motivated by the work on STAR-QAM in [4], we present the construction of H8QAM and it is shown that such codes with low PMEPR upper bounded by 3 (4.77dB).

The rest of the paper is organized as follows: Section II shows the construction of H8QAM constellation. Section III discusses the symbol error rate analysis and comparison with various 8-point constellations. Section IV describes the OFDM system model and defines PMEPR. Section V describes PMEPR for H8QAM Golay sequences using QPSK Golay sequences and conclusion is in Section VI.

II. CONSTRUCTION OF H8QAM

The QPSK constellation set, denoted by $Q$, can be realized as $Q \equiv j^x$ where $x \in \mathbb{Z}_4 \equiv \{0, 1, 2, 3\}$ and $j = \sqrt{-1}$. A unique n-tuple QPSK sequence $u = (u_0, u_1, \ldots, u_{n-1})$ can be associated with a unique sequence $x = (x_0, x_1, \ldots, x_{n-1})$, using $u_i = j^{x_i}$ where $x_i \in \mathbb{Z}_4$, $0 \leq i \leq n-1$. Let $\mathbb{Z}_4 \equiv \{0, 2\}$ as a subset of $\mathbb{Z}_4$ where we limit the possible values to be even. Another constellation QPSK set, denoted by $\hat{Q}$, can be realized as $\hat{Q} \equiv j^{y}$ where $y \in \hat{\mathbb{Z}}_4$.

Definition 1: Let $A$ and $B$ be the sets of complex numbers. Then the set sum of $A$ and $B$ is defined as

$$A \oplus B \equiv \{r + w | r \in A, w \in B\}. \quad (1)$$

Definition 2: Let $h$ be a complex number and $A$ be the set of complex numbers. Then the product of $h$ and $A$ is defined as

$$hA \equiv \{hr | r \in A\}. \quad (2)$$

Many solutions have been proposed to control PMEPR in OFDM [5]. One approach is designing codes that not only provide error correction but also reduce the PMEPR [6], [7], [8]. It has been known since the work of Popovic and Boyd [9], [10] that the use of Golay complementary sequences (GCS) [11] as codewords to control the modulation of carrier signals results in OFDM with PMEPR of at most 2. Davis and Jedwab [12] made a major theoretical advance, discovering that the large sets of binary length $2^n$ Golay complementary pairs described in [11] can be obtained from certain second-order cosets of the classical first order Reed-Muller Code. Special cases of these codes were given in [13], and the underlying theory was developed in [14]. However, the aforementioned codes are defined over the phase-shift keying (PSK) signal constellations. Codes based on popular constellations such as square-QAM were shown in [15], [16], [17]. Codes constructed from Golay complementary sequences are employed as pilot sequences by the European Telecommunications Standards Institute (ETSI) Broadband Radio Access Networks (BRAN) committee. The construction of low PMEPR codes for non-square QAM as a scaled set sum of two quadrature phase-shift keying (QPSK) constellations is given in [4]. Motivated by the work on STAR-QAM in [4], we present the construction of H8QAM and it is shown that such codes with low PMEPR upper bounded by 3 (4.77dB).

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Consequently, we can write 

\[ y = \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_{n-1} \end{pmatrix} \]

associated with two sequences \( u \) and \( H8QAM \), where \( u_k \in H, k = 0, 1, \ldots, n - 1 \) can be uniquely associated with two sequences \( x = (x_0, x_1, \ldots, x_{n-1}) \) and \( y = (y_0, y_1, \ldots, y_{n-1}) \) such that \( (x_k, y_k) \in Z_4 \times Z_4 \). Consequently, we can write

\[ u_k = \frac{1}{\sqrt{5}} j^{x_k} + \frac{2}{\sqrt{5}} j^{y_k} \]

According to these definitions, the H8QAM constellation is constructed as follows:

\[ H = \frac{1}{\sqrt{Q}} Q \oplus \frac{2}{\sqrt{Q}} Q \]

(3)

The corresponding constellation is shown in Fig. 1, where a rotation phasor (45 degrees) has been applied to (3) for the illustration purpose. A sequence \( u = (u_0, u_1, \ldots, u_{n-1}) \) of H8QAM, where \( u_k \in H, k = 0, 1, \ldots, n - 1 \), can be uniquely associated with two sequences \( x = (x_0, x_1, \ldots, x_{n-1}) \) and \( y = (y_0, y_1, \ldots, y_{n-1}) \) such that \( (x_k, y_k) \in Z_4 \times Z_4 \). Consequently, we can write

\[ u_k = \frac{1}{\sqrt{5}} j^{x_k} + \frac{2}{\sqrt{5}} j^{y_k} \]

(4)

III. SER of Various 8-point Constellations

In order to determine the error performance for any QAM, signal constellation must be specified. There are many possible configurations for 8-point constellation \((M = 8)\) in the literature [1], [2]. We shall consider six configurations including the proposed constellation, H8QAM, to compare. Let \( d_{ij} \) represent the Euclidean distance between \( i \)-th and \( j \)-th signal point in the constellation, then minimum distance of constellation is defined as \( d_{\text{min}} = \arg \min_{i \neq j} |d_{ij}| \). All constellations shown in Table. 1 are constructed such that \( d_{\text{min}} = 2 \). Let \( x + iy \) denote \( i \)-th point, then its power is calculated as, \( |A_i|^2 = x^2 + y^2 \). Assuming that all signal points are equally probable, the average transmitted signal power is given as:

\[ P_{av} = \frac{1}{M} \sum_{i=1}^{M} |A_i|^2. \]

(5)

The ratio of the average-signal-power to the average-noise-power can be expressed as follows (Eq.(6) in [1]):

\[ \rho = \frac{1}{8\sigma^2} \sum_{i=1}^{M} |A_i|^2, \]

(6)

where \( \sigma^2 \) is the single-sided power spectral density of the white Gaussian noise. Closed form expressions for symbol error rate (SER) for the all constellations in Table. I do not exist, therefore to compare we use a union bound of SER (Eq. (8) in [1]):

\[ P_e = \frac{\sigma}{8\sqrt{\pi}} \sum_{i=1}^{8} \sum_{k=1}^{8} \exp\left[-|A_i - A_k|^2/(4\sigma^2)\right] \]

(7)

The probability error performance curves obtained by (7) is very close to the exact SER when \( P_e < 10^{-2} \) [1]. The probability of error depends on minimum distance between pairs of signal points and average transmitter power. As can be seen from the figure H8QAM constellation outperforms 8-psk, 8-AMPM and square 8 QAM. H8QAM performance is outperformed by circle71QAM by a small margin. Between six
configurations for 8-point constellation given in Table. I only two configurations can be constructed as a set sum of two QPSK constellations namely H8QAM and 8-AMPM, which is the key to developing low PMEPR codes. Comparing these two, H8QAM certainly has better error performance.

Now we discuss characteristic that influence constellation design in terms of SNR efficiency. Assuming that all points are equally likely in a given constellation, the key parameter that determine the SNR efficiency are the minimum squared distance \( d_{\text{min}}^2 \) between its points, and its average power \( P_{\text{av}} \). We generally wish to maximize \( d_{\text{min}}^2 \) for a given \( P_{\text{av}} \), or to minimize \( P_{\text{av}} \) for a given \( d_{\text{min}} \). Therefore, defining constellation figure of merit (CFM), as the ratio [3]:

\[
\text{CFM} = \frac{d_{\text{min}}^2}{P_{\text{av}}} \quad (8)
\]

By keeping \( d_{\text{min}} \) constant, in Table. II we give values of CFM for all constellations under consideration. Again, H8QAM has a better CFM of 0.8 than 8-AMPM’s CFM, which is 0.667. Thus, our motivation to select H8QAM among various 8-point constellations can be summarized as (a) H8QAM can be constructed by set sum of two QPSK constellations (b) Given (a) is satisfied, better CFM and SER.

**IV. OFDM**

The basic block diagram of the OFDM system is shown in Fig. 3. Let the number of subcarriers be \( n \) for the OFDM system. Consider a sequence \( u = (u_0, u_1, \cdots, u_{n-1}) \) of length \( n \) consisting of elements of an \( M \)-point constellation. The collection of all \( M^n \) distinct sequences forms a code \( C \). The time domain OFDM signal after undergoing Inverse Fast Fourier Transform (IFFT) operation is given by

\[
S_u(t) = \sum_{i=0}^{n-1} u_i e^{j2\pi f_i t} \quad (9)
\]

for \( 0 \leq t \leq T \), where \( \Delta f = f_{i+1} - f_i \) is an integer multiple of time period of OFDM symbol \( T \). The value of \( \Delta f \) depends on the guard time and the cyclic prefix but we ignore this matter since it has no direct impact on the current work. This is chosen in order to maintain the orthogonality of subcarriers frequencies. The transmitted OFDM signal is the real part of the complex signal \( S_u(t) \). The real part of the transmitted complex envelope for a given codeword \( u \) is given by \( \Re(S_u(t)) \) and its instantaneous envelope power is equal to

\[
P_u(t) = \left| S_u(t) \right|^2 = \left( \sum_{i=0}^{n-1} u_i e^{j2\pi f_i t} \right)^2 = \sum_{i=0}^{n-1} \sum_{k=0}^{n-1} u_i^* u_k e^{-j2\pi (i-k) \Delta f} \quad (10)
\]

Thus, the mean power of \( S_u(t) \) during a symbol period is

\[
\frac{1}{T} \int_0^T P_u(t) dt = \left| u \right|^2 = \sum_{k=0}^{n-1} \left| u_k \right|^2.
\]

We define peak envelope power (PEP) of a sequence or codeword as the maximum instantaneous power of \( S_u(t) \) within \( T \) denoted by

\[
\text{PEP}(u) = \max_{0 \leq t \leq T} P_u(t).
\]

and the PMEPR of a code is

\[
\text{PMEPR}(C) = \max_{u \in C} \frac{\text{PEP}(u)}{P_{\text{av}}(C)}
\]

where \( P_{\text{av}}(C) \) is the mean envelope power of an OFDM signal in one symbol period averaged over all OFDM signals generated from \( C \).

\[
P_{\text{av}}(C) = \frac{1}{T} \sum_{u \in C} P(u) = \frac{1}{T} \sum_{u \in C} \int_0^T P_u(t) dt = \frac{1}{T} \sum_{u \in C} p(u) \left| u \right|^2
\]

where \( p(u) \) is the probability of transmitting the codeword \( u \). Since all codewords are assumed equally probable and for QPSK constellation in which all symbols have unit energy i.e., \( \left| u \right|^2 = n \). Therefore we get

\[
P_{\text{av}}(C) = n. \quad (11)
\]

**V. PMEPR OF H8QAM**

In this section, we find PMEPR of OFDM signal corresponding to H8QAM constellation. The transmitted OFDM signal corresponding to the sequence \( u \) of H8QAM is given by

\[
S_u(t) = \sum_{i=0}^{n-1} u_i e^{j2\pi f_i t} \quad \text{def} \quad S_{x,y}(t) = \frac{1}{\sqrt{5}} S_x(t) + \frac{2}{\sqrt{5}} S_y(t), \quad (11)
\]
\[ S_k(t) = \sum_{i=0}^{n-1} j^{x_i} e^{j2\pi i \Delta ft} \]

and

\[ S_y(t) = \sum_{i=0}^{n-1} j^{y_i} e^{j2\pi i \Delta ft} \]

such that \((x_i, y_i) \in \mathbb{Z}_4 \times \mathbb{Z}_4\). The instantaneous envelope power of the transmitted signal given by (11) is given as

\[ P_x(t) = \left| S_x(t) \right|^2 = \frac{1}{\sqrt{5}} S_x(t) + \frac{2}{\sqrt{5}} S_y(t) \]  

(12)

Consider two complex valued sequences \(x = (x_0, x_1, \ldots, x_{n-1})\) and \(y = (y_0, y_1, \ldots, y_{n-1})\) of length \(n\) satisfying

\[ C_x(\eta) + C_y(\eta) = (||x||^2 + ||y||^2) \delta(\eta) \]  

(13)

where \(C_x(\eta) = \sum_{i=0}^{n-1} x_i x_{i+\eta}^*\) is the aperiodic autocorrelation of sequence \(x\) at delay shift \(\eta\) and \(\delta(\eta)\) is the Kronecker function. Then, \(x\) and \(y\) are called Golay complementary sequences (GCS) [11]. We say that \(x\) is a Golay sequence (GS) if there exists a sequence \(y\) that is complementary to \(x\). Popovic made this important observation [18] and also reported in [14], [16], [17].

**Lemma 1:** Let \(x\) and \(y\) of length \(n\) be two sequences form a GCS such that any combination of \((x_k, y_k)\) where \(k = 0, 1, \ldots, n-1\) can be used to generate a H8QAM symbol as in (4). Consider the following two summations:

\[ S_k(t) = \sum_{k=0}^{n-1} j^{x_k} e^{-j2\pi k \Delta ft} \]

\[ S_y(t) = \sum_{k=0}^{n-1} j^{y_k} e^{-j2\pi k \Delta ft} \]

Then for any \(t\), the following inequality hold:

\[ |S_k(t)| \leq \sqrt{2n}, |S_y(t)| \leq \sqrt{2n} \]  

(14)

The proof of lemma 1 is based on the key property: \(|S_k(t)|^2 + |S_y(t)|^2 = 2n\).

Consider the generalized form of (12):

\[ P_{x, y}(t) = |S_{x, y}(t)|^2 = \frac{1}{\sqrt{5}} S_x(t) + \frac{2}{\sqrt{5}} S_y(t) \]  

(15)

For any sequence \(x \in \mathbb{Z}_4\), we define \(x + 2 = (x_0 + 2, x_1 + 2, \ldots, x_{n-1} + 2)\). Accordingly, we have

\[ S_{x+2}(t) = \sum_{k=0}^{n-1} j^{x_k} j^{2} e^{-j2\pi k \Delta ft} \]

\[ = -S_k(t) \]  

(16)

Based on the results obtained in Lemma 1, we derive a theorem for peak envelope power of \(P_{x, y}(t)\). **Theorem 1:** If \(x\) and \(y\) form a GCS, then \(P_{x, y}(t) \leq 4n\).

**Proof:** Expanding (15) and using (16), we get

\[ P_{x, y}(t) = \frac{1}{5} |S_x(t)|^2 + \frac{4}{5} |S_y(t)|^2 + \frac{2}{5} |S_x(t)S_y(t) + S_y(t)S_x(t)| \]

(17)

Adding (17) and (18), we get

\[ P_{x, y}(t) + P_{x, y+2}(t) = 2 \left( \frac{1}{5} |S_x(t)|^2 + \frac{4}{5} |S_y(t)|^2 \right) \]

(19)

applying the inequalities of (14) and since \(P_{x, y+2}(t) \geq 0\), we have

\[ P_{x, y}(t) \leq 4n. \]

VI. Conclusion

In this paper, we study the H8QAM constellation in terms of its constellation figure of merit (CFM) and initiated the study of H8QAM Golay complementary sequences. Constellation figure of merit of H8QAM is higher than other 8-point constellations. Construction of H8QAM Golay complementary sequences using two PSK Golay sequences is shown. When Golay complementary sequences are employed, it is shown that PMEPR is upper bounded by 3 (4.77 dB).

**References**


